TWO-PARAMETER NONCOMMUTATIVE GAUSSIAN PROCESSES

by

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Abstract

The reality of billion-user networks and multi-terabyte data sets brings forth the need for accurate and computationally tractable descriptions of large random structures, such as random matrices or random graphs. The modern mathematical theory of free probability is increasingly giving rise to analysis tools specifically adapted to such large-dimensional regimes and, more generally, *non-commutative probability* is emerging as an area of interdisciplinary interest.

This thesis develops a new non-commutative probabilistic framework that is both a natural generalization of several existing frameworks (viz. free probability, q-deformed probability) and a setting in which to describe a broader class of random matrix limits. From the practical perspective, this new setting is particularly interesting in its ability to characterize the behavior of large random objects that asymptotically retain a certain degree of commutative structure and therefore fall outside the scope of free probability. The type of commutative structure considered is modeled on the two-parameter families of generalized harmonic oscillators found in physics and the presently introduced framework may be viewed as a two-parameter deformation of classical probability. Specifically, we introduce (1) a generalized Non-commutative Central Limit Theorem giving rise to a two-parameter deformation of the classical Gaussian statistics and (2) a two-parameter continuum of non-commutative probability spaces in which to realize these statistics. The framework that emerges has a remarkably rich combinatorial structure and bears upon a number of well-known mathematical objects, such as a quantum deformation of the Airy function, that had not previously played a prominent role in a probabilistic setting. Finally, the present framework paves the way to new types of asymptotic results, by providing more general asymptotic theorems and revealing new layers of structure in previously known results, notably in the "correlated process version" of Wigner's Semicircle Law.

Thesis Supervisor: Todd Kemp
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"Live Free or Die"
– The state motto of New Hampshire.

Non-commutative probability theory is new field of mathematics that is progressively changing how we think, both as theorists and practitioners, of randomness at large scales. I was fortunate to have arrived to this research area, which aligned well with my intuition and stirred up my curiosity, but the path that led to it was certainly not a straightforward one. This thesis was shaped by the influence, encouragement, and support from great many people and the following is a faulty and, above all, incomplete attempt to acknowledge that.

My foremost thanks go to Todd Kemp and Philippe Biane.

Todd Kemp introduced me to free probability, and to non-commutative probability at large and I am fortunate to have had him as a mentor. He knew to give me both independence and support, so that my transition into this field happened quickly and naturally. He has taught me much and his enthusiasm for the subject has stuck with me ever since. He remains what I consider to be a model of a bright young faculty.

I had the privilege of spending the final year of my Ph.D. in Paris under the mentorship of Philippe Biane. His lucidity and breadth of perspective have had an enormous influence on me. I gained much from our conversations, above all the sense that if one is able to bore deep enough into the fabric of mathematics, new layers of unifying structure can emerge. Working with a polymath has changed my view of mathematics on the whole.

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To my mother, Ljubinka, for being positive, strong, and passionate.

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Every word of his thesis goes to them.

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Chapter 1

Introduction

Scientific and technological models routinely draw on the notion of *randomness* in order to abstract away any characteristics of the system deemed secondary or substitute for a lack of information. Thus, in progressively coming to face larger and more complex systems – either those, like the communication networks, that are of our own making or those, like the biological networks, which are not – we must also learn to account for more complex forms of uncertainty. The underlying issue is frequently that of "dimensionality", by which the computational complexity may render useless an otherwise perfectly satisfactory model of a large system. As an elementary example, taking two large matrices A and B (with A modeling, say, a communication channel and B the additive noise), computing the spectrum of the sum $A + B$ is conceptually straightforward, but computationally complex even when the individual spectra for A and B are known, as their relative eigenspaces must also be taken into account. Thus, regardless of the computational resources at one's disposal, for A and B sufficiently large, the operation becomes strictly prohibitive.

Whereas the simulations or numerical computations become prohibitive as the dimensionality of the system increases, emerging mathematical theories offer hope that the behaviors of "very large systems", that is, behaviors in the limit of the increasing system dimension, may in fact be described by powerful mathematical frameworks. In particular, the modern field of *free probability* [Voi86, VDN92] (see also introductory treatments [NS06, Bia03]) has so far shown much promise in capturing the behavior of certain classes of large random matrices. Returning to the matrix addition example, free probability has shown that for "almost any" pair of large enough matrices A and B, the spectrum of $A+B$ is well approximated by the so-called *free additive convolution* of the individual spectra of A and B. (For a precise statement, the reader is referred to [Voi91], as well as to the expository article [Bia03].) The free additive convolution is an algebraic procedure that is computationally efficient and the free probabilistic framework has indeed already lead to more powerful computational tools (see [RE08]). Moreover, characterization problems involving large systems in *wireless communications* have benefitted from advances in *random matrix theory* and its interface to free probability (see e.g. [AGZ10] for general theory and [TV04] for applications at hand) and, more recently, to emerging notions such as *operator-valued free probability* (see [RFOBS08] and the references therein).

Random matrix theory, free probability, and operator-valued free probability all belong to the broader field of *non-commutative probability.* Non-commutative probability takes root in *quantum probability* [HP84] (see also [Bia95, Mey93, Par92]), which broadly refers to a generalization of the classical probability theory to encompass *quantum observables*, represented as bounded linear operators on some separable Hilbert space. Recall that in the Kolmogorov's (classical) formulation, the probability triple $(\Omega, \Psi, \mathbb{P})$ can be seen as encoding the information available through an experiment on the space Ω, with the σ-field Ψ representing the measurable outcomes and the probability measure $\mathbb{P} : \Psi \to [0,1]$ the corresponding probabilities. In the non-commutative setting, the probability triple is replaced by the *non-commutative probability space* $(\mathcal{A}, \varphi)$, where $\mathcal{A}$ is a unital $*$-algebra whose elements are interpreted as *non-commutative random variables* and the unital linear functional φ plays the role of the expectation. For example, letting the algebra $\mathcal{A}$ contain $N \times N$ matrices (which are bounded linear operators on the Hilbert space $\mathbb{R}^N$) and the functional φ be their normalized trace $N^{-1}\mathrm{Tr}$, the notion of "matrix limits" becomes that of the convergence of their *moments*, i.e. of the sequences of complex numbers obtained by evaluating φ on elements of $\mathcal{A}$. The power of this general approach now comes from being able to realize these limits as elements of some other non-commutative probability space $(\mathcal{A}', \varphi')$ and, in turn, in representing whenever possible the elements of $\mathcal{A}'$ as operators on an infinite-dimensional Hilbert space. In this asymptotic regime, much of the structure that impeded the analysis in the finite-dimensional setting may vanish, yielding a framework in which sophisticated results may then be derived.

This "vanishing of structure" roughly described in the preceding paragraph is an attribute of matrix limits that fall into the framework of free probability. The latter can be seen as characterized by an *absence of commutation structure* (as made precise in e.g. [VDN92]). But, in practice, one must also deal with structures where the commutation structure does not vanish entirely and remains present, albeit in a potentially simpler form. After all, the world of classical probability is that in which the random variables *commute* and, at the opposite end, there is also the world of *anti-commutative* probability associated with the *fermionic* statistics in physics. Up until now, the most general non-commutative framework with well-developed probabilistic intuition built on explicit commutation structure was that of *q-commutation relations.* The latter are a single-parameter family of commutation relations encompassing the classical ($q = 1$), free ($q = 0$) and fermionic ($q = -1$) settings. This "q-deformed" probabilistic framework was introduced in the seminal 1991 paper of Bożejko and Speicher [BS91] and further developed by many authors over the course of the following two decades (a partial survey may be found in the present Section 1.2).

This thesis develops a more general non-commutative probabilistic framework centered around a second-parameter refinement of the q-commutation relations, viz. the (q,t)-commutation relations. Beyond comprising the classical, free, fermionic, and q-deformed frameworks, this generalized setting comes with richer combinatorial statistics, a more general version of the non-commutative *Central Limit Theorem* [Spe92], and a number of fundamental insights into the limits of certain random matrix ensembles (which may now include *time-correlated* random matrices) and the nature of certain *special functions* and *hypergeometric series.* Analogously to the approach taken by Bożejko and Speicher, which can also be used to formulate the classical, free, and fermionic probability theories, the framework developed in this thesis principally draws on the general structure of *Fock spaces* in order to construct Hilbert-space representations of operators (and operator limits) of interest and realize the desired commutativity structure. For this purpose, this thesis introduces the (q,t)-*Fock space*, a second-parameter deformation of the q-Fock space of Bożejko and Speicher [BS91]. In light of the physical intuition, the (q,t)-Fock space is found to provide a missing Hilbert-space realization of systems of certain deformed harmonic oscillators, introduced in the physics litterature by Chakrabarti and Jagannathan in 1991 [CJ91]. The remainder of this introduction reviews the physical and the probabilistic intuition behind classical and deformed Fock spaces, before providing a more detailed overview of the main contributions of this thesis.

1.1 Quantum fields and Fock spaces

Particles have either integer or half-integer spin: *bosons* are particles with spins $0, 1\hbar, 2\hbar, \ldots$, whereas *fermions* are particles with spins $\frac{1}{2}\hbar, \frac{3}{2}\hbar, \ldots$. The key aspect of the boson statistics is the fact that, according to the *spin-statistics theorem* of quantum field theory, the multi-particle wavefunction of a system with identical (undistinguishable) bosons is *symmetric*, that is, remains unchanged under exchange of any two particles. In contrast, the wave function of a system with fermions is *anti-symmetric*, in that an exchange of any two fermions incurs a change of phase by π.

The (deterministic, continuous) time-evolution of a quantum system is described by the *Schrödinger equation* $i\hbar\frac{\partial}{\partial t}\psi = \hat{H}\psi$, where $\hat{H}$ is a *Hamiltonian operator*, giving the total energy of the wavefunction. For a system with n particles, the corresponding n-body Schrödinger equation encodes the evolution of the system, but manipulations soon become unwieldy. As special relativity is inconsistent with single-particle quantum mechanics, the need for a convenient description of systems where particles could be created and destroyed lead to the formalism of *quantum field theory.* The amplitude of a field, such as the electro-magnetic field, is now quantized, with the corresponding quanta identified with individual partices. For this reason, the state of a quantum system containing identical bosons is described via *symmetric tensors* which generate the Hilbert space referred to as the *symmetric Fock space.* The state of a system containing identical fermions is described via *anti-symmetric tensors* which generate the *anti-symmetric Fock space.* The evolution of a field is now described by operators acting on a Hilbert space. In particular, the creation of a particle is represented by a *creation operator* and the destruction of a particle by an *annihilation operator. Observables* are represented by self-adjoint operators – for instance, the number of particles in a quantum state is obtained by the *number operator* – and the probabilistic aspects of quantum observables are generally encoded at the level of their *spectral measures.*

1.2 Quantum probability and beyond

The probability theory that can be naturally represented on the symmetric Fock space is simply the classical probability. In particular, the symmetric Fock space modeled on a *Gaussian Hilbert space* is closely tied to the *Wiener chaos decomposition* ([Wie38, Ito51, Seg56], see also [Jan97]). That the bosonic setting is inherently classical is readily seen from its commutative structure, as the annihilation operators a_i^b and creation operators $(a_j^b)^*$ satisfy the *canonical commutation relation (CCR)*:

$$(a_i^b)(a_j^b)^* - (a_j^b)^*(a_i^b) = \delta_{i,j}, \tag{CCR}$$

with the additional relations $a_i^b a_j^b = a_j^b a_i^b$ and $(a_i^b)^*(a_j^b)^* = (a_j^b)^*(a_i^b)^*$. In contrast, the fermionic setting is that of anti-commutative probability as the corresponding annihilation operators a_i^f and creation operators $(a_j^f)^*$ satisfy the *canonical anti-commutation relation (CAR)*:

$$(a_i^f)(a_j^f)^* + (a_j^f)^*(a_i^f) = \delta_{i,j} \tag{CAR}$$

with, additionally, $a_i^f a_j^f = -a_j^f a_i^f$ and $(a_i^f)^*(a_j^f)^* = -(a_j^f)^*(a_i^f)^*$. In the fermionic setting, the *field operator* $a_i^f + (a_i^f)^*$ can be seen to play the role analogous to the Gaussian random variable in classical probability (see e.g. [Bia95, Mey93, Par92]).

A q-deformation of the CCR and CAR relations given by $(a_i)(a_j)^* - q(a_j)(a_i) =$

$\gamma_{i,j}\delta_{i,j}$, for $q \in [-1,1]$ and $\gamma(i,j)$ some positive-definite function, was first considered in the quantum probabilistic context by Frisch and Bourret [FB70]. (The corresponding relation with $i = j$ and $\gamma(i,j) = 1$ already existed in the litterature, introduced by Arik and Coon [AC76] along with its *Bargmann-Fock* representation.) Frisch and Bourret were interested in the properties of the "parastochastic" random variable $a_i + a_i^*$. However, it was only two decades later, in Bożejko and Speicher's independent study of deformed canonical (anti-)commutation relations [BS91], that the existence question was resolved and such processes constructed. In particular, Bożejko and Speicher constructed a suitably deformed Fock space on which the creation and annihilation operators (analogues of those on the classical Fock spaces) satisfied the deformed commutation relation:

$$a_q(h)a_q(g)^* - q\, a_q(g)^* a_q(h) = \langle h, g\rangle_{\mathscr{H}} 1, \quad q \in [-1,1] \tag{q-CR}$$

for all h, g elements of some underlying Hilbert space $\mathscr{H}$.

Following much interest over the course of the two subsequent decades, the q-Fock space is now known to have a number of remarkable properties. Far from attempting to survey them all, we now focus on the probabilistic aspects of the algebras of bounded linear operators and point out a few key results. Namely, starting with the very motivation, it was shown that the deformed *field* operator $s_q(h) := a_q(h)^* + a_q(h)$ is a natural deformation, in the setting of (q-CR), of the Gaussian random variable. For that reason $s_q(h)$ is referred to as being a *q-Gaussian element* of the algebra of bounded linear operators on the q-Fock space. This interpretation is immediately apparent from the corresponding *Wick formula* and its orthogonalizing polynomial sequence [BS91, BKS97] (see [MN01, Kem05] for the q-analogue of the *complex* Gaussian). Furthermore, analogously to their classical counterparts, the q-Gaussians can indeed be represented as limits of a particularly natural form of a non-commutative Central Limit Theorem, due to Speicher [Spe92]. With regard to more advanced results, the *q-Brownian motion* was developed in [BKS97] and the q-deformed stochastic calculus in [DM03] (see also [Ans01] and [Śni01]), along with an *Itô formula* and relevant representation theorems. Analogues of sophisticated integration theorems of stochastic analysis were recently also shown to hold in the q-deformed setting [DNN].

There is currently also much interest in the q-Fock space that goes far beyond the probabilistic motivations. From an operator algebraic perspective, it had long been conjectured that the *q-Gaussian algebra*, namely the von Neumann algebra $\{s(h) \mid h \in \mathscr{H}\}''$ over an n-dimensional Hilbert space $\mathscr{H}$ is isomorphic to the free group factor with n generators, itself one of the most mysterious elementary objects in the theory of operator algebras. After several decades of increasingly supportive evidence (see e.g. [BKS97, Śni04, Ric05, Shl04, Dab10, KN11]), the conjecture was very recently verified by Guionnet and Shlyakhtenko [GS12] for all sufficiently small values of q.

1.3 Main contributions

The principal contributions of this thesis are the following:

1. The construction of a natural second-parameter refinement of the q-Fock space of Bożejko and Speicher [BS91], by which the "q-symmetrization" step given in [BS91] in terms of permutation inversions is now replaced by "(q,t)-symmetrization" that takes into account permutation *inversions and co-inversions.*

2. The introduction of a new two-parameter family of Gaussian statistics, viz. *the (q,t)-Gaussian statistics*, with the corresponding *Wick formula* refined as a function of *crossings* and *nestings* in pair-partitions.

3. The generalization of the non-commutative Central Limit Theorem of Speicher [Spe92] in which the underlying commutative structure is broadened to allow for real-valued commutation coefficients, rather than remaining restricted to commutation *signs.*

The q-Fock space of [BS91] and the corresponding q-Gaussian statistics are recovered from (1) and (2) by letting $t = 1$. Analogously to the q-Fock space setting, the construction in (1) is related to (2) as the (q, t)-Gaussian statistics are precisely those of the field operator on the (q, t)-Fock space. Furthermore, the construction in (2) is related to the theorem in (3) as the (q, t)-Gaussian statistics arise as the limits of the generalized non-commutative Central Limit Theorem. The framework formed by (1) through (3) therefore generalizes the q-deformed setting of Bożejko and Speicher in an extensive manner that nevertheless manages to preserve the inner consistency of the original framework.

In extending the operator algebraic setting of [BS91], the creation and annihilation on the (q, t)-Fock space now satisfy a two-parameter commutation relation that provides a unification of several types of well-known commutation relations:

4. A unification between the "math q-commutation relations" [AC76, FB70, BS91] and "physics q-commutation relations" [Bie89, Mac89].

5. A unification between the "q-commutation relations for $|q| \leq 1$" [BS91] and the "q-commutation relations for $|q| \geq 1$" [Boż07].

6. A Hilbert-space realization of the general form of the Chakrabarti-Jagannathan deformed oscillator algebra [CJ95].

Further contributions related to the general framework of (1) – (3) are the following:

7. The random matrix models for the (q, t)-Gaussian statistics, realized as a second-parameter refinement of the the extended Jordan-Wigner transform of [Bia97b].

8. The characterization of the (q, t)-Gaussian measure as the orthogonalizing measure for the (q, t)-*Hermite orthogonal polynomial sequence.*

9. A full description of the norms of the creation and annihilation operators on the (q, t)-Fock space (for all parameter ranges).

Finally, given the second degree of freedom, one may now consider various specializations. The $q = 0$ specialization, which in the q-deformed setting yields the full Fock space of free probability, now corresponds to a refinement of free probability that bears multiple surprising connections to other areas of mathematics. In particular:

i. The moments of the measure associated with the $(0,t)$-Gaussian element, aka the *t-semicircular*, are given by the deformed Catalan numbers of Carlitz and Riordan [FH85, CR64].

ii. The continued-fraction expansion of the moment-generating function associated with the t-semicircular element is the *Rogers-Ramanujan* continued fraction. The *Rogers-Ramanujan identities* (see e.g. [And98]) are used to derive an elegant form of the Cauchy transform of the t-semicircular measure.

iii. By the resulting connection to the work of Al Salam and Ismail [ASI83] and Ismail [Ism05], t-semicircular measure is characterized in terms of the zeros of a *deformed Airy function.*

iv. The $(0,t)$-Fock space provides a setting in which to realize Mazza and Piau's [MP02] generalization of *Wigner's semicircle law* ([Wig55], see also [AGZ10]) to *(time-correlated) Wigner processes.*

The remainder of this thesis is organized as follows. The structural aspects of Fock spaces and of the algebras of operators on these spaces are reviewed in Chapter 2, where the reader shall also find the relevant algebraic and Hilbert-space constructions. The role of Fock spaces as a natural setting for non-commutative probability is discussed in Chapter 3, focusing on the symmetric Fock space in classical probability, the full Fock space in free probability, and the q-Fock space as a setting for q-deformed probability. The (q,t)-Fock space is constructed Chapter 4, wich also studies the operators on the (q,t)-Fock space, the non-commutative probability spaces that emerge from these, and the relevant specializations. The generalized non-commutative Central Limit Theorem is derived in Chapter 5, along with the random matrix models for the (q,t)-Gaussian statistics. Since the background for this thesis is best described as a synthesis of ideas from algebra, functional analysis, and operator algebras, the three appendices provide the reader with a roadmap of the pre-requisite material, including the definitions and the statements of the essential results.

Chapter 2

Fock Spaces and Gaussian Algebras

Non-commutative probability, which provides the setting for this thesis and is the focus of the upcoming Chapter 3, broadly refers to a generalization of the classical probability theory to encompass quantum observables. While the probabilistic aspects of the bosonic quantum systems are described in terms of classical probability, the fermionic observables were instead the motivation for the development of *fermionic* or *anti-commutative* probability (e.g. [Bia95, Mey93, Par92]). The non-commutative probability theories of interest in this thesis – namely, the *free probability* [Voi86, VDN92], the *"q-deformed" probability* [BS91], and the two-parameter continuum of probability theories introduced in Chapter 4 – arise in a closely related context, namely, through the relevant deformations of the bosonic/fermionic structure. But, rather than concretely thinking of such deformations as violations of the Bose and Fermi statistics (e.g. [Gre91]), it will become apparent in Chapter 3 that there is an advantage to focusing instead on the abstract frameworks on which these deformations take place.

The sense in which one may speak of the deformations of the bosonic/fermionic structure is made concrete through the notion of a *Fock space*.[1] The latter is a type of Hilbert space serving as a quantum-field-theoretic construct used the encode possible states of a quantum system with a variable number of particles. In particular, the bosonic (classical) setting is realized on the *symmetric Fock space*, the fermionic

[1]Named after the physicist Vladimir Aleksandrovich Fock, who introduced the concept in [Foc32].

one on the *anti-symmetric Fock space*, whereas the free probability and q-deformed probability take place on the *full Fock space* and the *q-Fock space*, respectively.

Fock spaces, both as algebraic objects and as Hilbert spaces, are the subject of the present chapter. We will start by constructing in Section 2.1 an algebraic form of a Fock space, given by a *tensor algebra*. This algebraic Fock space will encode the relevant physical intuition, but only to a point. Indeed, in order to derive either sensible physical models or non-commutative probability theories, we will have to impose an inner product structure on the algebraic constructs and, completing with respect to the induced norm, come to work with Hilbert spaces. *Fock space* will then refer a Hilbert space arising as the completion of a tensor algebra with respect to a suitable inner product, and the various relevant Fock spaces will be the subject of Section 2.2. In the cases of interest, the probabilistic aspects of Fock spaces are principally based on the properties of the bounded linear operators on these spaces. Such operators, and the algebras they generate, are the focus of the remaining Section 2.3. Note that the discussion of the probabilistic aspects of Fock spaces is deferred until Chapter 3. For any background material relevant to the present chapter, the reader will be referred to Appendices A through C.

2.1 Algebraic Fock Space

The focus of this section is the *algebraic Fock space.* Starting with a vector space V over a field $\mathbb{F}$, taken throughout this thesis as a stand-in for $\mathbb{R}$ or $\mathbb{C}$, the algebraic Fock space is the *tensor algebra*

$$\bigoplus_{n\geq 0} V^{\otimes n}.$$

In the following, we attempt to give a concise treatment focused on helping the reader develop ease manipulating tensor algebras and the underlying algebraic constructs, namely tensor products and direct sums. Readers looking for a deeper understanding of the nature of the objects at hand will be referred to the supplementary material available in the appendices as well as the references therein.

2.1.1 Algebraic tensor product

Start with two vector spaces V and W over some field $\mathbb{F}$, with bases $E_1 = \{e_i\}_{i \in I}$ and $E_2 = \{\hat{e}_j\}_{i \in J}$, respectively. The *tensor product* $V \otimes W$ is the vector space over $\mathbb{F}$ with basis $\{e_i \otimes \hat{e}_j\}_{(i,j) \in I \times J}$ and with the following rules of addition and scalar multiplication:

$$(\alpha_1 v_1 + \alpha_2 v_2) \otimes w = \alpha_1 (v_1 \otimes w) + \alpha_2 (v_2 \otimes w), \tag{2.1}$$

$$v \otimes (\beta_1 w_1 + \beta_2 w_2) = \beta_1 (v \otimes w_1) + \beta_2 \otimes (v, w_2) \tag{2.2}$$

for all $v, v_1, v_2 \in V$, $w, w_1, w_2 \in W$, $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{F}$.

For the nature of the element $e_i \otimes \hat{e}_j$ to become clear, one must be willing to either consider explicit constructions of the tensor product (relying on the notion of a *quotient space*) or the *universal property* of the tensor product. For an in-depth discussion of both of these approaches, the reader is referred to Appendix A. In the meantime, if one admits each element $e_i \otimes \hat{e}_j$ as a formal object, relations (2.1) and (2.2) provide the necessary rules for defining every element of $V \otimes W$.

In particular, if $v = \sum_{i=1}^{n} \alpha_i e_i$ and $w = \sum_{j=1}^{m} \beta_j \hat{e}_j$, for $\alpha_i, \beta_j \in \mathbb{F}$, then $v \otimes w = \sum_{i=1}^{n} \sum_{j=1}^{m} \alpha_i \beta_j (e_i \otimes \hat{e}_j)$. Note that the element $v \otimes w$ of $V \otimes W$ is said to be a *pure tensor*. In general, $V \otimes W$ is composed of all finitely indexed sums $\sum_{i,j} \gamma_{i,j} (e_i \otimes \hat{e}_j)$ for $\gamma_{i,j} \in \mathbb{F}$, of which the pure tensors are those elements that can be written in the form $\sum_{i,j} \alpha_i \beta_j (e_i \otimes \hat{e}_j)$. The distinction is made clearer in the following example.

Example 1 Let V be a two-dimensional vector space over $\mathbb{F}$, spanned by a vectors e_1 and e_2. (For concreteness, one may take $\mathbb{F} = \mathbb{R}$, $e = (1, 0)$, $e_2 = (0, 1)$.) Then, $V \otimes V$ is the vector space of dimension 4, with basis

$$e_1 \otimes e_1, \quad e_1 \otimes e_2, \quad e_2 \otimes e_1, \quad e_2 \otimes e_2,$$

and vector-space operations given by (2.1) and (2.2). It is important to realize that the element $e_1 \otimes e_2$ is taken to be different than $e_2 \otimes e_1$. Furthermore, there are (many) elements of $V \otimes V$ that cannot be written as pure tensors. For example let $v = e_1 \otimes e_2 + e_2 \otimes e_1$ and suppose that v can be written as a pure tensor. Then,

$v = (ae_1 + be_2) \otimes (ce_1 + de_2)$ for some $a, b, c, d \in \mathbb{F}$. But, by bilinearity, $ac = 0 = bd$ while $ad = bc = 1$, which is impossible. •

It is therefore entirely clear how to formally manipulate the vector space that is the tensor product of two vector spaces, even if at the present level of exposition the nature of the underlying basis elements remains somewhat murky. Tensor products may be found to arise throughout various branches of mathematics, physics, and engineering. Indeed, as discussed in Appendix A, the tensor product $\mathbb{F}^n \otimes \mathbb{F}^n$ is isomorphic to the familiar *outer product* of $\mathbb{F}^n$ with $\mathbb{F}^n$, whereas the tensor product of vector spaces generated by $n \times n$ matrices over $\mathbb{F}$ is isomorphic to their *Kronecker product.*

The above approach analogously generalizes to any finite product of vector spaces. (There will be no need in this thesis to consider infinite tensor products, though such a thing is feasible.) First, an extension of bilinearity is called *multilinearity*, where for vector spaces $V_1, \ldots, V_n, Y$ over a field $\mathbb{F}$, a map $f : V_1 \times \ldots \times V_n \to Y$ is multilinear (or n-multilinear) if f is linear in each coordinate while keeping the other coordinates fixed. To form the (n-fold) tensor product $V_1 \otimes \ldots \otimes V_n$, it suffices to generalize the bilinearity relations (2.1) and (2.2) to n-multilinearity and identify the basis elements with the "n-fold tensors" (again, taken as formal objects) of the corresponding basis elements of $V_1, \ldots, V_n$.

An important example of an n-fold tensor products is the n^{th} *tensor power* $V^{\otimes n}$, defined as the tensor product of V with itself n times. Given a physical intepretation, if the elements of V are the possible states of a single-particle quantum system, then $V^{\otimes n}$ describes the state of the system with n particles of the same type.

From an algebraic point of view, tensor powers are additionally interesting in that they may be endowed with multiplicative structure as follows:

Proposition 1. *For any vector space V and positive integers n and m, there is a unique bijective bilinear map $V^{\otimes n} \times V^{\otimes m} \to V^{\otimes (n+m)}$ satisfying*

$$(v_1 \otimes \ldots \otimes v_n \ , \ v_1' \otimes \ldots \otimes v_m') \to v_1 \otimes \ldots \otimes v_n \otimes v_1' \otimes \ldots \otimes v_m'$$

for all $v_1, \ldots, v_n, v_1', \ldots, v_m' \in V$.

The reader may verify that the above multiplication map is well defined by passing to the basis elements of $V^{\otimes n}$, $V^{\otimes m}$, and $V^{\otimes(n+m)}$ and by multilinear extension the map

$$(e_{i_1} \otimes \ldots \otimes e_{i_n} \ , \ e_{j_1} \otimes \ldots \otimes e_{j_m}) \mapsto e_{i_1} \otimes \ldots \otimes e_{i_n} \otimes e_{j_1} \otimes \ldots \otimes e_{j_m}.$$

Note that the resulting operation is distributive over the addition, compatible with scalar multiplication, and associative. This observation will become useful shortly, when passing from a vector space to an algebra.

2.1.2 Algebraic direct sum

It is natural, for reasons which should become apparent over the course of the chapters that follow, to consider an algebraic object with the structure of a vector space that "contains" the various tensor powers of V. However, we have not defined any rules for adding elements of two distinct tensor powers of V. The solution is to distinguish the tensor powers by assigning them to distinct "coordinates" and form a vector space in which the usual operations are performed coordinate-wise.

Specifically, given two vector spaces W and Z over some field $\mathbb{F}$, the cartesian product $W \times Z$ can be given the structure of a vector space over $\mathbb{F}$ with the addition and the scalar multiplication as follows:

$$(w_1, z_1) + (w_2, z_2) = (w_1 + w_2\, ,\, z_1 + z_2), \tag{2.3}$$

$$\alpha(w, z) = (\alpha w, \alpha z), \tag{2.4}$$

for all $(w_1, z_1), (w_2, z_2) \in W \times Z$ and $\alpha \in \mathbb{F}$. This structure turns $W \times Z$ into a vector space denoted $V \oplus W$, referred to as *the (algebraic) direct sum of* W *and* Z. The definition extends analogously to any finite number of vector spaces, which, in turn, is used to define direct sums over arbitrary index sets as follows.

Definition 1. *Given a set J and a family of vector spaces $\{W_\gamma\}_{\gamma\in J}$ over a field $\mathbb{F}$, consider the set of J-tuples $(x_\gamma)_{\gamma\in J}$ with $x_\gamma = 0$ for all but finitely many γ. Then, the vector space generated by all such J-tuples with respect to the operations of component-wise addition and scalar multiplication is referred to as the* (algebraic) direct sum of $\{W_\gamma\}_{\gamma\in J}$ *and denoted* $\bigoplus_{\gamma\in J} W_\gamma$.

Returning to the tensor powers, the vector space $\bigoplus_{n\geq 0} V^{\otimes n}$ is clearly the desired object, containing and allowing us to manipulate the various tensor powers of V. For the convenience of notation, a typical element of this space may be written as

$$v_1 \otimes v_2 \ + \ v_3 \otimes v_4 \ + \ v_5 \otimes v_6 \otimes v_7 \otimes v_8$$

rather than

$$(0, \ \ 0, \ \ v_1 \otimes v_2 + v_3 \otimes v_4, \ \ 0, \ \ v_5 \otimes v_6 \otimes v_7 \otimes v_8, \ \ 0, \ \ 0, \ \ \ldots),$$

with the understanding that the addition is nevertheless component-wise.

It is principally the vector-space structure, and later the Hilbert-space structure, on $\bigoplus_{n\geq 0} V^{\otimes n}$ that is of interest to the remainder of this thesis. Still, we take a moment to consider next the multiplicative structure defined in Proposition 8, which will turn $\bigoplus_{n\geq 0} V^{\otimes n}$ into an *algebra*.

2.1.3 Tensor algebra

It is now straightforward to extend the multiplication $V^{\otimes n} \times V^{\otimes m} \to V^{\otimes(n+m)}$ defined in Proposition 8 to the whole of $\bigoplus_{n\geq 0} V^{\otimes n}$. It suffices to note that every element $w \in \bigoplus_{n\geq 0} V^{\otimes n}$ can be written as some (finite) sum of elements in $V^{\otimes n_1}, \ldots, V^{\otimes n_k}$, where the choice of k (when taken to be minimal) and of $n_1, \ldots, n_k$ is unique. Taking $w, w' \in \bigoplus_{n\geq 0} V^{\otimes n}$ and writing $w = \sum_i w_i, w' = \sum_j w'_j$, where $w_i \in V^{\otimes n_i}, w'_j \in V^{\otimes m_j}$, the resulting multiplication map on $\bigoplus_{n\geq 0} V^{\otimes n}$ satisfies

$$ww' = \sum_i w_i \sum_j w'_j = \sum_{i,j} w_i w'_j,$$

with the multiplication on $V^{\otimes n} \times V^{\otimes m}$ as given in Proposition 8 above. It is then straightforward to verify that this vector multiplication turns the corresponding vector space into an algebra. Specifically, given a vector space V, the *tensor algebra* on V is the vector space $\bigoplus_{n \geq 0} V^{\otimes n}$, with the multiplication law given as the above distributive extension of the multiplication given in Proposition 8.

The tensor algebra $\bigoplus_{n \geq 0} V^{\otimes n}$ will in the present context be referred to as the *algebraic Fock space.* The qualifier "algebraic" is meant to indicate that if the underlying vector spaces happen to be Hilbert spaces, one is purposefully ignoring the natural inner product structure on $\bigoplus_{n \geq 0} V^{\otimes n}$. Such structure is the subject of the following section. Prior to launching into a discussion of Hilbert spaces, however, we must take one final algebraic detour and address the nature of the field $\mathbb{F}$.

2.1.4 Complexification

Though the previous sections considered a general field $\mathbb{F}$, we will soon specialize it to a stand-in for $\mathbb{R}$ or $\mathbb{C}$. In fact, while the general theory to which we will refer to will largely concern complex Hilbert spaces and Banach algebras, the specific constructions of interest will be first carried out on *complexified* vector spaces, which admit a real basis. The notion of complexification arises in many areas of mathematics and it is in fact a natural one. For instance, the complexification of $\mathbb{R}^n$ will turn out to be isomorphic to $\mathbb{C}^n$ and the complexification of $\mathbb{R}[X]$ (the ring of real-valued polynomials in one variable) is isomorphic to $\mathbb{C}[X]$ (complex-valued polynomials in one variable). In the light of the previous developments, there are two sensible approaches to defining a complexification of a real vector space, namely via the direct sum or via the tensor product.

The idea of complexifying a vector space V via the direct sum comes from thinking of the pair $(v_1, v_2) \in V \times V$ as the formal sum $v_1 + iv_2$, with a suitably defined multiplication by complex scalars. Concretely, given a real vector space V, the *complexification* of V is the vector space $V_{\mathbb{C}}$ over $\mathbb{C}$ obtained by defining on $V \oplus V$ the multiplication by a complex scalar:

$$(\alpha + i\beta)(v_1, v_2) = (\alpha v_1 - \beta v_2, \beta v_1 + \alpha v_2). \tag{2.5}$$

Note that for $\beta = 0$, (2.5) recovers (A.6). Given now a real vector space V with basis $\{e_i\}$, it is clear that every element of $V_\mathbb{C}$ can be written as some linear combination of elements (e_i, e_j). Thus, the set $\{(e_i, e_j)\}$ spans $V_\mathbb{C}$, but it is not a basis! Indeed, by the multiplication rule, $i(0, e_j) = (e_j, 0)$. Instead, the reader may verify that if V is a real vector space with an $\mathbb{R}$-basis $\{e_j\}$, then $V_\mathbb{C}$ is a complex vector space with a $\mathbb{C}$-basis $\{(e_j, 0)\}$. One may thus can consider V as "living" inside $V_\mathbb{C}$, which is formally prescribed by the embedding $v \mapsto (v, 0)$.

Alternatively, given a real vector space V, one may wish to consider the complexification of V to be the vector space $\mathbb{C} \otimes_\mathbb{R} V$ (where the subscript on the tensor symbol is to make explicit the fact that the tensor product is taken over $\mathbb{R}$). In this setting, the reader may verify that there is again a natural identification of V as belonging to $\mathbb{C} \otimes_\mathbb{R} V$, by the embedding $v \to 1 \otimes v$. Similarly, any $\mathbb{R}$-basis $\{e_i\}$ of V gives rise to a $\mathbb{C}$-basis $\{1 \otimes e_j\}$.

The two approaches turn out to be equivalent, in that the resulting objects turn out to be isomorphic and this isomorphism extends to linear maps. For further details, the reader is referred to Appendix A.

2.2 Classical and Quantum Fock spaces

A *Fock space* is a particular type of Hilbert space that plays a prominent role in classical probability, non-commutative probability, and physics. The starting point for the discussion of Fock spaces is (1) a real, separable Hilbert space $\mathscr{H}$ whose complexification $\mathscr{H}_\mathbb{C}$ is taken to describe the quantum states of a single-particle system and (2) some unit vector Ω, disjoint from $\mathscr{H}$ and referred to as the *vacuum vector*. The algebraic tensor powers $\mathscr{H}_\mathbb{C}^{\otimes n}$ will be analogously taken to describe the state of an n-particle system, while the absence of particles will be encoded by $\mathscr{H}_\mathbb{C}^{\otimes 0}$, taken as the one-dimensional complex vector space spanned by Ω. The algebraic Fock space over $\mathscr{H}_{\mathbb{C}}$, previously defined as the tensor algebra $\bigoplus_{n=0}^{\infty} \mathscr{H}_{\mathbb{C}}^{\otimes n} := \mathscr{F}(\mathscr{H})$, now encodes the state of a quantum system with a variable number of particles. In essence, it may be said that this very shift from single-particle quantum mechanics to those of "fields" or "many-body systems" forms the premise of *quantum field theory.* One may now define suitable *creation operators* on $\mathscr{F}(\mathscr{H})$, injecting a particle into a given quantum state, *annihilation operators* destroying a particle, as well as all manner of quantum observables (e.g. field operators, particle number operator).

While the above purely algebraic structure may appear tentalizing on account of its simplicity and well-founded intuition, it is insufficient. Specifically, to be able to discuss operator spectra, their spectral decompositions (generalizations of the eigenvalue/eigenvector decompositions of matrices), and various transformations that inevitably arise in the physical context, one must leave the purely algebraic setting and transition to working with operators on Hilbert spaces. There is in fact a natural inner product structure on $\mathscr{F}(\mathscr{H})$ not taken into account at the algebraic level. It arises from the usual Hilbert-space versions of the direct sum and tensor product, which are reviewed in Appendix B. While the natural inner product structure provides a setting in which to define a rich probability theory (viz. the *free probability* of Voiculescu [Voi86, VDN92], discussed further in the following chapter), it does not yield the correct bosonic/fermionic framework. The present section therefore considers the various Hilbert spaces arising from the algebra $\bigoplus_{n=0}^{\infty} \mathscr{H}_{\mathbb{C}}^{\otimes n}$. These are obtained by either

- replacing the algebraic constructs by their Hilbert-space versions, or
- completing the tensor algebra $\mathscr{F}(\mathscr{H})$ with respect to a different choice of an inner product.

Such Hilbert spaces will be referred to as *Fock spaces* and will play a key role in the subsequent chapters.

2.2.1 Full (Boltzmann) Fock space

The *full Fock space*, also referred to as the *Boltzmann Fock space*, is the Hilbert space $\mathscr{F}_0(\mathscr{H})$ obtained by completing the tensor algebra $\mathscr{F}(\mathscr{H})$ with respect to the "usual" inner product. The latter will be denoted $\langle\, , \rangle_0$ and given by sesquiliner[2] extension of

$$\langle \Omega, \Omega \rangle_0 = 1 \tag{2.6}$$

$$\langle f_1 \otimes \ldots \otimes f_n, h_1 \otimes \ldots \otimes h_m \rangle_0 = \delta_{n,m} \langle f_1, h_1 \rangle \otimes \ldots \otimes \langle f_n, h_n \rangle, \tag{2.7}$$

where $\delta_{n,m} = 1$ iff $n = m$ and has value zero otherwise. Note that $\mathscr{F}_0(\mathscr{H})$ is the Hilbert space $\bigoplus_{n=0}^{\infty} \mathscr{H}_{\mathbb{C}}^{\otimes n}$ obtained by taking the usual Hilbert-space versions of the tensor product and direct sum (reviewed in Appendix B).

Despite of the fact that the remainder of this thesis will take place in the Hilbert-space setting, the purely algebraic object $\mathscr{F}(\mathscr{H})$ will remain very useful, as the latter forms a convenient dense domain in $\mathscr{F}_0(\mathscr{H})$ (as well as in other Fock spaces discussed shortly or introduced in the later chapters). In particular, many of the definitions and arguments will be formulated in the setting of $\mathscr{F}(\mathscr{H})$, whose elements are affine combinations of pure tensors, subsequently extending to the norm limit when appropriate.

Following the physical intuition, the fundamental operators on $\mathscr{F}_0(\mathscr{H})$ are the creation and annihilation operators, corresponding to the creation and destruction of a particle, respectively. For an element $h \in \mathscr{H}$, the *annihilation operator* $a_0(h)$ is defined on $\mathscr{F}(\mathscr{H})$ by linear extension of

$$a_0(h)\Omega = 0 \tag{2.8}$$

$$a_0(h) f_1 \otimes \ldots \otimes f_n = \langle h, f_1 \rangle_{\mathscr{H}} f_2 \otimes \ldots \otimes f_n. \tag{2.9}$$

The adjoint under $\langle\, , \rangle_0$ of $a_0(h)$ is referred to as the *creation operator* $a_0(h)^*$, and

[2] i.e. Linear in the first coordinate, conjugate-linear in the second.

the reader may verify that on $\mathscr{F}(\mathscr{H})$, the latter has the form

$$a_0(h)^* f_1 \otimes \ldots \otimes f_n = h \otimes f_1 \otimes f_2 \ldots \otimes f_n, \tag{2.10}$$

i.e. $a_0(h)^*$ *creates* h to the left of the tensor. For a worked-out calculation, the reader is referred to Chapter 4, which deals with these objects in a more general setting. The creation and annihilation operators are readily shown to extend uniquely to a bounded linear operator on the whole of $\mathscr{F}_0(\mathscr{H})$ (cf. Chapter 4). Finally, the *field operator* $s_0(h)$ given as the symmetrization of the annihilation operator, i.e. $s_0(h) = a_0(h) + a_0(h)^*$, will play a prominent role in the later chapters.

2.2.2 Symmetric Fock space

For $n \in \mathbb{N}$, let *the* n^{th} *symmetric power of* $\mathscr{H}_{\mathbb{C}}$, denoted $\mathscr{H}_{\mathbb{C}}^{\odot n}$, be the subspace of $\mathscr{H}_{\mathbb{C}}^{\otimes n}$ spanned by the elements

$$h_1 \odot \ldots \odot h_n = \frac{1}{\sqrt{n!}} \sum_{\sigma \in S_n} h_{\pi(1)} \otimes \ldots \otimes h_{\pi(n)}.$$

where S_n is the symmetric group of order n, that is, the set of all permutations on n letters. In other words, $\mathscr{H}_{\mathbb{C}}^{\odot n}$ is the natural symmetrization of $\mathscr{H}_{\mathbb{C}}^{\otimes n}$, with the normalization chosen so that

$$\langle f_1 \odot \ldots \odot f_n, h_1 \odot \ldots \odot h_m \rangle = \delta_{n,m} \sum_{\sigma \in S_n} \langle f_1, h_{\pi(1)} \rangle \otimes \ldots \otimes \langle f_n, h_{\pi(n)} \rangle. \tag{2.11}$$

Example 2 Let (X, Ψ, μ) be a σ-finite measure space and $\mathscr{H} = \mathscr{L}^2_{\mathbb{R}}(X, \Psi, \mu)$. For $n \in \mathbb{N}$, one may identify $\mathscr{H}_{\mathbb{C}}^{\odot n}$ with the subspace of symmetric functions in $\mathscr{L}^2_{\mathbb{C}}(X^n, \Psi^n, \mu^n)$ letting

$$h_1 \odot \ldots \odot h_n(x_1, \ldots, x_n) = \sum_{\sigma \in S_n} h_1(x_{\sigma(1)}) \ldots h_n(x_{\sigma(n)}). \tag{2.12}$$

In this case, one usually interprets $\mathscr{H}_{\mathbb{C}}^0$ as the one-dimensional vector space spanned by a single element, carrying measure 1. If $\mu(X) < \infty$, then $\mathscr{H}_{\mathbb{C}}^{0}$ is typically identified with the constant functions. •

The Hilbert-space direct sum $\bigoplus_{n=0}^{\infty} \mathscr{H}_{\mathbb{C}}^{\odot n}$ is referred to as the *symmetric Fock space* (over $\mathscr{H}$) and presently denoted $\mathscr{F}_1(\mathscr{H})$. Note that an equivalent construction is obtained by taking (2.11) to define a semi-inner product on $\mathscr{F}(\mathscr{H})$, taking the quotient (cf. Appendix A) of $\mathscr{F}(\mathscr{H})$ by the kernel of the map (2.11), and then completing the quotient space. Note that the symmetric Fock space is intimately connected to the *Wiener chaos decomposition* of an $\mathscr{L}^2$ space of a probability measure, as discussed in Chapter 3.

One may now define the *symmetric annihilation operators* $\{a_1(h)\}_{h\in\mathscr{H}}$ on the linear subspace $\mathscr{F}(\mathscr{H})$ formed by the symmetric tensors, by linear extension of

$$
\begin{aligned}
&a_1(h)\Omega = 0 \\
&a_1(h) f_1 \odot \ldots \odot f_n = \sum_{k=1}^{n} \langle h, f_k\rangle_{\mathscr{H}} f_1 \odot \ldots \odot \check{f}_k \odot \ldots \odot f_n,
\end{aligned}
\tag{2.13}
$$

where the subscript ˇ signifies that the corresponding element was deleted from the tensor. Similarly, define *symmetric creation operators* $\{a_1^*(h)\}_{h\in\mathscr{H}}$ by

$$
\begin{aligned}
&a_1(h)\Omega = 0 \\
&a_1(h)^* f_1 \odot \ldots \odot f_n = h \odot f_1 \odot f_2 \ldots \odot f_n.
\end{aligned}
\tag{2.14}
$$

One may again verify that the creation operators are indeed adjoints of the annihilation operators on the symmetric tensors and therefore on the dense domain $\mathscr{F}(\mathscr{H})$. However, unlike in the full Fock space setting, these operators are *not bounded*; indeed, for any unit vector $e \in \mathscr{H}$, one may readily check that $\|a_1(e)e^{\odot n}\|/\|e^{\odot n}\| = n+1$. Note that the creation and annihilation operators on $\mathscr{F}_1(\mathscr{H})$ are nevertheless *closable* and therefore retain sufficient structure to generally remain tractable (e.g. a well-defined spectrum and desirable spectral properties).

As previously remarked, given a quantum field theoretic interpretation, the symmetric Fock space is the *bosonic Fock space*. In particular, the symmetric power $\mathscr{H}_{\mathbb{C}}^{\otimes n}$

represents the system with n indistinguishable bosons, with the creation operator injecting a boson into the system and the annihilation operator destroying one.

2.2.3 Anti-symmetric Fock space

For $n \in \mathbb{N}$, let *the n^{th} anti-symmetric power of $\mathscr{H}_{\mathbb{C}}$*, denoted $\mathscr{H}_{\mathbb{C}}^{\wedge n}$, be the subspace of $\mathscr{H}_{\mathbb{C}}^{\otimes n}$ spanned by elements

$$h_1 \wedge \ldots \wedge h_n = \frac{1}{\sqrt{n!}} \sum_{\sigma \in S_n} (-1)^{\mathrm{inv}(\sigma)} h_{\pi(1)} \otimes \ldots \otimes h_{\pi(n)}, \tag{2.15}$$

where $\mathrm{inv}(\sigma)$ denotes the number of *inversions* of the permutation σ. That is, $\mathrm{inv}(\sigma)$ is the number of pairs $(i, j) \in [n] \times [n]$ such that $i < j$ but $\sigma(i) > \sigma(j)$. The quantity $(-1)^{\mathrm{inv}(\sigma)}$ is referred to as the *sign* or *signature* of σ, and the sign of a permutation is the *alternating character* of the symmetric group. Thus, (2.15) is a natural anti-symmetrization of the tensor product, with the normalization chosen so that the inner product of two anti-symmetric pure tensors becomes

$$\langle f_1 \wedge \ldots \wedge f_n, h_1 \wedge \ldots \wedge h_m \rangle = \delta_{n,m} \sum_{\sigma \in S_n} (-1)^{\mathrm{inv}(\sigma)} \langle f_1, h_{\sigma(1)} \rangle \otimes \ldots \otimes \langle f_n, h_{\sigma(n)} \rangle. \tag{2.16}$$

The *anti-symmetric Fock space* $\mathscr{F}_{-1}(\mathscr{H})$ is then obtained as the Hilbert-space direct sum $\mathscr{F}_{-1}(\mathscr{H}) = \bigoplus_{n=1}^{\infty} \mathscr{H}_{\mathbb{C}}^{\wedge n}$. Note that analogously to the symmetric case, one may equivalently take (2.16) as defining a semi-inner product and then proceed by separating and completing $\mathscr{F}(\mathscr{H})$ with respect to it.

The anti-symmetric Fock space is a very different object from its symmetric counterpart. In particular, note that the anti-symmetric tensor $h_1 \wedge \ldots \wedge h_n$ vanishes whenever $h_i = h_j$ for some $i \neq j$ in $\{1, \ldots, n\}$. Specifically, for every permutation $\sigma \in S_n$, there is a unique permutation $\hat{\sigma} \in S_n$ such that σ and $\hat{\sigma}$ differ only by the transposition $\tau_{i,j}$; the result then immediately follows by the fact that for each such pair, $(-1)^{\mathrm{inv}(\sigma)} h_{\sigma(1)} \otimes \ldots \otimes h_{\sigma(n)} + (-1)^{\mathrm{inv}(\hat{\sigma})} h_{\hat{\sigma}(1)} \otimes \ldots \otimes h_{\hat{\sigma}(n)} = 0$. Thus, if $\dim \mathscr{H} = n$, then $\mathscr{H}_{\mathbb{C}}^{\wedge k} = \{0\}$ for all $k > n$. The reader may easily verify that if $e_1, \ldots, e_n$ is an $\mathbb{R}$-basis for $\mathscr{H}$, a $\mathbb{C}$-basis for $\mathscr{F}_{-1}(\mathscr{H})$ is given by

$\{\Omega\} \cup \{e_{i_1} \wedge \ldots \wedge e_{i_m} \mid 1 \leq m \leq n, i_1 < \ldots < i_m\}$. It is then an easy counting exercise to verify that when $\dim \mathscr{H} = n$, the dimension of $\mathscr{F}_{-1}(\mathscr{H})$ is 2^n. In contrast, as long as $\mathscr{H}$ is non-degenerate, the *symmetric* Fock space $\mathscr{F}_1(\mathscr{H})$ is infinite-dimensional.

Note that in quantum field theory, the anti-symmetric Fock space is the *fermionic Fock space*. In particular, the vanishing of an anti-symmetric tensor containing a repeated element is precisely the *Pauli exclusion principle* according to which no two fermions may simultaneously occupy the same quantum state.

The antisymmetric annihilation operators $\{a_{-1}(h)\}_{h \in \mathscr{H}}$ defined by

$$\begin{aligned} &a_{-1}(h)\Omega = 0 \\ &a_{-1}(h) f_1 \wedge \ldots \wedge f_n = \sum_{k=1}^{n} (-1)^{k-1} \langle h, f_k \rangle_{\mathscr{H}} f_1 \wedge \ldots \check{f}_k \wedge \ldots \wedge f_n \end{aligned} \tag{2.17}$$

admit as adjoints the left creation operators $\{a_{-1}(h)^*\}_{h \in \mathscr{H}}$, where

$$\begin{aligned} &a_{-1}(h)^*\Omega = h \\ &a_{-1}(h)^* f_1 \wedge \ldots \wedge f_n = h \wedge f_1 \wedge f_2 \ldots \wedge f_n. \end{aligned} \tag{2.18}$$

Note that, unlike in the symmetric case, these operators uniquely extend to bounded linear operators on $\mathscr{F}_{-1}(\mathscr{H})$. Another relevant family in $\mathscr{B}(\mathscr{F}_{-1}(\mathscr{H}))$ is formed by the field operators $s_{-1}(h) = a_{-1}(h) + a_{-1}(h)^*$, which will later be interpreted as the Gaussian elements in the anti-commutative setting (cf. Chapter 3).

2.2.4 q-Fock space

That there exists a natural interpolation between the symmetric and anti-symmetric Fock spaces, passing through the full Fock space, was conjectured by physicists in the early 1970s, in [FB70]. While the more precise formulation of the conjecture will be discussed in Chapter 4 (at which point we will also mention some alternative existence proofs), it is at present fitting to consider the Hilbert space that provided the constructive answer. It was introduced by Bożejko and Speicher, who considered the question independently of [FB70], in their seminal paper [BS91].

For $\mathscr{H}$ a Hilbert space, define the sesquilinear form $\langle\,,\,\rangle_q$ on the tensor algebra $\mathscr{F}(\mathscr{H})$ by $\langle\Omega,\Omega\rangle_q = 1$ and

$$\langle f_1 \otimes \ldots \otimes f_n, h_1 \otimes \ldots \otimes h_n\rangle_q = \delta_{n,m} \sum_{\sigma \in S_n} q^{\mathrm{inv}(\sigma)} \langle f_1, h_{\sigma(1)}\rangle_{\mathscr{H}} \ldots \langle f_n, h_{\sigma(n)}\rangle_{\mathscr{H}}. \quad (2.19)$$

For $q \in [-1,1]$, (2.19) is clearly an interpolation between (2.11) for $q = 1$, (2.7) for $q = 0$, and (2.16) for $q = -1$. However, it is not a priori clear what geometric properties the above form may have outside of these special values of q – in particular, it is not obvious that $\langle\,\rangle_q$ should be an inner product for $q \in (-1,1)$. Nevertheless, Bożejko and Speicher managed to show that, indeed:

Theorem 1 ([BS91]). *For $q \in [-1,1]$, (2.19) defines a semi-inner product on $\mathscr{F}(\mathscr{H})$. If $q \in (-1,1)$, then (2.19) is an inner product.*

The *q-Fock space* $\mathscr{F}_q(\mathscr{H})$, defined for $q \in (-1,1)$ (or $q \in [-1,1]$ with additional care) is the completion of $\mathscr{F}(\mathscr{H})$ with respect to the inner product $\langle\,,\,\rangle_q$. The probabilistic structure that can be realized in the setting of the q-Fock space and its generalizations is at the core of this thesis. Prior to developing these, the remainder of this chapter considers the fundamental operator theoretic and operator algebraic properties of $\mathscr{F}_q(\mathscr{H})$ of use in the upcoming chapters.

Analogously to the classical Fock spaces, one may defne the left creation operators $\{a_q(h)^*\}_{h\in\mathscr{H}}$ by

$$\begin{aligned} &a_q(h)^*\Omega = h \\ &a_q(h)^* f_1 \otimes \ldots \otimes f_n = h \otimes f_1 \otimes \ldots \otimes f_n, \end{aligned} \quad (2.20)$$

along with the "twisted" annihilation operators

$$\begin{aligned} &a_q(h)\Omega = 0 \\ &a_q(h) f_1 \otimes \ldots \otimes f_n = \sum_{k=1}^{n} q^{k-1} \langle h, f_k\rangle_{\mathscr{H}} f_1 \otimes \ldots \otimes \check{f}_k \otimes \ldots \otimes f_n, \end{aligned} \quad (2.21)$$

When the underlying Hilbert space $\mathscr{H}$ is taken to be the complexification of some real Hilbert space and when h is taken as an element of the real Hilbert space, then $a_q^*(h)$ is indeed the adjoint of $a_q(h)$. To show this, it is helpful to rewrite the inner product $\langle\, , \rangle_q$ on $\mathscr{F}_q(\mathscr{H})$ recursively, as

$$\langle f_1 \otimes \ldots \otimes f_n, g_1 \otimes \ldots \otimes g_n \rangle_q = \sum_{k=1}^{n} q^{k-1} \langle f_1, g_k \rangle_{\mathscr{H}} \langle f_2 \otimes \ldots \otimes f_n, g_1 \otimes \ldots \otimes \breve{g}_k \otimes \ldots \otimes g_n \rangle_q . \tag{2.22}$$

One may also consider the field operator $s_q(h) = a_q(h) + a_q(h)^*$, which will be later associated with a deformation of the Gaussian measure (cf. Chapter 3). The creation, annihilation, and field operators uniquely extend to bounded linear operators on $\mathscr{F}_q(\mathscr{H})$ as follows.

Theorem 2 ([BS91, BKS97]). *For any $f \in \mathscr{H}$, the operators $a_q(f), s_q(f)$ on $\mathscr{F}_q(\mathscr{H})$ are bounded for $q \in [-1, 1)$, with norm*

$$\|a_q(f)\| = \begin{cases} \|f\|_{\mathscr{H}} & q \in [-1, 0], \\ \\ \frac{1}{\sqrt{1-q}} \|f\|_{\mathscr{H}} & q \in [0, 1) \end{cases} \tag{2.23}$$

$$\|s_q(f)\| = \frac{1}{\sqrt{1-q}} \|f\|_{\mathscr{H}} \tag{2.24}$$

The mathematical interest in the classical and deformed Fock spaces principally stems from (1) their operator algebraic properties, or, (2) their probabilistic aspects. Whereas the present thesis is principally focused on the latter, the former are reviewed in the following section.

2.3 Algebras of operators on Fock spaces

We now turn to the properties of the algebras generated by the creation, annihilation and field operators on the Fock spaces, introduced in the previous section. It is a general fact that the bounded linear operators on a given Hilbert space form an algebra,[3] with the with scalar multiplication on the underlying field and addition

[3]An *algebra* $\mathcal{A}$ is a vector space endowed with a bilinear multiplication map $\mathcal{A} \times \mathcal{A} \to \mathcal{A}$.

given in the obvious way and the multiplicative structure given by map composition. Given a Hilbert space $\mathscr{H}$, the algebra of bounded linear operators on $\mathscr{H}$ is typically denoted $\mathscr{B}(\mathscr{H})$. For concretness, one may consider a Hilbert space of dimension n (e.g. the Euclidean space $\mathbb{R}^n$), in which case $\mathscr{B}(\mathscr{H})$ is the algebra of $n \times n$ matrices over the underlying field.

The algebra $\mathscr{B}(\mathscr{H})$ is an instance of a *Banach algebra*, which is an abstract algebra endowed with a norm with respect to which it is a Banach space. When passing from a single operator to an algebra, the operator-*theoretic* questions give way to operator-*algebraic* questions. In this latter vein, the first part of the present section considers the *commutation relations* that happen to be satisfied by the creation and annihilation operators on the familiar Fock spaces.

In the probabilistic context, a particularly relevant class of Banach algebras are the *von Neumann algebras*, which provide a setting for a "non-commutative measure theory", in the sense made precise in Appendix C. We will presently make fairly little use of the general theory of von Neumann algebras, but will instead focus on the von Neumann algebra generated by the field operators on the q-Fock space, termed the *q-Gaussian algebra.* Indeed, the q-Gaussian algebra, along with a certain expectation functional, forms the framework of "q-deformed probability", discussed in Chapter 3. The remainder of this section will review a handful of fundamental properties of this algebra in preparation for the probabilistic treatment of the following chapter. For any pre-requisite background material, the reader is referred to the Appendices B and C.[4]

2.3.1 Commutation relations

The classical and quantum Fock spaces considered in this thesis all share the property that the creation and annihilation operators satisfy a commutation relation. In the bosonic/fermionic cases, the underlying commutation relations are precisely certain

[4]Basic properties of Banach spaces, Hilbert spaces, and of $\mathscr{B}(\mathscr{H})$ are reviewed in Appendix B, whereas the general properties of Banach algebras and the more advanced properties of $\mathscr{B}(\mathscr{H})$ are the focus of Appendix C. In particular, for a definition and further discussion of some key aspects of von Neumann algebras, the reader is referred to see Section C.3 of Appendix C.

fundamental relations of quantum mechanics, whereas the corresponding relation on the q-Fock space can be viewed as an interpolation between these two classical cases.

Specifically, starting with the symmetric (bosonic) Fock space $\mathscr{F}_1(\mathscr{H})$, a straightforward calculation using the definitions (2.14) and (2.13) of the previous chapter yields that for all $g, h \in \mathscr{H}$,

$$\begin{aligned} &a_1(h)a_1(g)^* - a_1(g)^*a_1(h) = \langle h, g\rangle_{\mathscr{H}} 1, \qquad \text{(CCR)} \\ &a_1(h)a_1(g) = a_1(g)a_1(h), \\ &a_1(h)^*a_1(g)^* = a_1(g)^*a_1(h)^*, \end{aligned}$$

where 1 denotes the identity on $\mathscr{F}_1(\mathscr{H})$. The above commutation relations are referred to as the *canonical commutation relations (CCR)*.

Remark 1 For readers already familiar with the *CCR algebra*, it is worthwhile to pause here and consider the nature of its representation on the symmetric Fock space. As the operators $\{a_1(h)\}_{h\in\mathscr{H}}$ are unbounded, one may expect the general problem of starting with the CCR and seeking a Hilbert-space representation of the corresponding $*$-algebra (cf. Appendix C) to be ill-posed. In a sense, this was one of the original problems with the early formalism of quantum mechanics as developed by Heisenberg [Hei25], Born and Jordan [BJ25], Dirac [Dir25] and others. Specifically, letting x be the *position operator* acting on a wavefunction ψ as $(x\psi)(x) = x\psi(x)$ and p be the *momentum operator* acting as $(p\psi)(x) = -i\hbar\,(\partial\psi/\partial x)$, an easy computation yields the canonical commutation relation $[x, p] = i\hbar$. However, x and p cannot be represented as operators on a finite-dimensional Hilbert space, as readily seen by taking traces. Following an approach attributed to Hermann Weyl (see e.g. [Wey31]), one may instead take (complex) exponentials, turning the problem into that of representing a jointly irreducible pair of one-parameter groups satisfying the corresponding form, viz. the *Weyl form*, of the CCR. Then, by the *Stone-von Neumann Theorem* [Sto32, vN31] (see also expository article [Ros04]), all such pairs turn out to be unitarily equivalent. However, note that the Stone-von Neumann Theorem only holds for a system with finitely many degrees of freedom, i.e. for $[x_k, p_j] = \delta_{k,j} i\hbar$, $x_k x_j = x_j x_k$,

$p_k p_j = p_j p_k$ with $k, j \in \{1, \dots, n\}$. For a discussion of the representations of the CCR with infinitely many degrees of freedom, the reader may consult Section 7.2 of [Ros04]. Returning to the Fock space setting, letting $\hat{x}_k = (a(e_k) + a(e_k)^*)/4$ and $\hat{p}_j = (a(e_j) - a(e_j)^*)/(4i)$ yields the relation $[\hat{x}_k \hat{p}_j, \hat{p}_j \hat{x}_k] = i\delta_{k,j}$. Thus, at least when dealing with finitely many degrees of freedom, the symmetric Fock space representation of the CCR is equivalent to all others. •

Similarly, for the anti-symmetric (fermionic) Fock space $\mathscr{F}_{-1}(\mathscr{H})$, one may easily show using (2.18) and (2.17) that for all $g, h \in \mathscr{H}$,

$$\begin{aligned} &a_{-1}(h)a_{-1}(g)^* + a_{-1}(g)^* a_{-1}(h) = \langle h, g\rangle_{\mathscr{H}} 1 \\ &a_{-1}(h)a_{-1}(g) = -a_{-1}(g)a_{-1}(h), \\ &a_{-1}(h)^* a_{-1}(g)^* = -a_{-1}(g)^* a_{-1}(h)^*, \end{aligned} \tag{CAR}$$

with the above commutation relations referred to as the *canonical anti-commutation relations (CAR)*. Analogously to Remark 1, there is a fermionic analogue of the Stone-von Neumann theorem (see e.g. Theorem 7.1 in [Ros04]) ensuring the equivalence of the representations of the CAR algebra with finitely many degrees of freedom. In the fermionic setting, the field operators $s_{-1}(h) = a_{-1}(h) + a_{-1}(h)^*$ are easily shown to satisfy

$$s_{-1}(h)s_{-1}(g) + s_{-1}(g)s_{-1}(h) = 2\langle h, g\rangle_{\mathscr{H}} 1, \tag{2.25}$$

which are the *Clifford relations.* In particular, the von Neumann algebra (cf. Appendix C) generated by the fermionic field operators, namely $\Gamma_{-1}(\mathscr{H}) := vN\{s_{-1}(h) \mid h \in \mathscr{H}\}$, is the (complex von Neumann) Clifford algebra over $\mathscr{H}$.

Finally, note that for the full Fock space $\mathscr{F}_0(\mathscr{H})$, no such commutation relations can be shown to hold (for reasons apparent in Section 2.3.4). Instead, using (2.9) and (2.10) yields

$$a_0(h)a_0(g)^* = \langle h, g\rangle_{\mathscr{H}} 1, \tag{0-CR}$$

for all $g, h \in \mathscr{H}$.

2.3.2 q-CR and subalgebras of $\mathscr{B}(\mathscr{F}_q(\mathscr{H}))$

As discussed in the previous chapter, the q-Fock space $\mathscr{F}_q(\mathscr{H})$ provides an interpolation between the symmetric, anti-symmetric, and full Fock spaces. This interpolation also occurs at the level of commutation relations. Specifically, using (2.20) and (2.21),

$$a_q(h)a_q(g)^* - qa_q(g)^*a_q(h) = \langle h, g\rangle_{\mathscr{H}}1. \tag{q-CR}$$

The above is typically referred to as the *q-commutation relation.* The existence of operators on a Hilbert space satisfying q-CR was first conjectured in the physics litterature, by Frisch and Bourret [FB70]. In addition to the (explicit) Fock-space realization provided by Bożejko and Speicher [BS91], there also exist various existence proofs of the realizability of these relations [BS94, Fiv90, Gre91, Spe92, Spe93, YW94, Zag92].

The present section will deal with the algebra of bounded linear operators on the q-Fock space $\mathscr{F}_q(\mathscr{H})$ for $1 < q < -1$. As mentioned in the previous chapter, the creation and annihilation operators on $\mathscr{F}_1(\mathscr{H})$ are unbounded, and one would instead need to consider an *affiliated von Neumann algebra.*[5] Though the anti-symmetric Fock space suffers from no such ailment, further discussion of operators on $\mathscr{F}_1(\mathscr{H})$ and $\mathscr{F}_{-1}(\mathscr{H})$ is deferred to the following chapter, which will provide a more natural setting for these objects.

Consider the subalgebras of $\mathscr{B}(\mathscr{F}_q(\mathscr{H}))$ generated by the creation and annihilation operators, namely the C^* algebra (cf. Appendix C)

$$\mathcal{A}_q := C^*\{a_q(h) \mid h \in \mathscr{H}\}, \tag{2.26}$$

and the von Neumann algebra

$$\mathcal{W}_q := vN\{a_q(h) \mid h \in \mathscr{H}\}. \tag{2.27}$$

[5]If $\mathcal{A}$ is a von Neumann algebra acting on a Hilbert space $\mathscr{H}$ and D a closed and densely defined operator on $\mathscr{H}$, then D is said to be *affiliated with* $\mathcal{A}$ if D commutes with every unitary operator in $\mathcal{A}'$.

The C^* algebra $\mathcal{A}_q$ was studied by Dykema and Nica. In [DN93], they constructed an explicit unitary $U_q \in \mathscr{B}(\mathscr{F}_q(\mathscr{H}))$ for which $U_q\mathcal{A}_qU_q^* \supseteq \mathcal{A}_0$ and showed that, at least for small values of q, it is the case that $U_q\mathcal{A}_qU_q^* = \mathcal{A}_0$. This result was very recently improved by Kennedy and Nica [KN11], who constructed a unitary $\hat{U}_q$ with the property that $\hat{U}_q\mathcal{A}_q\hat{U}_q^* \subseteq \mathcal{A}_0$ for the full range $q \in (-1, 1)$.

Passing to von Neumann algebras, it is well known that for the full Fock space, $\mathcal{W}_0$ is the whole of $\mathscr{B}(\mathscr{F}_0)$. Using the result of Dykema and Nica, the same observation then extends to the q-Fock space, namely:

Theorem 3 (e.g. Theorem 5 in [Kem05]). *For $-1 < q < 1$, $\mathcal{W}_q = \mathscr{B}(\mathscr{F}_q(\mathscr{H}))$.*

For the probabilistic considerations, further discussed in Chapter 3, the more natural setting is provided by the *q-Gaussian algebra* Γ_q which is the von Neumann algebra generated by the field operators on $\mathscr{F}_q$, i.e.

$$\Gamma_q := vN\{a_q(h) + a_q(h)^* \mid h \in \mathscr{H}\}. \tag{2.28}$$

The q-Gaussian algebra will be the focus of the remainder of this chapter.

2.3.3 The vacuum expectation state

In the probabilistic setting of Chapter 3, the bounded linear operators on the familiar Fock spaces will be interpreted as certain non-commutative random variables. For instance, as anticipated, the elements of the q-Gaussian algebra Γ_q will turn out to be natural generalizations of classical Gaussian random variables. In the framework of non-commutative probability, the role of the classical expectation is played by a linear functional defined on the algebra containing the random variables, so that one may evaluate the expectation of sums and products of random variables (and, in particular, their "moments"). In the Fock space setting, the natural notion of expectation on the algebra containing the bounded linear operators is the functional $a \mapsto \langle a\Omega, \Omega\rangle$, referred to as the *vacuum expectation state*[6] and denoted φ_q.

[6]Note that in compliance with the physics conventions, vacuum expectation state is instead frequently given as $a \mapsto \langle \Omega, a\Omega\rangle$. In that case, the inner product would be taken as conjugate-linear

The vacuum expectation state has the form of a *vector state* (with respect to the vacuum vector), which is an object that arises naturally in the operator algebraic setting. A vector state is a key ingredient in the representation theorem known as the Gelfand-Naimark-Segal (GNS) construction, which yields explicit Hilbert-space representations of certain classes of Banach algebras (namely, of C^* algebras). For further details, the reader is referred to the Appendix C. At present, we take a moment to review some technical aspects of φ_q, which will be of use in Chapter 3.

Any linear functional φ on a Banach space $\mathscr{X}$ is said to be a *trace* if $\varphi(x_1x_2) = \varphi(x_2x_1)$ for all $x_1, x_2 \in \mathscr{X}$. It is a general fact that there cannot exist a trace on $\mathscr{B}(\mathscr{F}_q(\mathscr{H}))$, and it is easy to see that the vacuum expectation state in particular is non-tracial. For example, note that for any vector e in $\mathscr{H}$ with unit norm,

$$\varphi_q(a_q(e)a_q(e)^*) = 1 \neq \varphi_q(a_q(e)^*a_q(e)) = 0.$$

The above example also serves to show that φ_q is not *faithful*, in that $\varphi_q(a_q(e)^*a_q(e)) = 0$ even though $a_q(e)^*a_q(e) \neq 0$. However, the restriction of φ_q to the q-Gaussian algebra Γ_q is both tracial and faithful. The traciality can be easily seen by adopting an appropriate combinatorial representation, such as the labeled Dyck path and chord-crossing diagram representations used in Chapter 4, but we instead refer the reader to an algebraic proof that works in a more general setting (see Theorem 4.4 of [BS94]). The faithfulness of φ_q follows from the fact that the vacuum vector Ω is cyclic and separating for Γ_q (see Theorem 4.3 of [BS94]).

That Ω is *cyclic* for Γ_q is equivalent to the statement that for any vector $h \in \mathscr{H}$, there is an element $\Psi(h) \in \Gamma_q$ such that $\Psi(h)\Omega = h$. The cyclicity of Ω for Γ_q is then readily shown by induction: in particular, to realize any tensor $h_1 \otimes \ldots \otimes h_n$, it suffices to write

$$h_1 \otimes \ldots \otimes h_n = s_q(h_1) \ldots s_q(h_n)\Omega + \eta, \tag{2.29}$$

where η now belongs to $\oplus_{i=0}^{n-1} \mathscr{H}_{\mathbb{C}}^{\otimes i}$. That Ω is *separating* for Γ_q means that for every $a \in \Gamma_q$, we have $a\Omega = 0$ only if $a = 0$. Therefore, if Ω is both cyclic and separating

in the first coordinate and linear in the second.

for Γ_q, then for every $h \in \mathscr{H}$, there is a *unique* operator $\Psi(h) \in \Gamma_q$ with the property that $\Psi(h)\Omega = h$. In fact, one can show (see Corollary 2.8 in [BKS97]) that

$$\Psi(f^{\otimes n}) = \sum_{k=0}^{n} \binom{n}{k}_q (a_q(f)^*)^k a_q(f)^{n-k}, \tag{2.30}$$

where

$$\binom{n}{k}_q := \frac{[n]_q[n-1]_q \ldots [1]_q}{[n-k]_q[n-k-1]_q \ldots [1]_q\,[k]_q[k-1]_q \ldots [1]_q} \qquad \text{and} \qquad [n]_q := \frac{1-q^n}{1-q}, \tag{2.31}$$

for all $n, k \in \mathbb{N}$. An equivalent manner of writing down the element $\Psi(f^{\otimes n})$ makes use of the *q-Hermite polynomials.* The latter are an orthogonal polynomial sequence $\{H_n(z;q)\}_{n\in\mathbb{N}_0}$ given by the three-term recurrence

$$zH_n(z;q) = H_{n+1}(z;q) + [n]_q H_{n-1}(z;q) \tag{2.32}$$

with initial conditions

$$H_0(z;q) = 1, \qquad H_1(z;q) = z. \tag{2.33}$$

For general properties of the q-Hermite orthogonal polynomials, the reader is referred to [ISV87]. In the meantime, note that one may write (cf. Proposition 2.9 in [BKS97]):

$$\Psi(f^{\otimes n}) = H_n^{(q)}(s_q(f)). \tag{2.34}$$

Note that $\Psi(f_1 \otimes \ldots \otimes f_n)$ also has an explicit form given in terms of *Feynman diagrams* [EP03].

Remark 2 For the interested reader, we pause here to show that Ω is indeed separating for Γ_q. Making use of the notion of the *commutant* of an algebra (see Appendix C), we show the equivalent statement, which is that Ω is cyclic for the commutant Γ_q'.

Following [BS94], define the anti-linear conjugation operator $J : \mathscr{F}_q \to \mathscr{F}_q$ by $Ja\Omega = a^*\Omega$ for all $a \in \Gamma_q$. To see that J is well defined, suppose that $a_1^*\Omega = a_2^*\Omega$ for some $a_1, a_2 \in \Gamma_q$ and using traciality of the vacuum expectation, note that

$$0 = \|(a_1^* - a_2^*)\Omega\|^2 = \langle (a_1 - a_2)(a_1^* - a_2^*)\Omega, \Omega\rangle = \langle (a_1^* - a_2^*)(a_1 - a_2)\Omega, \Omega\rangle = \|(a_1 - a_2)\Omega\|^2,$$

implying that $a_1\Omega = a_2\Omega$. Since $(a_1a_2)^* = a_2^*a_1^*$, one may verify that every element of $J\Gamma_qJ$ commutes with all of Γ_q, i.e. $J\Gamma_qJ \subseteq \Gamma_q'$. Analogously to the proof that Ω is cyclic for Γ_q, Ω is shown to be cyclic for $J\Gamma_qJ$, and therefore (by inclusion) for Γ_q'. By the previously mentioned equivalence, Ω is therefre separating for Γ_q.

Note that the conjugation operator J used above has a meaningful explicit form. Specifically, it can be shown to act on pure tensors by "reversing the order":

$$J(f_{i(1)} \otimes \ldots \otimes f_{i(n)}) = f_{i(n)} \otimes \ldots \otimes f_{i(1)},$$

for all $f_{i(1)}, \ldots, f_{i(n)} \in \mathscr{H}$. Moreover, Γ_q' can be shown to be nothing else but the algebra of the *right field operators* on $\mathscr{F}_q$. Specifically, for $f \in \mathscr{H}$, define the *right creation operator* $a_{q,R}(h)^*$ as

$$a_{q,R}(h)^*\Omega = h, \qquad a_{q,R}(h)^* f_1 \otimes \ldots \otimes f_n = f_1 \otimes \ldots \otimes f_n \otimes h, \tag{2.35}$$

and the corresponding *right annihilation operator* $a_{q,R}(h)$ as

$$a_{q,R}(h)\Omega = 0, \qquad a_{q,R}(h) f_1 \otimes \ldots \otimes f_n = \sum_{k=1}^{n} q^{n-k} \langle h, f_k\rangle_{\mathscr{H}} f_1 \otimes \ldots \check{f}_k \otimes \ldots \otimes f_n. \tag{2.36}$$

To show that $a_{q,R}(h)^*$ is indeed the adjoint of $a_{q,R}(h)$ in the usual inner product $\langle\, , \rangle_q$ on $\mathscr{F}_q$, it is helpful to rewrite the latter recursively, as

$$\langle f_1 \otimes \ldots \otimes f_n, h_1 \otimes \ldots \otimes h_n\rangle_q = \sum_{k=1}^{n} q^{n-k} \langle f_n, h_k\rangle_{\mathscr{H}} \langle f_1 \otimes \ldots \otimes f_{n-1}, h_1 \otimes \ldots \otimes \check{h}_k \otimes \ldots \otimes h_n\rangle_q. \tag{2.37}$$

The reader may verify that the above reformulation (2.37) is equivalent to the formulations (2.22) and (2.19) of the previous section. With these elements, one can now show that $\Gamma_q' = vN\{a_{q,R}(h)^* + a_{q,R}(h) \mid h \in \mathscr{H}\}$. In the case of the (q,t)-*Gaussian* *algebras*, which will be introduced and studied in Chapter 4, the commutant will be more complicated than the right field operators. In fact, on account of the vacuum expectation state $\varphi_{q,t}$ no longer being a trace, we will not have such convenient access to a well-defined conjugation operator. •

2.3.4 Factoriality

We conclude the present chapter with an aside on certain more advanced aspects of the q-Gaussian algebra over a Hilbert space $\mathscr{H}$, i.e. of $\Gamma_q(\mathscr{H}) =: \Gamma_q$. Beyond its probabilistic interpretations, discussed in the upcoming Chapter 3, Γ_q is an interesting object in its own right and remains subject of a certain amount of speculation and ongoing research in the field of von Neumann algebras.

For $q \in [-1, 1)$, Γ_q is a highly non-commutative object – it is a *factor*, that is, a von Neumann algebra with a trivial center. In other words, there is no element of Γ_q other a scalar multiple of the identity that commutes with all of Γ_q. Proving the factoriality Γ_q for the same parameter range was, however, not a trivial matter.

It was long known that Γ_0 is a factor of isomorphism type II_1 (cf. Appendix C). Moreover, as previously mentioned, $\Gamma_{-1}(\mathscr{H})$ is the Clifford algebra over $\mathscr{H}$, and therefore a hyperfinite II_1 factor. For the full range $q \in [1, 1)$, the factoriality of Γ_q and its isomorphism type[7] were first established in [BKS97], but only in the case when $\mathscr{H}$ infinite-dimensional. The nature of Γ_q for finite-dimensional Hilbert spaces remained elusive for several years, though it was highly suspected that Γ_q should indeed be a factor as long as $\dim \mathscr{H} \geq 2$. (If $\dim \mathscr{H} = 1$, then $\mathscr{H}$ is the one-dimensional vector space over $\mathbb{R}$ and, on account of the multilinearity of the tensor product, Γ_q becomes a commutative algebra.) In [Śni04], Śniady weakened the dimensionality assumption to allow for the case where the dimension of $\mathscr{H}$ exceeds a fixed function of q. Finally, in the remarkably concise paper [Ric05], Ricard successfully proved the factoriality of Γ_q in full generality.

Theorem 4 (Theorem 1 in [Ric05]). *For $q \in (-1, 1)$ and $\dim \mathscr{H} \geq 2$, Γ_q is a factor.*

[7]As expected, for $q \in [1, 1)$, Γ_q is also of type II_1, which stems from the fact that the vacuum expectation state on Γ_q is an ultraweakly continuous normalized faithful trace (cf. Appendix C).

There also happens to be a second, as of yet unfinished chapter on the intrigue surrounding the q-Gaussian algebras. It was long known that when $\dim \mathscr{H} = n$, Γ_0 is is isomorphic to the *free group factor of order* n (cf. Appendices A and C) and it is presently conjectured that the isomorphism extends to the full range $q \in [-1, 1)$. The free group factors remain one of the most mysterious, yet also one of the most assiduously studied objects in the field of operator algebras. One of the main outstanding conjectures in this field is the so-called *free group factor isomorphism problem*, which asks a seemingly simple question:

Is the free group factor of order n isomorphic to the free group factor of order m, for all $n, m \in \mathbb{N}$?

It was, in fact, through his work on the free group factor isomorphism problem that Voiculescu recognized the depth of the probabilistic structure that arises when Γ_0 is equipped with the vacuum expectation state – the result was the theory of *free probability* [Voi86, VDN92] – and this a priori purely operator algebraic question keeps giving rise to new techniques that draw on diverse areas of mathematics. Such a fertilization can also be observed at the level of an a priori simpler question, namely that of ascertaining whether q-Gaussian algebra over an n-dimensional Hilbert space is isomorphic to the free group factor of order n for general values $q \in [-1, 1)$.

An increasing amount of similarity was progressively shown to hold between Γ_q and the free group factor. For instance, both are II_1 factors, both are also non-injective ([BKS97, Nou04]) and solid [Shl04]. In a very recent breakthrough, representing a culmination of over two decades of work by various authors, Guionnet and Shlyakhtenko [GS12] have shown that for sufficiently small values of q (with bounds depending on the dimension of $\mathscr{H}$), Γ_q is indeed isomorphic to the free group factor. The proof of Guionnet and Shlyakhtenko is based on new methods. In particular, as an answer to an operator algebraic question of recognizing free group factors, it draws on the probabilistic perspective in a novel way in that it hinges on the solution of the free analogue of a certain classical stochastic differential equation (viz. the *Monge-Ampére equation*). Understanding the structure of Γ_q for the full range $q \in [-1, 1)$ may require further innovations, testifying to the richness and the depth of this object.

Chapter 3

Non-commutative Probability

The setting of this thesis is principally that of *non-commutative probability*[1], which may be viewed as the theory that arises when operator-algebraic frameworks are endowed with probabilistic interpretations. Non-commutative probability takes root in *quantum probability* [HP84] (see also [Bia95, Mey93]), which describes the probabilistic aspects of quantum observables. The fundamental bosonic observables are governed by the canonical commutation relations (CCR) and the underlying probabilistic framework can be cast in the setting of classical probability. In particular, the symmetric Fock space introduced in the previous chapter is intimately tied to the so-called *Wiener chaos decomposition*, discussed shortly. In contrast, the analogous fermionic observables are governed by the canonical *anti*-commutation relations (CAR). Due to this inherent anti-commutativity, fermionic probability cannot be fully described in the setting of classical probability – one must instead generalize the notion of a random variable from a measurable function to an operator on a Hilbert

[1]Due to the subject's diversity and its relative youth, the bibliography for the present chapter is spread between a number of essential references. Quantum probability is the focus of a St-Flour course given in 1995 by Biane [Bia95] (in French) and of the monograph *Quantum probability for probabilists* by Meyer [Mey93]. (See also Parthasarathy's original text [Par92].) Wiener chaos and the symmetric Fock space are discussed in Janson's *Gaussian Hilbert Spaces* [Jan97]. Two excellent texts for free probability are Voiculescu, Dykema, Nica's *Free Random Variables* [VDN92] and Nica and Speicher's *Lectures in the Combinatorics of Free Probability* [NS06], while Biane's expository paper *Free probability for probabilists* [Bia03] offers what is likely the most vibrant introduction to the field. The present chapter does not discuss random matrix theory, and the reader is instead encouraged to consult *An introduction to random matrices* by Anderson, Guionnet, and Zeitouni [AGZ10]. The material on the q-Fock spaces is drawn from the various in-line references, with the article [BKS97] by Bożejko, Kümmerer, and Speicher as the recommended introductory reference.

space. The resulting framework may be considered as that of *anti-commutative probability*, whose probabilistic depth as the natural counterpart of classical probability is well understood (e.g. [Bia95, Mey93, Par92]).

In its modern form, non-commutative probability is an abstract theory broadly concerned with the properties of a state acting on an operator algebra. In particular, Kolmogorov's probability triple is replaced by a *non-commutative probability space* $(\mathcal{A}, \varphi)$, where $\mathcal{A}$ is a unital $*$-algebra (cf. Appendix C) whose elements are interpreted as *non-commutative random variables* and the unital linear functional $\varphi : \mathcal{A} \to \mathbb{C}$ plays the role of the expectation. As will be seen shortly, many classical scenarios can be recast in this setting, though the real strength of the theory comes in ascribing probabilistic intuition to a far broader class of objects. While the the non-commutative probability is significantly more general than the quantum probabilistic setting from which it arose, it is nevertheless the case that the richest non-commutative probability theories known today can be viewed as arising from some deformation of the bosonic/fermionic structure. In particular, this is the case of both the free probability of Voiculescu [Voi86, VDN92] and of the q-deformed probability of Bożejko and Speicher [BS91]. It is also the case of the two-parameter continuum of probability theories introduced in this thesis. Specifically, all of the aforementioned theories (including also the bosonic and the fermionic settings) are realized on non-commutative probability spaces where $\mathcal{A}$ is an algebra of bounded linear operators on a suitable Fock space and φ is the vacuum expectation state.

The relevant Fock spaces, the algebras of operators on these spaces, and the properties of the vacuum expectation state were the subject of the previous chapter. At present, we focus on the probabilistic aspects of these objects. The discussion will begin with the role of the symmetric (bosonic) Fock space in classical probability, and the chaos decomposition to which it gives rise. Further on in the chapter, we will see that unlike in the bosonic case, the probabilistic intuition in the fermionic, free, and q-deformed setting will not be carried by the Fock space itself, but by the bounded linear operators on this space. Following a review of general results in non-commutative probability and a discussion of "non-commutative independence", the remainder of the chapter will focus on the non-commutative probability theory that arises from the algebra of bounded linear operators on the q-Fock space (comprising the fermionic and the free cases) when equipped with the vacuum expectation state.

Note that combinatorics will make their first appearance in this chapter, through the classical and quantum *Wick formulas.* At this point, the objects we will need are *pair-partitions*, which are partitions of a set into parts of size two. The collection of pair-partitions of $[2n]$ will be denoted $\mathscr{P}_2(2n)$ with $\mathscr{V} \in \mathscr{P}_2(2n)$ usually written as $\mathscr{V} = \{\{w_1, z_1\}, \ldots, \{w_n, z_n\}\}$. In the non-commutative setting, we will need to impose an order on the elements of each part and, frequently, also on the parts themselves. In that case, $\mathscr{V}$ will be written as $\{(w_1, z_1), \ldots, (w_n, z_n)\}$, ordered so that $w_i < z_i$, or even $((w_1, z_1), \ldots, (w_n, z_n))$ with the understanding that, additionally, $w_1 < \ldots < w_n$. Despite the fact that the proofs of many of the results presented in this chapter are in essence largely combinatorial, we will not discuss the combinatorial point of view much further until Chapter 4. At that point, the view of combinatorics as one of the tools of choice in non-commutative probability will become more apparent.

3.1 Symmetric Fock space and the Wiener chaos

Recall that the classical Gaussian random variable can be viewed as an element of the Hilbert space $\mathscr{L}^2_{\mathbb{R}}(\mathbb{R}, \mathcal{B}, \lambda)$, for $\mathcal{B}$ the Borel σ-field on $\mathbb{R}$ and λ the Lebesgue measure; in particular, for given $\mu \in \mathbb{R}$ and $\sigma^2 \geq 0$, the Gaussian density is the function $(2\pi\sigma^2)^{-1/2} \exp(-(x-\mu)/2\sigma^2)$. A Gaussian has as characteristic function (Fourier transform) the map $t \mapsto \exp(i\mu t - \sigma^2 t^2/2)$. Of course, the desired characteristic function can be realized on a broad class of measure spaces and $(\mathbb{R}, \mathcal{B}, \lambda)$ will from now on be replaced by whichever suitable measure space $(\Omega, \Psi, \mathbb{P})$.

A Gaussian element of mean zero (i.e. $\mu = 0$) is said to be *centered.* If $(\xi_1, \ldots, \xi_m)$ is a centered *Gaussian family,* i.e. a zero-mean multivariate Gaussian, the joint moments of the ξ_i's are given as the sum of all possible factorizations as a product of covariances, i.e.

$$\mathbb{E}(\xi_{i_1}\dots\xi_{i_{2n-1}}) = 0 \tag{3.1}$$

$$\mathbb{E}(\xi_{i_1}\dots\xi_{i_{2n}}) = \sum_{\mathscr{V}\in\mathscr{P}_2(2n)} \mathbb{E}(\xi_{i_{z_1}}\xi_{i_{w_1}})\dots\mathbb{E}(\xi_{i_{z_n}}\xi_{i_{w_n}}), \tag{3.2}$$

for $2n \leq m$, where each pair-partition $\mathscr{V} \in \mathscr{P}_2(2n)$ is written as $\mathscr{V} = \{\{z_1, w_1\}, \dots, \{z_n, w_n\}\}$. In physics, (3.2) is typically referred to as the *Wick formula.*

The more general setting in which to discuss classical interpretations of "Gaussian elements" is provided by the *Gaussian Hilbert spaces.* These also happen to be the central objects connecting the symmetric Fock spaces to the setting of classical probability. As we will for the sake of simplicity be working with *real* Gaussian random variables, the symmetric Fock spaces appearing in this section will not be taken over a *complexification* of a real Hilbert space, but over the real Hilbert space *itself.* However, the results are readily adaptable to the case of complex Gaussian random variables, in which case the complexification of the underlying Hilbert space will appear. For the statements of the relevant results in the complex case, the reader is referred to [Jan97].

Definition 2. *Given a probability space $(\Omega, \Psi, \mathbb{P})$ and V some linear subspace of $\mathscr{L}^2_{\mathbb{R}}(\Omega, \Psi, \mathbb{P})$ consisting of centered Gaussian elements, the closure $\mathscr{H}$ of V with respect to the norm on $\mathscr{L}^2_{\mathbb{R}}(\Omega, \Psi, \mathbb{P})$ is said to be a (real)* Gaussian Hilbert space.

In particular, if $(\xi_1, \dots, \xi_m)$ is a centered Gaussian family, then the closure of their linear span $\{\sum_{i=1}^m \alpha_i\xi_i \mid \alpha_i \in \mathbb{R}\}$ is a finite-dimensional Gaussian Hilbert space. Conversely, any finite-dimensional Gaussian Hilbert space is easily seen to be of this form.

From now on, fix $\mathscr{H}$ to be a Gaussian Hilbert space defined on a probability space $(\Omega, \Psi, \mathbb{P})$. Let $\mathcal{P}_n(\mathscr{H})$ be the linear space of all elements of the form $p(h_1, \dots, h_m)$, taken over polynomials p of degree at most n and all $h_1, \dots, h_m \in \mathscr{H}$ $(m \in \mathbb{N})$. Since the elements of $\mathscr{H}$ belong to $\mathscr{L}^p_{\mathbb{R}}(\Omega, \Psi(\mathscr{H}), \mathbb{P})$ for all $p \geq 1$, Hölder's inequality implies that any finite product of elements of $\mathscr{H}$ belongs to $\mathscr{L}^2_{\mathbb{R}}(\Omega, \Psi(\mathscr{H}), \mathbb{P})$. Then, letting $\overline{\mathcal{P}}_n(\mathscr{H})$ be the closure in $\mathscr{L}^2_{\mathbb{R}}(\Omega, \Psi(\mathscr{H}), \mathbb{P})$ of $\mathcal{P}_n(\mathscr{H})$, set

$$H^{:n:} := \overline{\mathcal{P}}_n(\mathscr{H}) \cap \overline{\mathcal{P}}_{n-1}(\mathscr{H})^{\perp}. \tag{3.3}$$

The subspace $\mathscr{H}^{:n:}$ is referred to as the *homogeneous chaos of order n*. One may readily verify that $\mathscr{H}^{:0:} = \mathbb{R}$ and $\mathscr{H}^{:1:} = \mathscr{H}$.

As a recapitulative example, consider the probability space $(\mathbb{R}, \mathcal{B}, \gamma)$, where $\mathcal{B}$ is the Borel σ-field and γ a non-degenerate centered Gaussian measure (i.e. $\mu = 0$, $\sigma^2 > 0$). Then, the map $x \mapsto x$ is a non-degenerate centered Gaussian element of $\mathscr{L}^2_{\mathbb{R}}(\mathbb{R}, \gamma)$ and $\mathscr{H} = \{tx \mid t \in \mathbb{R}\}$ is a Gaussian Hilbert space. The subspace $\overline{\mathcal{P}}_n(\mathscr{H})$ is the space of polynomials of degree at most n. Since the dimension of $\overline{\mathcal{P}}_n(\mathscr{H})$ is $n+1$, it follows from the definition (3.3) that $\mathscr{H}^{:n:}$ is one-dimensional. In fact, $\mathscr{H}^{:n:}$ can be seen to be spanned by H_n, the n^{th} term in the Hermite orthogonal polynomial sequence, introduced in the previous chapter. As it happens, $H_1, H_2, \dots$ form an orthonormal basis of $\mathscr{L}^2_{\mathbb{R}}(\mathbb{R}, \mathcal{B}, \gamma)$, but this is not a coincidence.

More generally , given a Gaussian Hilbert space $\mathscr{H}$ over a probability space $(\Omega, \Psi, \mathbb{P})$, the spaces $\mathscr{H}^{:n:}$ $(n \in \mathbb{N}_0)$ are readily seen to be mutually orthogonal, closed subspaces of $\mathscr{L}^2_{\mathbb{R}}(\Omega, \Psi, \mathbb{P})$. Moreover, $\{\overline{\mathcal{P}}_n(\mathscr{H})\}_{n \geq 0}$ is an increasing sequence of closed subspaces in $\mathscr{L}^2_{\mathbb{R}}(\Omega, \Psi, \mathbb{P})$ and one can show that $\overline{\mathcal{P}}_n(\mathscr{H}) = \oplus_{k=0}^{n} \mathscr{H}^{:k:}$. It follows that $\oplus_{k=0}^{\infty} \mathscr{H}^{:k:}$ equals the closure in $\mathscr{L}^2_{\mathbb{R}}(\Omega, \Psi, \mathbb{P})$ of $\cup_{n=0}^{\infty} \overline{\mathcal{P}}_n(\mathscr{H})$. Now let $\Psi(\mathscr{H})$ be the (coarser) σ-field generated by the elements of $\mathscr{H}$. Clearly, $\overline{\mathcal{P}}_n(\mathscr{H}) \subseteq \mathscr{L}^2_{\mathbb{R}}(\Omega, \Psi(\mathscr{H}), \mathbb{P})$ and therefore $\oplus_{k=0}^{n} \mathscr{H}^{:k:} \subseteq \mathscr{L}^2_{\mathbb{R}}(\Omega, \Psi(\mathscr{H}), \mathbb{P})$. But, it is also possible to show the reverse inclusion (cf. Theorem 2.6 in [Jan97]). This fact is at the core of the following celebrated result.

Theorem 5 ([Wie38, Ito51, Seg56], see also Corollary 2.8 in [Jan97]). *If $\mathscr{H}$ is a Gaussian Hilbert space and $\Psi(\mathscr{H})$ the σ-field generated by $\mathscr{H}$, then $\mathscr{L}^2_{\mathbb{R}}(\Omega, \Psi(\mathscr{H}), \mathbb{P})$ has the orthogonal decomposition*

$$\mathscr{L}^2_{\mathbb{R}}(\Omega, \Psi(\mathscr{H}), \mathbb{P}) = \bigoplus_{n=0}^{\infty} \mathscr{H}^{:n:} \tag{3.4}$$

The above result is referred to as the *Wiener chaos decomposition.* The chaos decomposition can be also carried out at the level of individual elements. For that, let π_n be the orthogonal projection of $\mathscr{L}^2(\Omega, \Psi(\mathscr{H}), \mathbb{P})$ onto $\mathscr{H}^{:n:}$. This allows us to write down the chaos decomposition of a random variable $\xi \in L^2(\Omega, \Psi(\mathscr{H}), \mathbb{P})$, as

$$\xi = \sum_{n=0}^{\infty} \pi_n(\xi),$$

with the sum converging in the $\mathscr{L}^2$ sense. Note that $\pi_0(\xi) = \mathbb{E}(\xi)$.

Next, the *Wick product* of $\xi_1, \ldots, \xi_n \in \mathscr{H}$ is given by

$$: \xi_1 \ldots \xi_n := \pi_n(\xi_1, \ldots, \xi_n), \tag{3.5}$$

where π_n is again the orthogonal projection of $\mathscr{L}^2(\Omega, \Psi(\mathscr{H}), \mathbb{P})$ onto $\mathscr{H}^{:n:}$. (Also, let $: := 1 \in H^{:0:}$.) There is a general expression for $: \xi_1 \ldots \xi_n :$ that can be written down as a sum of *Feynman diagrams* or *incomplete pair partitions.* For now, we simply point out a few examples, namely:

$$\begin{aligned}
&: h_1 := h_1 \\
&: h_1 h_2 := h_1 h_2 - \mathbb{E}(h_1 h_2) \\
&: h_1 h_2 h_3 := h_1 h_2 h_3 - h_1 \mathbb{E}(h_2 h_3) - \mathbb{E}(h_1 h_3) h_2 - \mathbb{E}(h_1 h_2) h_3
\end{aligned}$$

One may show that the Wick product $: \xi_1 \ldots \xi_n :$ is the same for every Gaussian Hilbert space $\mathscr{H}$ containing $\xi_1, \ldots, \xi_n$. It is the Wick products that now yield the desired correspondence between the symmetric Fock spaces and the Hilbert spaces of classical probability.

Theorem 6 ([Seg56, DM77]). *If $\mathscr{H}$ is a real Gaussian Hilbert space, then the map*

$$h_1 \odot \ldots \odot h_n \mapsto : h_1 \ldots h_n :$$

defines a Hilbert-space isometry of $\mathscr{H}^{\odot n}$ onto $\mathscr{H}_{\mathbb{R}}^{:n:}$; this extends to the tensor algebras and then further to an isometry of the symmetric Fock space $\oplus_{n=1}^{\infty} \mathscr{H}^{\odot n}$ onto

$\oplus_{n=0}^{\infty}\mathscr{H}^{:n:} = \mathscr{L}^2_{\mathbb{R}}(\Omega, \Psi(\mathscr{H}), \mathbb{P})$.

The correspondence between the symmetric Fock space and the spaces of classical probability can be interpreted much further. For instance, working with the function space $\mathscr{L}^2_{\mathbb{R}}(\mathbb{R}_+, \mathcal{B}, \lambda)$ as a setting for classical Brownian motion, the annihilation operator $a_1(h)$ can be shown to correspond in this framework to the *Malliavin derivative* (also known as the *Cameron-Gross-Malliavin derivative*) in the direction of h. We will not pursue these analogies further, but the interested reader is referred to [Bia95, Mey93].

3.2 Non-commutative probability spaces

The classical random variables are measurable functions, and the algebras formed by classical random variables are therefore commutative. In *non-commutative probability*, measurable functions are replaced by operators on Hilbert spaces or, more generally, elements of a (non-commutative) algebra. The classical probability triple is therefore replaced by a pair consisting of an algebra, containing the "non-commutative random variables", and of a linear functional playing the role of expectation. More precisely:

Definition 3. *A* non-commutative probability space *is a pair $(\mathcal{A}, \varphi)$, where $\mathcal{A}$ is a unital algebra over $\mathbb{C}$ and $\varphi : \mathcal{A} \to \mathbb{C}$ a linear functional such that $\varphi(1_{\mathcal{A}}) = 1$. The elements of $\mathcal{A}$ are referred to as* non-commutative random variables.

Given an element $a \in \mathcal{A}$, applying φ to the monomials $a^n \in \mathcal{A}$ yields the analogue of the moments of a classical random variable. In the setting of $(\mathcal{A}, \varphi)$, the natural notion of a "distribution" of a non-commutative random variable is therefore a description (in whichever form) of its moment sequence $\{\varphi(a^n)\}_{n=1}^{\infty}$. If $\mathcal{A}$ is an $*$-algebra, the corresponding notion must be extended to φ evaluated on all non-commutative monomials in a and a^*. More generally, at the level of joint distributions (i.e. joint moments), the relevant notions are the following.

Definition 4. *Given a non-commutative probability space $(\mathcal{A}, \varphi)$, the* joint moments of $a_1, \ldots, a_n$ *are all expressions of the form*

$$\varphi(a_{i_1} \ldots a_{i_k}), \qquad \text{with } k \geq 0 \text{ and } i_1, \ldots, i_k \in [n].$$

If $\mathcal{A}$ is an $*$-algebra, the joint $*$-moments *are of the form*

$$\varphi(a_{i_1}^{\epsilon_1} \ldots a_{i_k}^{\epsilon_1}), \qquad \text{with } k \geq 0,\ i_1, \ldots, i_k \in [n] \text{ and } \epsilon_1, \ldots, \epsilon_k \in \{1, *\}.$$

In the non-commutative setting, it is important to realize that the product $a_{i_1} \ldots a_{i_k}$ is a non-commutative *word*; in particular, repetitions of indices are allowed and the ordering of the letters is fixed. Of course, at this point there is no rule preventing us from taking $\mathcal{A}$ to be a commutative algebra[2] and, in fact, much of the classical probability can be cast in this framework. But, so can various algebraic objects that carry no classical probabilistic intuition.

Example 3 Let $(\Omega, \Phi, \mathbb{P})$ be a classical probability space. Letting $\mathcal{A} = \mathscr{L}^\infty(\Omega, \Phi, \mathbb{P})$ and $\varphi = \mathbb{E}$ encompasses all essentially bounded classical random variables. A more useful setting is instead obtained by letting $\mathcal{A}$ be the algebra of classical random variables with finite moments of all orders, i.e. $\cap_{p\geq 1} L^p(\Omega, \Phi, \mathbb{P})$ (frequently denoted $\mathscr{L}^{\infty-}(\Omega, \Phi, \mathbb{P})$). (The reader should check that $\mathcal{A}$ is indeed closed under multiplication.) It is certainly not the case that *all* classical random variables of interest have moments of all orders and such exceptions will not be comprised within the non-commutative framework. •

Example 4 Consider the non-commutative probability space formed by an algebra $\mathcal{M}_n(\mathbb{C})$ of $n \times n$ complex matrices (under usual matrix multiplication) and the normalized trace $\mathcal{M}_n(\mathbb{C})$, i.e. let φ be the map

$$a \mapsto \frac{1}{n} \sum_{i=1}^{n} \alpha_{i,i}, \qquad \text{for } a = (\alpha_{i,j}) \in \mathcal{M}_n(\mathbb{C}). \tag{3.6}$$

[2]The picture will be different when we get to the notion of non-commutative independence. Specifically, free independence will turn out to be powerful for dealing with matricial limits, but of little use in the classical setting.

It follows that the $n \times n$ matrices form a natural non-commutative probability space.

Combined with the previous example, one may also consider the algebra $\mathcal{M}_n(\mathscr{L}^{\infty-}(\Omega, \Phi, \mathbb{P}))$ of *random matrices over* $(\Omega, \Phi, \mathbb{P})$ and the linear functional

$$a \mapsto \int tr(a(\omega))d\mathbb{P}(w), \qquad \text{for } a = (\alpha_{i,j}) \in \mathcal{M}_n(\mathscr{L}^{\infty-}(\Omega, \Phi, \mathbb{P})). \tag{3.7}$$

The non-commutative probability space is now that of *random $n \times n$ matrices.* •

Of course, most of the examples considered in the previous chapters give rise to non-commutative probability spaces. Here are two more particularly relevant ones.

Example 5 Let $\mathscr{H}$ be a Hilbert space and set $\mathcal{A}$ to be $\mathscr{B}(\mathscr{H})$. A natural choice for φ (but, by far not the only one) is the *vector-state*, i.e. the linear functional $a \mapsto \langle ae, e\rangle$, for some (fixed) unit vector e. If the Hilbert space in question is a Fock space, the unit vector is taken to be the vacuum vector Ω and the vector-state becomes the usual vacuum expectation state (cf. Section 2.3.3). •

Example 6 If G is a group, let $\mathcal{A}$ be the group algebra $\mathbb{C}G$ and φ be the functional τ_G given by $\tau_G(\sum \alpha_g g) = \alpha_e$. For further details, the reader is referred to Example 33 of Appendix A. •

Of course, additional assumptions on $\mathcal{A}$ and φ will yield a richer theory. For example, the algebras given in the previous examples are $*$ algebras, with the involution given as complex conjugation (Example 3), conjugate (Hermitian) transposition (Example 4), taking operator adjoints (Example 3), and $(\sum \alpha_g g)^* = (\sum \overline{\alpha}_g g^{-1})$ in Example 6. The functionals φ in Examples 3, 4, and 6 are positive, faithful traces. The additional structure that comes with such refinements is discussed in the remainder of this section.

3.2.1 C^* probability spaces

In the framework of non-commutative probability spaces, there will be a distinct advantage to working in the setting of a C^* probability space.

Definition 5 (Definition 3.7 in [NS06]). *A non-commutative probability space* $(\mathcal{A}, \varphi)$ *is a* C^* probability space *if* $\mathcal{A}$ *is a* C^* *algebra and the functional* φ *is positive, i.e.* $\varphi(a^*a) \geq 0$ *for all* $a \in \mathcal{A}$.

In a sense, the C^* probability spaces provide the simplest non-commutative framework with a probabilistic intuiton starting to resemble that found in the classical setting. For instance, if x is a classical random variable, functional composition and chain rule lend meaning to $f(x)$ for any measurable function f. For any non-commutative random variable a, one may analogously consider the random variable $g(a)$, where g is now any continuous function on $\mathrm{sp}(a)$ and $g(a)$ is defined by functional calculus (cf. Appendix C). Moreover, via functional calculus and the fact that every C^* algebra has a Hilbert-space representation, one can show that every normal element of $\mathcal{A}$ is now associated with a bona fide measure that encodes its moments:

Theorem 7 (e.g. Proposition 3.13 in [NS06]). *Let* $(\mathcal{A}, \varphi)$ *be a* C^* *probability space and let* a *be a normal element of* $\mathcal{A}$. *Then, there exists a compactly supported probability measure* μ *on* $\mathbb{C}$ *such that*

$$\varphi(a^k(a^*)^\ell) = \int z^k \overline{z}^\ell \, d\mu(z), \qquad \forall\, k, \ell \in \mathbb{N}. \tag{3.8}$$

The support of μ *is contained in the spectrum of* a. *Furthermore, for* f *any complex-valued continuous function on the spectrum of* a,

$$\varphi(f(a)) = \int f \, d\mu. \tag{3.9}$$

The reader may verify that above proposition follows immediately from the developments of the previous chapter. Indeed, letting $X = \mathrm{sp}(a)$ and $\rho : C(X) \to \mathcal{A}$

be the usual functional calculus, then $\varphi \circ \rho$ is a positive linear functional on $C(X)$. By the result of Example 17, it follows that there exists a positive measure μ with support in X and such that $\varphi \circ \rho(f) = \int_X f \, d\mu$ for every $f \in C(X)$. To obtain (3.8), it then suffices to take $f(z) = z^k \overline{z}^\ell$. Note that if φ additionally happens to be faithful, i.e. if $\varphi(a^*a) = 0$ only if $a = 0$, then one may show (e.g. Proposition 3.15 in [NS06]) that the support of μ *equals* $\mathrm{sp}(a)$.

Using the fact that φ is positive, one can show (e.g. Proposition 3.8 in [NS06]) that $|\varphi(a)| \leq \|a\|$ for all $a \in \mathcal{A}$. If φ is additionally faithful, there is an alternative description for the operator norm, that is often useful:

Proposition 2 (e.g. Proposition 3.17 in [NS06]). *If $(\mathcal{A}, \varphi)$ is a C^* probability space and φ is faithful, then for evey $a \in \mathcal{A}$,*

$$\|a\| = \lim_{n \to \infty} \varphi((a^*a)^n)^{1/2n}. \tag{3.10}$$

3.3 Non-commutative independence

As previously stated, in the non-commutative setting, the notion of a joint distribution of a collection of random variables is given by their joint moments. In particular, the idea that the marginal laws of "independent" random variables should completely characterize the joint law translates into a description of a rule on the factoring of the joint moments. That is, we are looking for possible rules by which an expression of the type $\varphi(a_{r_1} \ldots a_{r_m})$ can be uniquely written as a product of the individual (marginal) moments of $a_{r_1}, \ldots, a_{r_m}$.

In the general (non-commutative) setting, there are many possible notions of independence and the choice of a definition seems to be largely driven by the nature of the framework at hand. For example, the *free probability* of Voiculescu [Voi86, VDN92], which has shown extraordinary depth and has re-invigorated the field of non-commutative probability, arises naturally in settings where there is an absence of commutative structure (in a sense that will be made precise shortly). In contrast, as further discussed in Section 3.4, there is presently no natural notion of independence in the presence of q-commutation relations, but there are different specialized approaches that provide useful substitutes.

The quest for general notions of non-commutative independence is, in a concrete sense, doomed from the get-go. Specifically, as shown by Speicher in [Spe97], there are only three possible general moment-factoring frameworks that behave as a *universal product*; these correspond to *classical, free, and boolean independence.* Later, Muraki [Mur96, Mur97] and Lu and De Giosa [Lu97, DGL97, DGL98] relaxed one of Speicher's assumptions (an inherent ordering assumption) and showed that there are two more sensible general notions of independence, viz. the *monotone and anti-monotone independence.*

The underlying idea behind the notion of a *universal product* is that one should be able to describe an associative factoring rule that, when applied to the joint moment $\varphi(a_1 \dots a_n)$, only takes into account the structural properties of the string $a_1 \dots a_n$. For example, if a_1, a_2, and a_3 are all "independent" and identically distributed, then the moment $\varphi(a_1 a_2 a_1)$ should be the same as $\varphi(a_3 a_1 a_3)$ (which need not be the same as $\varphi(a_3 a_3 a_1)$). More generally, let π be the partition of $[n]$ that determines the repetition pattern in $a_1 \dots a_n$, where two elements i_1, i_2 of $[n]$ will belong to the same part (block) in π if and only if $a_{i_1} = a_{i_2}$. Furthermore, letting $\mathscr{P}(n)$ denote the partially ordered set (in fact, lattice) of partitions of $[n]$ with the ordering relation given by the reverse inclusion order, one may further refine π as a partition $\sigma \leq \pi$. Now, let $\varphi_\sigma(a_1 \dots a_n)$ denote the factoring of φ over the product $a_1 \dots a_n$ according to the blocks in σ in a manner that respects the natural ordering. (For a precise definition, the reader is referred to [Spe97] or to Chapter 5, which uses the factoring of moments over interval partitions.) How much each $\varphi_\sigma(a_1 \dots a_n)$ contributes to the moment $\varphi(a_1 \dots a_n)$ is controlled by the weight $t(\sigma; \pi)$ and one wishes to write

$$\varphi(a_1 \dots a_n) = \sum_{\substack{\sigma \in \mathscr{P}(n) \\ \sigma \leq \pi}} t(\sigma; \pi) \varphi_\sigma(a_1 \dots a_n).$$

One can then formalize the notion of a universal product as follows.

Definition 6 ([Spe97]). *Given a family of non-commutative probability spaces*

$\{(\mathcal{A}_i, \varphi_i)\}_{i\in I}$, *a* universal product *of two non-commutative probability spaces,* $(\mathcal{A}_i, \varphi_i)$ *and* $(\mathcal{A}_j, \varphi_j)$*, is formed by an algebra* $\mathcal{A}$ *given as the free product (with identification of units if* $\mathcal{A}_i, \mathcal{A}_j$ *are unital) and the expectation denoted* $\varphi_i \cdot \varphi_j$ *such that:*

i. For all triples $(\mathcal{A}_i, \varphi_i)$, $(\mathcal{A}_j, \varphi_j)$, $(\mathcal{A}_k, \varphi_k)$,

$$\varphi_i \cdot (\varphi_j \cdot \varphi_k) = (\varphi_i \cdot \varphi_j) \cdot \varphi_k.$$

ii. For each $n \in \mathbb{N}$ *and partitions* π, σ *of* $[n]$ *with* $\sigma \leq \pi$*, there are constants* $t(\sigma;\pi)$ *such that for all* $a_{\ell_1} \dots a_{\ell_n}$ *(with* $a_{\ell_m} \in \mathcal{A}_{\ell_m}$ *and* $\ell_1, \dots, \ell_n \in \{i, j\}$*) with repetition pattern* π,

$$\varphi(a_{\ell_1} \dots a_{\ell_n}) = \sum_{\substack{\sigma \in \mathscr{P}(n) \\ \sigma \leq \pi}} t(\sigma;\pi)\, \varphi_\sigma(a_{\ell_1} \dots a_{\ell_n}), \tag{3.11}$$

where $\mathscr{P}(n)$ *denotes the collection (in fact, lattice) of partitions of* $[n]$ *with the ordering relation given by the reverse inclusion order.*

Different choices of the weight function $t(\sigma;\pi)$ then give rise to different notions of independence. As discussed previously, there are only five choices available, but we will focus on two: the classical (tensor) independence and the free independence. As previously discussed, the classical framework can be realized on the symmetric Fock space, whereas free framework will correspond to the full Fock space. The reader may wish to note that the Boolean independence and the two forms of monotone independence also have representations on suitable Fock spaces (see [Mur97, Lu97] and [BS04], respectively).

3.3.1 Classical (tensor) independence

Consider a family of subalgebras $\mathcal{A}_i$ of $\mathcal{A}$ indexed over some set I. Subalgebras $\{\mathcal{A}_i\}$ are said to be *tensor independent* if

- $\{\mathcal{A}_i\}$ commute with each other, i.e. $[a_i, a_j] = 0$ if $i \neq j$ for every $a_i \in \mathcal{A}_i$ and $a_j \in \mathcal{A}_j$;

- $\varphi(a_{i_1} \dots a_{i_n}) = \varphi(a_{i_1}) \dots \varphi(a_{i_n})$ whenever $a_{i_k} \in \mathcal{A}_{i_k}$ $(k = 1, \dots, n)$ and there are no repeated elements in $i_1, \dots, i_n \in I$.

The terminology comes from the fact that given two non-commutative probability spaces $(\mathcal{A}_1, \varphi_1)$ and $(\mathcal{A}_2, \varphi_2)$, their tensor product $(\mathcal{A}_1 \otimes \mathcal{A}_2, \varphi_1 \otimes \varphi_2)$ (where $\mathcal{A}_1 \otimes \mathcal{A}_2$ is the tensor product of algebras and $\varphi_1 \otimes \varphi_2$ a tensor product of linear maps – cf. Appendix A) naturally contains a pair of independent subalgebras, namely $\mathcal{A}_1 \otimes 1$ and $1 \otimes \mathcal{A}_2$. If $\mathcal{A}_i$ happen to be algebras of classical random variables with moments of all orders, then the tensor independence is readily seen to be equivalent to classical independence.

In the framework of universal products, tensor independence is given by (3.11) with weighting

$$t(\pi;\sigma) = \begin{cases} 1, & \pi = \sigma \\ 0 & \sigma < \pi. \end{cases} \tag{3.12}$$

Indeed, each partition π gives the repetition patterns of elements in the product $a_1 \dots a_n$, but φ will not factor further over each underlying product of identical elements.

3.3.2 Free independence

A family of subalgebras $\mathcal{A}_i \subseteq$ of $\mathcal{A}$ indexed over some set I is said to be *freely independent* (or *free*) if $\varphi(a_{i_1} \dots a_{i_n}) = 0$ whenever $a_{i_k} \in \mathcal{A}_{i_k}$ $(k = 1, \dots, n)$ and $i_1 \neq i_2 \neq i_3 \neq \dots \neq i_n$. (Note that the condition allows for the case where $i_1 = i_3$ or $i_1 = i_n$, as it concerns only the adjoining indices.)

If $(\mathcal{A}, \varphi)$ is a C^* probability space $\{A_i\}_{i \in I}$ are free as $*$-algebras, one may easily check that the C^* algebras $\{C^*(A_i)\}_{i \in I}$ are free as well. Similarly, if $\{A_i\}_{i \in I}$ and $\mathcal{A}$ are von Neumann algebras and $\mathcal{A}_i$ are self-adjoint, then one may show (e.g. Proposition 2.5.6 in [VDN92]) that $\{A_i\}$ being free implies that $\{A_i''\}$ are free. Note that if $\mathcal{A}$ is a von Neumann algebra, $(\mathcal{A}, \varphi)$ is usually said to be a W^* *probability space.*

To compare the tensor to free independence, fix a non-commutative probability space $(\mathcal{A}, \varphi)$ and consider two subalgebras algebras $\mathcal{A}_1$ and $\mathcal{A}_2$. If $\mathcal{A}_1$ and $\mathcal{A}_2$ are tensor-independent in $(\mathcal{A}, \varphi)$, then for every $a_1, a_1' \in \mathcal{A}_1$ and $a_2, a_2' \in \mathcal{A}_2$, we have

$$\varphi(a_1 a_2 a_1' a_2') = \varphi(a_1 a_1')\varphi(a_2 a_2'). \tag{3.13}$$

In contrast, one may easily verify that if two algebras $\hat{\mathcal{A}}_1$ and $\hat{\mathcal{A}}_2$ are instead freely independent,

$$\varphi(\hat{a}_1\hat{a}_2\hat{a}_1'\hat{a}_2') = \varphi(\hat{a}_1\hat{a}_1')\varphi(\hat{a}_2)\varphi(\hat{a}_2') + \varphi(\hat{a}_1)\varphi(\hat{a}_1')\varphi(\hat{a}_2\hat{a}_2') - \varphi(\hat{a}_1)\varphi(\hat{a}_1')\varphi(\hat{a}_2)\varphi(\hat{a}_2'). \tag{3.14}$$

In the framework of universal products, the weight function $t(\pi, \sigma)$ is clearly somewhat more complicated to describe, though one may easily compute it for $\pi = \sigma$:

$$t(\sigma; \sigma) = \begin{cases} 1, & \sigma \text{ is non-crossing} \\ 0 & \text{otherwise.} \end{cases} \tag{3.15}$$

In the above example, the partition π for $a_1 a_2 a_1' a_2'$ is $\{\{1,3\},\{2,4\}\}$, which is not a non-crossing partition. From (3.14), it is clear that the moment $\varphi(a_1 a_2 a_1' a_2')$ indeed factors over refinements of π and that, furthermore, the factoring $\varphi(a_1 a_1')\varphi(a_2 a_2)'$ (corresponding to $\sigma = \pi$) is not present.

Returning to the above example of two algebras $\hat{\mathcal{A}}_1$ and $\hat{\mathcal{A}}_2$ that are free, note that if $\varphi(\hat{a}_1) = \varphi(\hat{a}_2) = 0$, then $\varphi(\hat{a}_1\hat{a}_2\hat{a}_1^*\hat{a}_2^*) = 0$. If $\hat{\mathcal{A}}_1$ and $\hat{\mathcal{A}}_2$ happen to be algebras of classical random variables and φ is the usual expectation, it then follows that $\mathbb{E}(|\hat{a}_1\hat{a}_2|^2) = 0$, i.e. the random variable $\hat{a}_1\hat{a}_2$ is degenerate. In other words, free independence is not a useful notion for classical random variables.

In contrast, freeness arises naturally in the "absence of commutative structure". Specifically, one may show that given a *group free product* $G = *_{i\in I} G_i$ (cf. Appendix A), the corresponding group von Neumann algebras $\{L(G_i)\}_{i\in I}$ are free in $(L(G), \psi_G)$. (For notations and details, the reader is referred to Proposition 5.11 and Remark 5.12 in [NS06].) Freeness also arises naturally in the Fock space setting. Namely, letting $\mathscr{F}_0(\mathscr{H})$ denote the full Fock space and $\{\mathscr{H}_i\}_{i\in I}$ be a family of mutually orthogonal subspaces of $\mathscr{H}$, then the operator algebras $\{\mathscr{B}(\mathscr{H}_i)\}_{i\in I}$ are free in $(\mathscr{B}(\mathscr{H}), \varphi_0)$ (e.g.

Proposition 7.15 in [NS06]), where φ_0 is again the vacuum expectation state on the full Fock space. Finally, freeness also arises in familiar asymptotic settings; in particular, finite collections of independent self-adjoint Gaussian matrices can shown to be *asymptotically free* in the limit of increasing matrix size (see Lecture 22 in [NS06]). Perhaps more surprisingly, there is also asymtotic freeness between two sequences of *constant* matrices which are "rotated against each other" via a Haar unitary random matrix (see Lecture 23 in [NS06]). From the point of view of matrix spectra, the consquences of this fact are astounding as free probability (via the machinery of *free additive convolution*) now provides means of approximating the spectra of sums of large matrices (see [Bia03] and concrete implementations in [RE08]).

3.3.3 Non-universal notions

The five notions of universal products can be cast as instances of more general frameworks [Len98, Fra03]. In particular, in [Fra03], a general framework is used to describe *Lévy processes*, viz. processes with independent and stationary increments, for the five forms of the universal product.

But, depending on the application at hand, the full strength of a universal product is frequently not needed. As further discussed shortly, in the setting of operators on the q-Fock space, there is no natural notion of independence that arises (despite the fact that the five universal products have natural Fock-space representations); nevertheless, there is a natural form of Lévy processes based on suitably deformed cumulants.[3] To define Brownian motion in the q-Fock space setting, one may also adopt a different approach, relying on a natural notion of filtration on the q-Gaussian algebra. The result is surprisingly natural, as the analogy to classical Brownian motion carries through on many levels. Both the q-Lévy processes and the q-Brownian motions are discussed later in this chapter.

[3] We will not be discussing cumulants much other than to mention that analogously to the classical cumulants, one can define non-commutative cumulants. As in the classical setting, the basic idea is that "independence" should linearize the cumulants of products of non-commutative random variables. For the mechanics of the free cumulants, the reader is referred to [NS06] and to [Ans01] for the broader notion of *partition-dependent cumulants*.

Finally, in the setting of Chapter 5, yet another notion of non-commutative independence will be of use. There, all elements of a sequence will obey a commutation relation of the type $a_i^\epsilon a_j^{\epsilon'} = \mu_{\epsilon',\epsilon}(j,i) a_i^\epsilon a_j^{\epsilon'}$, where the $\mu_{\epsilon',\epsilon}(j,i)$ will be real numbers. To factor a moment of the type $\varphi(a_1 \dots a_n)$, one will need to apply the commutation relation (incurring in the process some product of commutation coefficients) until the underlying π becomes an interval-partition (as defined in Chapter 5), and then factor according to intervals. If a product $a_1 \dots a_n$ already corresponds to an interval partition, the factoring rule will match that of tensor independence as well as free independence; otherwise, there is some product of commutation coefficients that was incurred in the process and the rule is therefore generally different from the familiar notions.

3.4 Probability on the q-Fock space

The two natural non-commutative probability spaces arising from the q-Fock space $\mathscr{F}_q(\mathscr{H})$ are $(\mathscr{B}(\mathscr{F}_q(\mathscr{H})), \varphi_q)$ and $(\Gamma_q(\mathscr{H}), \varphi_q)$, where φ_q is the vacuum expectation taking an operator a to $\langle a\Omega, \Omega\rangle_q$ and $\Gamma_q(\mathscr{H})$ is the von Neumann algebra generated by the q-Gaussian elements $\{s_q(h) \mid h \in \mathscr{H}\}$. Both algebras are C^* algebras and φ_q is easily seen to be positive on $\mathscr{B}(\mathscr{F}_q(\mathscr{H}))$, and therefore also when restricted to $\mathscr{F}_q(\mathscr{H})$. However, φ_q is not faithful on $\mathscr{B}(\mathscr{F}_q(\mathscr{H}))$, as $\varphi_q(a(h)^* a(h)) = 0$ for any $h \in \mathscr{H}$. Since the vacuum vector Ω is separating for $\Gamma_q(\mathscr{H})$ (cf. Appendix C), it can be easily seen that φ_q is faithful on the q-Gaussian algebra $\Gamma_q(\mathscr{H})$. What is more, the restriction of φ_q to $\Gamma_q(\mathscr{H})$ was also previously shown to be tracial. The distinction between $(\mathscr{B}(\mathscr{F}_q(\mathscr{H})), \varphi_q)$ and $(\Gamma_q(\mathscr{H}), \varphi_q)$ will not matter from the point of view of discussing moments of elements, but $(\Gamma_q(\mathscr{H}), \varphi_q)$ will provide the natural setting in which to define Brownian motion and stochastic calculus.

3.4.1 Joint $*$-moments and measures

The joint $*$-moments of the creation and annihilation operators on the q-Fock space are given by a Wick-type formula derived in Bożejko and Speicher's original article

[BS91]. Summing these moments over all $*$-types then readily yields the *q-Wick formula* for the q-Gaussian elements. The latter differs from its classical counterpart in that the pair-partitions are now refined to count the number of *crossings*. In particular, given a pair-partition $\mathscr{V} \in \mathscr{P}_2(2n)$ written as $\mathscr{V} = \{(w_1, z_1), \ldots, (w_n, z_n)\}$, the *number of crossings in* $\mathscr{V}$, denoted $\mathrm{cross}(\mathscr{V})$, is the cardinality of the set $\{1 \leq i < j \leq n \mid w_i < w_j < z_i < z_j\}$. Crossings, as well as certain natural counterparts thereof, will play an important role in Chapters 4 and 5, where they are discussed in more detail. In the present notation, the joint $*$-moments of interest are the following:

Lemma 1. *For all $n \in \mathbb{N}$ and $\epsilon(1), \ldots, \epsilon(2n) \in \{1, *\}^{2n}$,*

$$\varphi_q(a_q(h_1)^{\epsilon(1)} \ldots a_q(h_{2n-1})^{\epsilon(2n-1)}) = 0$$
$$\varphi_q(a_q(h_1)^{\epsilon(1)} \ldots a_q(h_{2n})^{\epsilon(2n)}) = \sum_{\mathscr{V} \in \mathscr{P}_2(2n)} q^{cross(\mathscr{V})} \prod_{i=1}^{n} \varphi_q(a_q(h_{w_i})^{\epsilon(w_i)} a_q(h_{z_i})^{\epsilon(z_i)}),$$

where each $\mathscr{V}$ is (uniquely) written as a collection of pairs $\{(w_1, z_1), \ldots, (w_n, z_n)\}$ with $w_1 < \ldots < w_n$ and $w_i < z_i$.

Similarly, the joint and marginal moments of the q-Gaussian elements can be expressed as follows.

Lemma 2 (The q-Wick Formula). *For all $n \in \mathbb{N}$ and $h_1, \ldots, h_{2n} \in \mathscr{H}$,*

$$\begin{aligned}
\varphi_q(s_q(h_1) \ldots s_q(h_{2n-1})) &= 0 \\
\varphi_q(s_q(h_1) \ldots s_q(h_{2n})) &= \sum_{\mathscr{V} \in \mathscr{P}_2(2n)} q^{cross(\mathscr{V})} \prod_{i=1}^{n} \langle h_{w_i}, h_{z_i} \rangle_{\mathscr{H}}.
\end{aligned}$$

Lemma 3. *The moments of the q-Gaussian element $s_q(h)$ are given by*

$$\varphi_q(s_q(h)^{2n-1}) = 0 \tag{3.16}$$

$$\varphi_q(s_q(h)^{2n}) = \|h\|_{\mathscr{H}}^{2n} \sum_{\mathscr{V}\in\mathscr{P}_2(2n)} q^{cross(\mathscr{V})} = \|h\|_{\mathscr{H}}^{2n}\, [z^n] \cfrac{1}{1-\cfrac{[1]_q z}{1-\cfrac{[2]_q z}{1-\cfrac{[3]_q z}{\cdots}}}}, \tag{3.17}$$

where $[z^n](\cdot)$ denotes the coefficient of the z^n term in the power series expansion of $(\cdot)$.

Since the classical probability can be realized on the symmetric Fock space, the moments of the classical Gaussian element, given in (3.2), match those in (3.17) for $\sigma = \|h\|$ and $q = 1$. Barring the scalar $\|h\|^2$, the even moments simply count all pair-partitions of $[2n]$, which are readily seen to be given as $(2n-1)!!$. For general q, the generating function of crossings in pair-partitions is well known in the combinatorial litterature, and the moment $\varphi_q(s_q(h)^{2n})$ can therefore be written in (mostly) closed-form. The corresponding expression was implicit in the work of Touchard [Tou52, Tou50a, Tou50b] and distilled by Riordan [Rio75]. The resulting expression for $\sum_{\mathscr{V}\in\mathscr{P}_2(2n)} q^{\mathrm{cross}(\mathscr{V})}$ is known as the *Touchard-Riordan formula.* For the modern proofs of the formula, the reader is referred to [Rea79, Pen95, Jos08].

Lemma 4 (Follows from [Rio75, Tou52]). *For $n \in \mathbb{N}$,*

$$\varphi_q(s_q(h)^{2n}) = \|h\|_{\mathscr{H}}^{2n} \frac{1}{(1-q)^n} \sum_{k=-n}^{n} (-1)^k q^{k(k-1)/2} \binom{2n}{n+k}. \tag{3.18}$$

The fact that the elements of Γ_q are normal (in fact, self-adjoint) provides means of identifying the elements of the resulting non-commutative random variables with bona fide probability measures. By Theorem 7, for element of $x \in \Gamma_q$, there exists a compactly-supported measure μ on the Borel subsets of $\mathbb{R}$ with the property that

$\varphi_q(x^n) = \int_{\mathbb{R}} x^n \, d\mu(x)$. Letting $e \in \mathscr{H}$ be a unit vector, one typically refers to the measure of $s_q(e)$ as the *(normalized) q-Gaussian measure*, denoted μ_q. The q-Gaussian measure can be shown to be absolutely continuous with respect to the Lebesgue measure; in fact, its density is known to be given by

$$d\mu_q(x) = \frac{1}{\pi}\sqrt{1-q}\,\sin\theta \prod_{n=1}^{\infty}(1-q^n)|1-q^n e^{i2\theta}|^2 \, dx, \tag{3.19}$$

with

$$x = \frac{2}{\sqrt{1-q}}\cos\theta \qquad \text{for } \theta \in [0,\pi]. \tag{3.20}$$

The above density is also that of the orthogonalizing measure for the q-Hermite orthogonal polynomial sequence discussed in the previous chapter. The link between the moments of the q-Gaussian elements and the q-Hermite orthogonal polynomials can also be inferred from the continued-fraction expansion (3.17) (see [Vie85, ISV87]). For further discussion of the q-Gaussian measure, the reader is referred to [BKS97] and the references therein.

3.4.2 q-Brownian motion and stochastic calculus

The framework for a stochastic calculus on the q-Fock space was laid out by Bożejko, Kümmerer and Speicher in [BKS97]. The basic idea is the following. Let $\mathscr{H} = \mathscr{L}^2(\mathbb{R}_+)$ with respect to Lebesgue measure on the Borel sets and set

$$x_t = s_q(1_{[0,t]}). \tag{3.21}$$

The *q-Brownian motion* on $(\Gamma_q(\mathscr{H}), \varphi_q)$ is then the process $(x_t)_{t\geq 0}$.

Next, one must also define a suitable conditional expectation. For that, let $\mathcal{A}_{t]}$ be the von Neumann algebra generated by $\{x_u \mid u \leq t\}$ and let $P_{t]}$ be the orthogonal projection $\mathscr{H} \to \mathscr{H}_{t]}$, where $\mathscr{H}_{t]}$ is the closed linear span of $\{1_{[0,u]} \mid u \leq t\}$. The conditional expectation with respect to $\mathcal{A}_{t]}$ should then (somehow) correspond, at the level of $\Gamma_q(\mathscr{H})$, to the passage from $h_1 \otimes \ldots \otimes h_n \in \mathscr{H}^{\otimes n}$ to $(P_{t]}h_1) \otimes \ldots \otimes (P_{t]}h_n) \in (\mathscr{H}_{t]})^{\otimes n} \subset \mathscr{H}^{\otimes n}$. Thinking more generally of $P_{t]}$ as a contraction T from $\mathscr{H}$ to some

Hilbert space $\mathscr{H}'$, consider first the map $\mathcal{F}(T) : \mathscr{H} \otimes \ldots \otimes \mathscr{H} \to \mathscr{H} \otimes \ldots \otimes \mathscr{H}$ given by $\mathcal{F}(T)(f_1 \otimes \ldots \otimes f_n) = Tf_1 \otimes \ldots \otimes Tf_n$. Since T is a contraction, one may verify that $\mathcal{F}(T)$ extends to a bounded linear operator on $\mathscr{F}_q(\mathscr{H})$

Recall that for every $n \in \mathbb{N}$ and any tensor $h_1 \otimes \ldots \otimes h_n \in \mathscr{H}^{\otimes n}$, there is a unique element $\Psi(h_1 \otimes \ldots \otimes h_n)$ of $\Gamma_q(\mathscr{H})$ with the property that $\Psi(h_1 \otimes \ldots \otimes h_n)\Omega = h_1 \otimes \ldots \otimes h_n$. Then, Ψ extends to an injective map $\Gamma_q(\mathscr{H})\Omega \to \Gamma_q(\mathscr{H})$. Conversely, since φ_q is separating, the map $\Gamma_q(\mathscr{H}) \to \Gamma_q(\mathscr{H})\Omega$ given by $x \mapsto x\Omega$ is also injective. It is now clear how to "translate" the map $\mathcal{F}(T)$ (in particular, $\mathcal{F}(P_{t]})$) to a map $\Gamma_q(T) : \Gamma_q(\mathscr{H}) \to \Gamma_q(\mathscr{H})$. Namely, one should use the above correspondence and require that

$$(\Gamma_q(T)x)\Omega = \mathcal{F}_q(T)(x\Omega),$$

for all $x \in \Gamma_q(\mathscr{H})$. Indeed, $\Gamma_q(T)$ is well defined whenever T is a contraction and it fulfills the intended goal:

Theorem 8 (Theorem 2.11 in [BKS97]). *Let $T : \mathscr{H} \to \mathscr{H}'$ be a contraction between real Hilbert spaces.*

i. There exists a unique map $\Gamma_q(T) : \Gamma_q(\mathscr{H}) \to \Gamma_q(\mathscr{H}')$ such that

$$(\Gamma_q(T)x)\Omega = \mathcal{F}_q(T)(x\Omega),$$

for all $x \in \Gamma_q(\mathscr{H})$. The map $\Gamma_q(\mathscr{H})$ is linear, bounded, completely positive, and preserves the canonical trace φ_q.

ii. If T is an orthogonal projection onto a subspace, $\Gamma_q(T)$ is a conditional expectation.

One may now set

$$\varphi_q(x \mid \mathcal{A}_t) = \Gamma_q(P_{t]})x, \tag{3.22}$$

for all $x \in \Gamma_q(\mathscr{H})$.

To pass from q-Brownian motion to stochastic integrals, consider first the following result.

Proposition 3 (Corollary 4.7 in [BKS97]). *The process* $M_n^{(q)}(t) := t^{n/2}H_n(x_t/\sqrt{t};q) = \Psi(1_{[0,t]}^{\otimes n})$ *has the property that* $\varphi_q(M_n^{(q)}(t) \mid \mathcal{A}_{s]}) = M_n^{(q)}(s)$ *for all* $s \leq t$. *(That is,* $M_n^{(q)}(t)$ *is a martingale with respect to* $\varphi_q(\cdot \mid \mathcal{A}_t)$*.)*

Analogously to the special case of $q = \pm 1$, i.e. to the *quantum stochastic calculus* [HP84] (see also the introductory treatments in [Bia95, Mey93, Par92]), it would be reasonable to expect $M_n^{(q)}$ to be the value of the multiple stochastic integral

$$\int_{\substack{0 \leq t_1, \ldots, t_n \leq t \\ t_i \neq t_j \, (i \neq j)}} \cdots \int dx_{t_1} \ldots dx_{t_n}.$$

Of course, developing stochastic integration in this setting is a delicate task, as the integrand need not generally commute with the increments $dx_{t_1} \ldots dx_{t_1}$. For the $q = 0$ case, the definitive version of free stochastic calculus was developped by Biane and Speicher in [BS98]. Passing to the framework of *bioperators* (elements of $\mathcal{A} \otimes \mathcal{A}$) and of the *simple adapted biprocesses* (piecewise constant functions $\mathbb{R}_+ \to \mathcal{A} \otimes \mathcal{A}$ that vanish at large enough time), one can multiply the integrand on both sides by the increments. This choice ends up being natural, allowing for the free analogues of familiar representation theorems and an Itô formula (an integration by parts formula).

Building on these ideas, Anshelevich [Ans01] and Śniady [Śni01] independently defined two versions of a q-deformed stochastic calculus with respect to four basic processes: annihilation, creation, gauge and time. The framework of Anshelevich can be seen as more natural as the corresponding gauge operator is self-adjoint. A general framework was introduced in Donati-Martin [DM03], along with the fundamental properties of the Wiener-space analysis in this setting, viz. decomposition in chaos and related representation theorems. Recently, Deya, Noreddine, and Nourdin [DNN] further developed the resulting stochastic calculus, proving the analogues of the Fourth Moment Theorem, transfer principle, and q-version of the Breuer-Major formula (see [NP05, BM83, Nua06] for the corresponding results in the classical setting and [KNPS] for those of free probability).

We will not further discuss the stochastic calculus in the setting of q-Gaussian algebras, other than to emphasize the fact that the conditional expectation $\varphi_q(x \mid \mathcal{A}_t)$ was defined relying heavily on the fact that Ω was a separating vector for $\Gamma_q(\mathscr{H})$, which, in turn, was previously shown using the traciality of the vacuum expectation. Chapter 6 considers the question of conditional expectation in the non-tracial setting, in the relation to the two-parameter family of Fock spaces introduced in Chapter 4.

Chapter 4

The (q, t)-Gaussian Process

The goal of this chapter is to introduce a second-parameter refinement of the q-Fock space, formulated as the (q,t)*-Fock space* $\mathscr{F}_{q,t}$. The (q,t)-Fock Space is constructed via a direct refinement of Bożejko and Speicher's framework [BS91], yielding the q-Fock space when $t = 1$. Before overviewing the details of the construction and the main results, we take a moment to point out why the present refinement is, in fact, particularly natural.

- At the structural level, the permutation inversions underlying the q-Fock space construction are now replaced by the joint statistics of permutation inversions *and co-inversions.* Accordingly, the crossings in pair partitions that index the relevant moment formulae now become the joint statistics of crossings *and nestings.* While the permutation co-inversions are the complements of the permutation inversions, a satisfying characterization of the joint distribution of crossings and nestings in pair partitions is an open problem in combinatorics, of relevance to broader combinatorial questions, e.g. [KZ06, Kla06, CDD$^+$07].

- The commutation relation satisfied by the creation and annihilation operators on the (q,t)-Fock space is that of the *Chakrabarti-Jagannathan* deformed oscillator algebra [CJ91], which encompasses both the q-CR and the "physics" q-relations [Bie89, Mac89]. The special functions and orthogonal polynomials arising in the setting of the q-CR become their natural two-parameter generalizations in the present framework. In particular, the q-Hermite orthogonal polynomials given by the three-term recurrence $zH_n(z;q,t) = H_{n+1}(z;q,t) + [n]_q H_{n-1}(z;q,t)$ become the *(q,t)-Hermite orthogonal polynomials* $zH_n(z;q,t) = H_{n+1}(z;q,t) + [n]_{q,t} H_{n-1}(z;q,t)$. Note that the latter were previously considered in the setting of [CJ91], and are also a specialization of the octabasic Laguerre family [SS96].

- The $q = 0 < t$ case corresponds to a new single-parameter deformation of the full Boltzmann Fock space of free probability [Voi86, VDN92] and of the corresponding semicircular operator. The corresponding measure is encoded, in various forms, via the Rogers-Ramanujan continued fraction (e.g. [And98]), the Rogers-Ramanujan identities (e.g. [And98]), the t-Airy function [Ism05], the t-Catalan numbers of Carlitz-Riordan [FH85, CR64], and the first-order statistics of the reduced Wigner processes [Kho01, MP02].

At this point, it is also important to note that the (q,t)-deformed framework independently arises in a more general asymptotic setting, through non-commutative central limit thms and random matrix models. This is the subject of Chapter 5. In particular, analogously to the q-annihilation and creation operators which arise as weak limits in the non-commutative Central Limit Theorem introduced by Speicher [Spe92], the (q,t)-annihilation and creation operators appear in the generalized non-commutative Central Limit Theorem of Chapter 5. In broad terms, this second-parameter refinement is a consequence of the passage from a commutation structure built around commutation *signs*, taking values in $\{-1, 1\}$, to a more general structure based on commutation coefficients taking values in $\mathbb{R}$.

However, it should also be remarked that, despite many analogies with the original formulation, the (q,t)-Fock space is an altogether different object from the q-Fock space when $t \neq 1$. For instance, the vacuum expectation state φ (cf. Section 4.3) is not tracial on the $*$-algebra generated by the field operators $\{s_{q,t}(h)\}_{h \in \mathscr{H}}$.

4.1 Main results

The following is an overview of the main results in this chapter, encompassing an overview of the (q,t)-Fock space construction.

Definition/Theorem 1. *Let $\pi \mapsto U_\pi^{(n)}$ denote the unitary representation of the symmetric group S_n on $\mathscr{H}^{\otimes n}$ given by $U_\pi^{(n)}\, h_1 \otimes \ldots \otimes h_n = h_{\pi(1)} \otimes \ldots \otimes h_{\pi(n)}$ and, for every permutation $\pi \in S_n$, let $inv(\pi)$ and $cinv(\pi)$ respectively denote the inversions and co-inversions of a permutation (cf. Section 4.2). Given the tensor algebra $\mathscr{F} := \mathscr{F}(\mathscr{H})$, consider the "projection" operator $P_{q,t} : \mathscr{F} \to \mathscr{F}$ given by $P_{q,t} = \bigoplus_{n=0}^{\infty} P_{q,t}^{(n)}$ with $P_{q,t}^{(n)} : \mathscr{H}^{\otimes n} \to \mathscr{H}^{\otimes n}$, with*

$$P_{q,t}^{(n)} := \sum_{\pi \in S_n} q^{inv(\pi)} t^{cinv(\pi)}\, U_\pi^{(n)}. \tag{4.1}$$

Consider, further, the sesquilinear form $\langle\ ,\ \rangle_{q,t}$ on $\mathscr{F}$ given, via the usual inner product $\langle\ ,\ \rangle_0$ on the full Fock space (cf. Section 4.3), by $\langle \eta, \xi \rangle_{q,t} = \langle \xi, P_{q,t}\, \eta \rangle_0 \quad \forall \eta, \xi \in \mathscr{F}$.

Then, for all $n \in \mathbb{N}$, $P_{q,t}^{(n)}$ is strictly positive definite for all $|q| < t$, $\langle\ ,\ \rangle_{q,t}$ is an inner product, and the (q,t)-Fock space *is the completion of $\mathscr{F}$ with respect to the norm induced by $\langle\ ,\ \rangle_{q,t}$.*

Definition/Theorem 2. *Define the* creation operators $\{a_{q,t}(f)^* \mid f \in \mathscr{H}\}$ *and the* annihilation operators $\{a_{q,t}(f) \mid f \in \mathscr{H}\}$ *on $\mathscr{F}$ by linear extension of:*

$$a_{q,t}(f)^*\Omega = f, \qquad a_{q,t}(f)^* h_1 \otimes \ldots \otimes h_n = f \otimes h_1 \otimes \ldots \otimes h_n \tag{4.2}$$

and

$$a_{q,t}(f)\Omega = 0, \qquad a_{q,t}(f) h_1 \otimes \ldots \otimes h_n = \sum_{k=1}^{n} q^{k-1} t^{n-k} \langle f, h_k \rangle_{\mathscr{H}}\, h_1 \otimes \ldots \otimes \check{h}_k \otimes \ldots \otimes h_n. \tag{4.3}$$

In the following parameter ranges, (4.2) *and* (4.3) *extend to bounded linear operators* *on $\mathscr{F}_{q,t}$, with norm*

$$\|a_{q,t}(f)\| = \begin{cases} \|f\|_{\mathscr{H}}, & -t \leq q \leq 0 < t \leq 1 \\ \frac{1}{\sqrt{1-q}}\,\|f\|_{\mathscr{H}}, & 0 < q < t = 1 \\ \sqrt{(n_* + 1)\, t^{n_*}}\,\|f\|_{\mathscr{H}}, & 0 < q = t < 1 \\ \sqrt{\frac{t^{\hat{n}+1} - q^{\hat{n}+1}}{t-q}}\,\|f\|_{\mathscr{H}}, & 0 < q < t < 1 \end{cases} \tag{4.4}$$

for

$$n_* = \lfloor t/(1-t) \rfloor, \qquad \hat{n} = \left\lfloor \frac{\log(1-q) - \log(1-t)}{\log(t) - \log(q)} \right\rfloor, \tag{4.5}$$

with $a_{q,t}(f)^*$ becoming the adjoint of $a_{q,t}(f)$ on $\mathscr{F}_{q,t}$.

For all $f, g \in \mathscr{H}$, the creation and annihilation operators satisfy the (q,t)-commutation relation

$$a_{q,t}(f)a_{q,t}(g)^* - q\, a_{q,t}(g)^* a_{q,t}(f) = \langle f, g \rangle_{\mathscr{H}}\, t^N, \tag{(q,t)-CR}$$

where t^N is the operator on $\mathscr{F}_{q,t}$ defined by the linear extension of $t^N\Omega = \Omega$ and $t^N h_1 \otimes \ldots \otimes h_n = t^n h_1 \otimes \ldots \otimes h_n$ for all $h_1, \ldots, h_n \in \mathscr{H}$.

Finally, for all $n \in \mathbb{N}$ and $\epsilon(1), \ldots, \epsilon(2n) \in \{1, *\}$, the corresponding mixed moments of the creation and annihilation operators on $\mathscr{F}_{q,t}$ are

$$\varphi_{q,t}(a_{q,t}(h_1)^{\epsilon(1)} \ldots a_{q,t}(h_{2n-1})^{\epsilon(2n-1)}) = 0 \tag{4.6}$$

$$\begin{aligned} \varphi_{q,t}(a_{q,t}(h_1)^{\epsilon(1)} \ldots a_{q,t}(h_{2n})^{\epsilon(2n)}) = \sum_{\mathscr{V} \in \mathscr{P}_2(2n)} & q^{cross(\mathscr{V})} t^{nest(\mathscr{V})} \varphi(a_{q,t}(h_{w_1})^{\epsilon(w_1)} a_{q,t}(h_{z_1})^{\epsilon(z_1)}) \ldots \\ & \ldots \varphi(a_{q,t}(h_{w_n})^{\epsilon(w_n)} a_{q,t}(h_{z_n})^{\epsilon(z_n)}), \end{aligned} \tag{4.7}$$

where $P_2(2n)$ is the collection of pair partitions of $[2n]$ (cf. Section 4.2), each $\mathscr{V} \in P_2(2n)$ is (uniquely) written as a collection of pairs $\{(w_1, z_1), \ldots, (w_n, z_n)\}$ with $w_1 <$

$\ldots < w_n$ *and* $w_i < z_i$, *with* $cross(\mathscr{V})$ *and* $nest(\mathscr{V})$ *denoting the numbers of crossings and nestings, respectively, in* $\mathscr{V}$ *(cf. Section 4.2).*

Definition/Theorem 3. *For* $h \in \mathscr{H}$, *the* (q,t)-Gaussian *element* $s_{q,t}(h) \in \mathscr{B}(\mathscr{F}_{q,t})$ *is given by* $s_{q,t}(h) := a_{q,t}(h) + a_{q,t}(h)^*$. *The* (q,t)*-Gaussian is self-adjoint, with moments given by*

$$\varphi_{q,t}(s_{q,t}(h)^{2n-1}) = 0 \tag{4.8}$$

$$\varphi_{q,t}(s_{q,t}(h)^{2n}) = \|h\|_{\mathscr{H}}^{2n} \sum_{\mathscr{V}\in\mathscr{P}_2(2n)} q^{cross(\mathscr{V})}\, t^{nest(\mathscr{V})} = \|h\|_{\mathscr{H}}^{2n}\, [z^n] \cfrac{1}{1-\cfrac{[1]_{q,t}z}{1-\cfrac{[2]_{q,t}z}{1-\cfrac{[3]_{q,t}z}{\cdots}}}} \tag{4.9}$$

where $[z^n](\cdot)$ *denotes the coefficient of the* z^n *term in the power series expansion of* $(\cdot)$ *and for all* $n \in \mathbb{N}$,

$$[n]_{q,t} := \sum_{i=1}^{n} q^{i-1}t^{n-i} = \frac{t^n - q^n}{t-q}. \tag{4.10}$$

Furthermore, the distribution of $s_{q,t}(e)$, *with* $\|e\|_{\mathscr{H}} = 1$, *is the unique real probability measure that orthogonalizes the* (q,t)-Hermite orthogonal polynomial sequence *given by the recurrence*

$$zH_n(z;q,t) = H_{n+1}(z;q,t) + [n]_{q,t}H_{n-1}(z;q,t), \tag{4.11}$$

with $H_0(z;q,t) = 1,\ H_1(z;q,t) = z$.

Definition/Theorem 4. *For* $q = 0 < t \leq 1$, $\mathscr{B}(\mathscr{F}_{0,t})$ *is the von Neumann algebra generated by* $\{a_{q,t}(h)\}_{h\in\mathscr{H}}$. *The* t-semicircular *element is the corresponding specialization of the* (q,t)*-Gaussian element* $s_{0,t}(h) \in \mathscr{B}(\mathscr{F}_{0,t})$. *Its moments are*

$$\varphi_{0,t}(s_{0,t}(h)^{2n-1}) = 0$$

$$\varphi_{0,t}(s_{0,t}(h)^{2n}) = \|h\|_{\mathscr{H}}^{2n} \sum_{\mathscr{V}\in NC_2(2n)} t^{nest(\mathscr{V})} = \|h\|_{\mathscr{H}}^{2n}\, C_n^{(t)}$$

where $NC_2(2n)$ denotes the lattice of non-crossing pair-partitions and $C_n^{(t)}$ are referred to as the Carlitz-Riordan t-Catalan numbers [FH85, CR64], determined by the recurrence

$$C_n^{(t)} = \sum_{k=1}^{n} t^{k-1} C_{k-1}^{(t)} C_{n-k}^{(t)}, \tag{4.12}$$

with $C_0^{(t)} = 1$. The moments of the normalized t-semicircular element $s_{0,t} := s_{0,t}(e)$, for $\|e\|_{\mathscr{H}} = 1$, are encoded by the generalized Rogers-Ramanujan continued fraction

$$\sum_{n\geq 0} \varphi_{q,t}(s_{q,t}(h)^n) z^n = \cfrac{1}{1 - \cfrac{t^0\, z^2}{1 - \cfrac{t^1\, z^2}{1 - \cfrac{t^2\, z^2}{\cdots}}}}. \tag{4.13}$$

The Cauchy transform of the corresponding normalized t-semicircular measure $\mu_{0,t}$ is

$$\int_{\mathbb{R}} \frac{1}{z-\eta}\, d\mu_{0,t}(\eta) = \frac{1}{z} \frac{\sum_{n\geq 0}(-1)^n \frac{t^{n^2}}{(1-t)(1-t^2)\ldots(1-t^n)} z^{-2n}}{\sum_{n\geq 0}(-1)^n \frac{t^{n(n-1)}}{(1-t)(1-t^2)\ldots(1-t^n)} z^{-2n}} = \frac{1}{z} \frac{A_t(1/z^2)}{A_t(1/(z^2 t))}, \tag{4.14}$$

where A_t denotes the t-Airy function of [Ism05], given by

$$A_t(z) = \sum_{n\geq 0} \frac{t^{n^2}}{(1-t)\ldots(1-t^n)} (-z)^n. \tag{4.15}$$

Letting $\{z_j\}_{j\in\mathbb{N}}$ denote the sequence of zeros of the rescaled t-Airy function $A_t(z/t)$, the measure $\mu_{0,t}$ is a discrete probability measure with atoms at

$$\pm 1/\sqrt{z_j}, \quad j \in \mathbb{N}$$

with corresponding mass

$$-\frac{A_t(z_j)}{2\, z_j\, A_t'(z_j/t)},$$

where $A_t'(z) := \frac{d}{dz} A_t(z)$. The only accumulation point of $\mu_{0,t}$ is the origin.

The normalized t-semicircular measure $\mu_{0,t}$ is also the unique probability measure *orthogonalizing the* t-Chebyshev II orthogonal polynomial sequence $\{U_n(z;t)\}_{n\geq 0}$, *a specialization of the orthogonal polynomials of Al-Salam and Ismail [ASI83] and determined by the three-term recurrence*

$$zU_n(z;t) = U_{n+1}(z;t) + t^{n-1}U_{n-1}(z;t),$$

with $U_0(z;t) = 1,\ U_1(z;t) = z.$

Finally, t-semicircular measure $\mu_{0,t}$ *is the weak limit of the first-order statistics of the reduced Wigner process [Kho01, MP02]. Specifically, for all* $\rho \in [0,1]$ *and* $n \in \mathbb{N}$,

$$\lim_{N\to\infty} \varphi_N \left(\frac{W_{N,\rho}(1)}{N} \frac{W_{N,\rho}(2)}{N} \cdots \frac{W_{N,\rho}(n)}{N} \right) = \rho^{n/2} \varphi_{0,t}(s_{0,t}^n) \qquad \text{for} \quad t = \rho^2, \tag{4.16}$$

where $\varphi_N = \frac{1}{N} Tr \otimes \mathbb{E}$ *and* $\{W_{N,\rho}(k)\}_{k\in\mathbb{N}}$ *is a sequence of Wigner matrices with correlations*

$$\begin{aligned} \mathbb{E}(w_{i,j}(k)w_{i',j'}(k)) &= \begin{cases} 1, & (i,j) = (i',j') \text{ or } (i,j) = (j',i') \\ 0, & \text{otherwise} \end{cases} && (4.17) \\ \mathbb{E}(w_{i,j}(k)w_{i,j}(m)) &= \rho^{m-k} \quad \text{for } m > k, && (4.18) \end{aligned}$$

with $w_{i,j}(k)$ *denoting the* $(i,j)^{th}$ *entry of* $W_{N,\rho}(k)$.

Finally, recall that the crux of the upcoming Chapter 5 is the extension of Speicher's *Non-commutative Central Limit Theorem* [Spe92], giving rise to an *an asymptotic model for operators satisfying the commutation relation* (q,t)*-CR*. It should be emphasized that this asymptotic model provides an existence proof, independent of the explicit construction of Section 4.3, and provides an alternative reason for the fundamental ordering bound present throughout, namely $|q| < t$.

4.1.1 Deformed Quantum Harmonic Oscillators

In physics, the oscillator algebra of the quantum harmonic oscillator is generated by elements $\{1, a, a^*, N\}$ satisfying the canonical commutation relations

$$[a, a^*] = 1, \quad [N, a] = -a, \quad [N, a^*] = a^*,$$

where a^*, a, and N can be identified with the creation, annihilation, and number operators on the Bosonic Fock space. Physicists may also speak of *generalized deformed oscillator algebras*, which are instead generated by elements satisfying the deformed relations

$$a^*a = f(N), \quad aa^* = f(N+1), \quad [N, a] = -a, \quad [N, a^*] = a^*,$$

where f is typically referred to as the *structure function of the deformation.* While an in-depth review of single-parameter deformations of the quantum oscillator algebra is available in [Dod02], of particular interest are the so-called Arik-Coon q-deformed oscillator algebra [AC76] given by

$$aa^* - qa^*a = 1, \quad [N, a] = -a, [N, a^*] = a^*, \quad f(n) = \frac{1-q^n}{1-q}, \quad q \in \mathbb{R}_+$$

and the Biedengarn-Macfarlane q-deformed oscillator algebra [Bie89, Mac89] given by

$$aa^* - qa^*a = q^{-N}, \quad [N, a] = -a, \quad [N, a^*] = a^*, \quad f(n) = \frac{q^{-n} - q^n}{q^{-1} - q}, \quad q \in \mathbb{R}_+.$$

For $q \in [-1, 1]$, the Hilbert space representation of the Arik-Coon algebra, generalized to an infinite-dimensional setting, is given by the q-Fock space of Bozejko and Speicher [BS91]. Manipulating Definition/Theorem 1 and 2, the reader may readily verify that for $q \in (0, 1]$, a Hilbert space realization of the Biedengarn-Macfarlane algebra is similarly given by the (q, t)-Fock space specialized to $t = q^{-1}$. More generally, the

Chakrabarti-Jagganathan oscillator algebra [CJ91] is given by

$$aa^* - qa^*a = p^{-N}, \quad [N, a] = -a, \quad [N, a^*] = a^*, \quad f(n) = \frac{p^{-n} - q^n}{p^{-1} - q}, \quad q, p \in \mathbb{R}_+.$$

For $0 \leq |q| < p^{-1}$, the reader may also verify that the (q, t)-Fock space again provides the desired Hilbert space realization for for $t = p^{-1}$. A Bargmann-Fock representation of this algebra was previously considered in [IB93] for the case of a single oscillator, but no explicit underlying Hilbert space was constructed nor shown to exist in the parameter range considered. Instead, an explicit representation was provided in the space of analytic functions via suitably deformed differential operators. Note that the two-parameter deformation of the Hermite orthogonal polynomial sequence given by (4.11) also appears as recurrence (15) in [IB93].

4.1.2 General Brownian Motion

From a high-level perspective, the (q, t)-Gaussian processes fall under the framework of *Generalized Brownian Motion* [BS96]. The latter is described by families of self-adjoint operators $G(f)$, where f belongs to some real Hilbert space $\mathscr{H}$, and a state φ on the algebra generated by the $G(f)$, given by

$$\varphi(G(f_1) \ldots G(f_{2n})) = \sum_{\mathscr{V} \in \mathscr{P}_2(2n)} \tau(\mathscr{V}) \prod_{(i,j) \in \mathscr{V}} \langle f_i, f_j \rangle_{\mathscr{H}},$$

for some positive definite function $\tau : \mathscr{P}_2(2n) \to \mathbb{R}$. Thus, the q-Brownian motion is given by $\tau_q(\mathscr{V}) = q^{\mathrm{cross}(\mathscr{V})}$, whereas in the present context, the (q, t)-Brownian motion corresponds to $\tau_{q,t}(\mathscr{V}) = q^{\mathrm{cross}(\mathscr{V})} t^{\mathrm{nest}(\mathscr{V})}$. Many other known τ functions exist. Most generally, a beautiful framework by Guţă and Maassen [GM02a, GM02b], proceeding via the combinatorial theory of species of structures [Joy81], encompasses the familiar deformations (Bosonic, Fermionic, free, as well as [BS91, BS96]). It is foreseeable, though not presently clear, whether (or how) the current formulation can be encompassed within the same framework.

Arguably the most closely-related framework to that of the (q, t)-Gaussians is provided by Bożejko in [Boż07], introduced with the goal of extending the q-commutation relations beyond the $q \in [-1, 1]$ parameter range. In particular, Bożejko studied operators satisfying two types of deformed commutation relations, namely

$$A(f)A^*(g) - A^*(g)A(f) = q^N \langle f, g \rangle_{\mathscr{H}} \quad \text{for } q > 1$$

and

$$B(f)B^*(g) + B^*(g)B(f) = |q|^N \langle f, g \rangle, \quad \text{for } q < -1.$$

Comparing Bożejko's creation and annihilation operators and his q-deformed inner product with the present defns, Bożejko's setting turns out to be that of the (q, t)-Fock space for $q \mapsto 1$ an $t \mapsto q$. The difficulty encountered by Bożejko, pertaining to the fact that the Gaussian element is no longer self-adjoint, is consistent with the fact that for $t > 1$, the creation and annihilation operators are no longer bounded.

As later found, a relevant two-parameter deformation of the canonical commutation and anti-commutation relations was previously studied by Bożejko and Yoshida [BY06], as part of a more general framework. In [BY06], a general version of the n-dimensional "projection" operator $P_{q,t}^{(n)}$ was given by $\prod_{i=1}^{n} \tau_i \sum_{\pi \in S_n} q^{\mathrm{inv}(\pi)}$, where $\{t_n\}$ is any sequence of positive numbers. The reader may readily verify that the (q, t)-projection operator $P_{q,t}^{(n)}$ is recovered for $\tau_n = t^{n-1}$ and by substituting $q \mapsto q/t$. Instead, Bożejko and Yoshida focus on the specialization $\tau_n = s^{2n}$ and $q \mapsto q$, yielding the so-called (q, s)-*Fock space.* The corresponding combinatorial structure (now given in terms of crossings and *inner points*), the Wick-type formulas, and continued fractions are considered in [BY06].

In the context of the generalized Brownian motion, the point of the present pair of articles, which construct and describe various aspects of the (q, t)-Fock space, is to argue that the framework at hand is ultimately a highly natural refinement of the q-Fock space. This argument is based on the structural depth of the (q, t)-deformed framework, as evidenced by its intimate ties to various fundamental mathematical objects, as well as to the correspondence with the natural generalizations of non-commutative asymptotic frameworks (developped in Chapter 5).

4.2 Combinatorial Preliminaries

The present section briefly reviews the key combinatorial constructs that will, in the subsequent sections, be used to encode the structure of the (q,t)-Fock space, the creation, annihilation, and field operators on this space, as well as the mechanics of the relevant limit theorems. In broad terms, the objects of interest are *set partitions*, *permutations*, *lattice paths*, and certain combinatorial statistics thereof.

4.2.1 Partitions

Denote by $\mathscr{P}(n)$ the set of partitions of $[n] := \{1, \ldots, n\}$ and by $\mathscr{P}_2(2n)$ the set of *pair partitions* of $[2n]$, that is, the set of partitions of $[2n]$ with each part containing exactly two elements. (Note that the pair partitions are also referred to as *pairings* or *perfect matchings.*) It will be convenient to represent a pair partition as a list of ordered pairs, that is, $\mathscr{P}_2(2n) \ni \mathscr{V} = \{(w_1, z_1), \ldots, (w_n, z_n)\}$, where $w_i < z_i$ for $i \in [n]$ and $w_1 < \ldots < w_n$. Of particular interest are the following two statistics on $\mathscr{P}_2(2n)$.

Definition 7 (Crossings and Nestings). *For $\mathscr{V} = \{(w_1, z_1), \ldots, (w_n, z_n)\} \in \mathscr{P}_2(2n)$, pairs (w_i, z_i) and (w_j, z_j) are said to* cross *if $w_i < w_j < z_i < z_j$. The corresponding* crossing *is encoded by (w_i, w_j, z_i, z_j) with*

$$\begin{aligned} \mathit{Cross}(\mathscr{V}) &:= \{(w_i, w_j, z_i, z_j) \mid (w_i, z_i), (w_j, z_j) \in \mathscr{V} \text{ with } w_i < w_j < z_i < z_j\}, \\ \mathit{cross}(\mathscr{V}) &:= |\mathit{Cross}(\mathscr{V})|. \end{aligned}$$

For $\mathscr{V} = \{(w_1, z_1), \ldots, (w_n, z_n)\} \in \mathscr{P}_2(2n)$, pairs (w_i, z_i) and (w_j, z_j) are said to nest *if $w_i < w_j < z_j < z_i$. The corresponding* nesting *is encoded by (w_i, w_j, z_j, z_i) with*

$$\begin{aligned} \mathit{Nest}(\mathscr{V}) &:= \{(w_i, w_j, z_j, z_i) \mid (w_i, z_i), (w_j, z_j) \in \mathscr{V} \text{ with } w_i < w_j < z_j < z_i\}, \\ \mathit{nest}(\mathscr{V}) &:= |\mathit{Nest}(\mathscr{V})|. \end{aligned}$$

The two concepts are illustrated in Figures 4-1 and 4-2, by visualizing the pair parti-

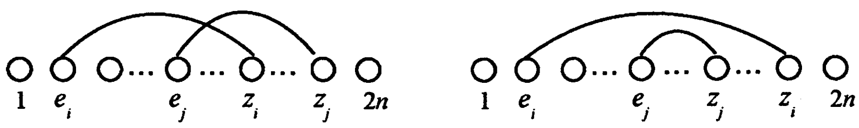

Figure 4-1: An example of a crossing [left] and nesting [right] of a pair partition $\mathscr{V} = \{(e_1, z_1), \ldots, (e_n, z_n)\}$ of $[2n]$.

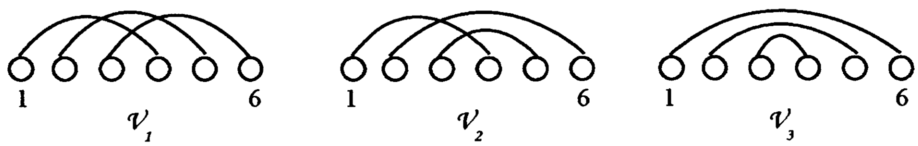

Figure 4-2: Example of three pair partitions on $[2n] = \{1, \ldots, 6\}$: $\text{cross}(\mathscr{V}_1) = 3$, $\text{nest}(\mathscr{V}_1) = 0$ [left], $\text{cross}(\mathscr{V}_2) = 2$, $\text{nest}(\mathscr{V}_2) = 1$ [middle], $\text{cross}(\mathscr{V}_3) = 0$, $\text{nest}(\mathscr{V}_3) = 3$ [right].

tions as collections of disjoint chords with end-points labeled (increasing from left to right) by elements in $[2n]$.

Note that the above terminology is becoming relatively standard (e.g. [KZ06, Kla06, CDD+07]), though nestings may also appear in the literature as "pair embracings" (e.g. [dMV94, Nic96, Boż07]). Next, let $[n]_{q,t}$ denote the (q,t)-analogue of a positive integer n, given by

$$[n]_{q,t} := t^{n-1} + qt^{n-2} + \ldots + q^{n-1} = (t^n - q^n)/(t-q). \tag{4.19}$$

Then, $t = 1$ recovers the usual q-analogue of integers, i.e. $[n]_q = n_{q,1}$. In this notation, both the generating functions of crossings in $\mathscr{P}_2(2n)$ and the joint generating function of crossings and nestings in $\mathscr{P}_2(2n)$ admit elegant continued fractions, given by

$$\sum_{\substack{n\in\mathbb{N},\\ \mathscr{V}\in\mathscr{P}_2(2n)}} q^{\text{cross}(\mathscr{V})} z^n = \cfrac{1}{1 - \cfrac{[1]_q z}{1 - \cfrac{[2]_q z}{1 - \cfrac{[3]_q z}{\cdots}}}}, \qquad \sum_{\substack{n\in\mathbb{N},\\ \mathscr{V}\in\mathscr{P}_2(2n)}} q^{\text{cross}(\mathscr{V})} t^{\text{nest}(\mathscr{V})} z^n = \cfrac{1}{1 - \cfrac{[1]_{q,t} z}{1 - \cfrac{[2]_{q,t} z}{1 - \cfrac{[3]_{q,t} z}{\cdots}}}}.$$

The above continued fractions can be obtained via a classic encoding of weighted

Dyck paths (see [Fla80], also in a more relevant context [Bia97b, KZ06, KD95]).

The generating function of crossings in $\mathscr{P}_2(2n)$ admits an interesting explicit (even if not closed-form) expression, namely

$$\sum_{\mathscr{V}\in\mathscr{P}_2(2n)} q^{\mathrm{cross}(\mathscr{V})} = \frac{1}{(1-q)^n}\sum_{k=-n}^{n}(-1)^k q^{k(k-1)/2}\binom{2n}{n+k},$$

known as the *Touchard-Riordan formula* [Tou52, Tou50a, Tou50b, Rio75]. No analogue of the above expression is presently known for the joint generating function of crossings and nestings.

4.2.2 Permutations

Let S_n denote the permutation group on n letters. .

Definition 8 (Inversions and Coinversions). *Given $\sigma \in S_n$, for $n \geq 2$, the pair $i,j \in [n]$ with $i < j$ is an* inversion in σ *if $\sigma(i) > \sigma(j)$. The corresponding inversion is encoded by (i,j) and the set of inversions of σ is given by $Inv(\sigma) := \{(i,j) \mid i,j \in [n], i < j, \sigma(i) > \sigma(j)\}$ with cardinality $inv(\sigma) := |Inv(\sigma)|$. Analogously, the pair $i,j \in [n]$ with $i < j$ is a* coinversion in σ *if $\sigma(i) < \sigma(j)$. The corresponding coinversion is encoded by (i,j) and contained in the set $Cinv(\sigma) := \{(i,j) \mid i,j \in [n], i < j, \sigma(i) < \sigma(j)\}$ with cardinality $cinv(\sigma) := |Cinv(\sigma)|$. For $n = 1$, the sets of inversions and coinversions are taken to be empty.*

It is well known that the generating function of the permutation inversions is the so-called q-factorial, namely

$$\sum_{\sigma\in S_n} q^{\mathrm{inv}(\sigma)} = \prod_{i=1}^{n}[i]_q. \tag{4.20}$$

The above expression is in fact readily obtained as a product of the generating functions of the crossings incurred by the "chord" $i \mapsto \sigma(i)$ from the chords $j \mapsto \sigma(j)$ for $j > i$. By adapting this reasoning to coinversions, or by realizing that

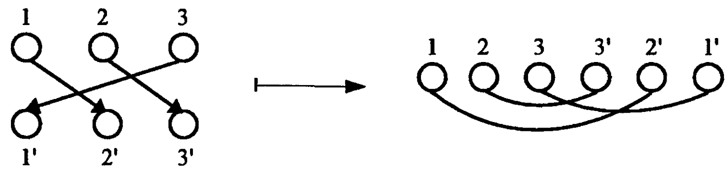

Figure 4-3: Example of transforming the permutation $\sigma = (231)$ into a pair-partition, with permutation inversions corresponding to crossings and coinversions to nestings.

$\mathrm{cinv}(\sigma) = \binom{n}{2} - \mathrm{inv}(\sigma)$, the reader may verify that

$$\sum_{\sigma \in S_n} q^{\mathrm{inv}(\sigma)} t^{\mathrm{cinv}(\sigma)} = \prod_{i=1}^{n} [i]_{q,t}. \tag{4.21}$$

For $n \geq 2$ and $\sigma \in S_n$, it is convenient to visually represent σ in a two-line notation. Then, σ corresponds to a bipartite perfect matching and the inversions correspond to crossings in the diagram and the coinversions to "non-crossings". For instance, in adopting this representation, it becomes clear that a permutation and its inverse have the same number of inversions (and therefore also of coinversions).

Remark 3 Representing a permutation $\sigma \in S_n$ in a two-line notation, aligning the two rows, and relabeling yields a unique pair-partition $\pi \in \mathscr{P}_2(2n)$. By additionally reversing the order of the bottom line, as illustrated in Figure 4-3, the inversions will correspond to crossings of the pair partition and the co-inversions to the nestings. Indeed, given an inversion $i < j$, $\sigma(i) > \sigma(j)$, this transformation yields the pairs $(i, 2n+1-\sigma(i))$ and $(j, 2n+1-\sigma(j))$. Since $i < j < 2n+1-\sigma(i) < 2n+1-\sigma(j)$, the two pairs now form a crossing in π. Similarly, given a co-inversion $k < m$, $\sigma(k) < \sigma(m)$, the transformation yields the pairs $(k, 2n+1-\sigma(k))$ and $(m, 2n+1-\sigma(m))$, resulting in the ordering $k < m < 2n+1-\sigma(m) < 2n+1-\sigma(k)$ and yielding an inversion in π.

Naturally, this transformation of a permutation into a pair partition is by no means surjective, and is illustrative of the reasons why the set partitions (with the crossings and nestings) and the permutations (with inversions and coinversions) will both be found to feature in the algebraic structure of the (q, t)-Fock space. •

4.2.3 Paths

In the present context, *paths* will refer to finite sequences of coordinates in the lattice $\mathbb{Z} \times \mathbb{Z}$. A North-East/South-East (NE/SE) path will refer to a path $(x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n)$ where $(x_0, y_0) = (0, 0)$, $x_i = x_{i-1} + 1$ and $y_{i+1} \in \{y_{i+1} + 1, y_{i+1} - 1\}$ for $i \in [n]$. Indeed, interpreting the coordinates as vertices and introducing an edge between (x_{i-1}, y_{i-1}) and (x_i, y_i) for $i \in [n]$ yields a trajectory of a walker in the plane, starting out at the origin and moving, at each step of length $\sqrt{2}$, either in the NE or SE direction. A *Dyck path of length* $2n$ is NE/SE path $(x_0, y_0), (x_1, x_2), \ldots, (x_n, y_n)$ where $y_i \geq 0$ for all $i \in [n-1]$ and $y_n = 0$. Given $n \in \mathbb{N}$, the set of Dyck paths of length $2n$ will be denoted by D_n. The reader is referred to Figure 4-5 of Section 4.4 for an illustration.

Dyck paths of length $2n$ are counted by the Catalan number $C_n = (n+1)^{-1}\binom{2n}{n}$ and are found to be in bijective correspondence with a surprising number of combinatorial objects. In the present context, in Section 4.4, weighted Dyck paths will be found to encode the moments of the creation and field operators on the (q,t)-Fock space.

4.3 The (q, t)-Fock Space

The present section constructs the (q,t)-Fock space as a refinement of the construction introduced in [BS91].

Consider a real, separable Hilbert space $\mathscr{H}$ and some distinguished vector Ω disjoint from $\mathscr{H}$. Let $\mathscr{F} = (\mathbb{C}\Omega) \oplus \bigoplus_{n\geq 1} \mathscr{H}_{\mathbb{C}}^{\otimes n}$, where $\mathscr{H}_{\mathbb{C}}$ is the complexification of $\mathscr{H}$ and both the direct sum and tensor product are understood to be algebraic. In particular, $\mathscr{F}$ can be viewed as the vector space over $\mathbb{C}$ generated by $\{\Omega\} \cup \{h_1 \otimes \ldots \otimes h_n\}_{h_i \in \mathscr{H}, n \in \mathbb{N}}$.

For $f \in \mathscr{H}$ and $q, t \in \mathbb{R}$, define the operators $a(f)^*$ and $a(f)$ on $\mathscr{F}$ by linear extension of:

$$a(f)^*\Omega = f, \qquad a(f)^* h_1 \otimes \ldots \otimes h_n = f \otimes h_1 \otimes \ldots \otimes h_n \tag{4.22}$$

and

$$a(f)\Omega = 0, \qquad a(f)h_1 \otimes \ldots \otimes h_n = \sum_{k=1}^{n} q^{k-1} t^{n-k} \langle f, h_k \rangle_{\mathscr{H}} \, h_1 \otimes \ldots \otimes \check{h}_k \otimes \ldots \otimes h_n, \tag{4.23}$$

where the superscript $\check{h}_k$ indicates that h_k has been deleted from the product. Note that the creation operator $a(f)^*$ is defined identically to the operator $c^*(f)$ in [BS91], whereas the "twisted" annihilation operator $a(f)$ is the refinement of the operator $c(f)$ in [BS91] by a second parameter, t. Next, define the operator t^N on $\mathscr{F}$ by linear extension of

$$t^N \Omega = \Omega, \qquad t^N h_1 \otimes \ldots \otimes h_n = t^n \, h_1 \otimes \ldots \otimes h_n. \tag{4.24}$$

Since $t > 0$, the operator can be written in a somewhat more natural form as $e^{\alpha N}$, where $\alpha = \log(t)$.

Lemma 5. *For all $f, g \in \mathscr{H}$, the operators $a(f), a(g)^*$ on $\mathscr{F}$ fulfill the relation*

$$a(f)a(g)^* - q\, a(g)^* a(f) = \langle f, g \rangle_{\mathscr{H}} \, t^N.$$

Proof. For any $n \in \mathbb{N}$ and $g, h_1, \ldots, h_n \in \mathscr{H}$,

$$\begin{aligned}
&a(f)a(g)^* h_1 \otimes \ldots \otimes h_n = a(f) g \otimes h_1 \otimes \ldots \otimes h_n \\
&= t^n \langle f, g \rangle_{\mathscr{H}} \, h_1 \otimes \ldots \otimes h_n + \sum_{k=2}^{n+1} q^{k-1} t^{n+1-k} \langle f, h_{k-1} \rangle_{\mathscr{H}} g \otimes h_1 \otimes \ldots \otimes \check{h}_{k-1} \otimes \ldots \otimes h_n \\
&= t^n \langle f, g \rangle_{\mathscr{H}} \, h_1 \otimes \ldots \otimes h_n + g \otimes \left(\sum_{k=1}^{n} q^k t^{n-k} \langle f, h_k \rangle_{\mathscr{H}} h_1 \otimes \ldots \otimes \check{h}_k \otimes \ldots \otimes h_n \right) \\
&= t^n \langle f, g \rangle_{\mathscr{H}} \, h_1 \otimes \ldots \otimes h_n + q\, a(g)^* a(f) h_1 \otimes \ldots \otimes h_n
\end{aligned}$$

$\square$

Define the sesquilinear form $\langle , \rangle_{q,t}$ on $\mathscr{F}$ by

$$\langle g_1 \otimes \ldots \otimes g_n, h_1 \otimes \ldots \otimes h_m \rangle_{q,t} = 0 \quad \text{for } m \neq n \tag{4.25}$$

and otherwise recursively by

$$\langle g_1 \otimes \ldots \otimes g_n, h_1 \otimes \ldots \otimes h_n \rangle_{q,t} = \sum_{k=1}^{n} q^{k-1} t^{n-k} \langle g_1, h_k \rangle_{\mathscr{H}} \langle g_2 \otimes \ldots \otimes g_n, h_1 \otimes \ldots \otimes \check{h}_k \otimes \ldots \otimes h_n \rangle_{q,t}. \tag{4.26}$$

Remark 4 For all q, setting $t = 1$ recovers the usual inner product on the q-Fock space of [BS91]. Letting $t = 0$ yields the following sesquilinear form on the full Fock space:

$$\langle g_1 \otimes \ldots \otimes g_n, h_1 \otimes \ldots \otimes h_n \rangle_{q,0} = q^{\binom{n}{2}} \langle g_1, h_n \rangle_{\mathscr{H}} \ldots \langle g_n, h_1 \rangle_{\mathscr{H}},$$

which does not generally satisfy the positivity requirement of an inner product. •

The range of t required for $\langle , \rangle_{q,t}$ to be an inner product will be characterized shortly, in Lemma 8. But, first, note that for all $f \in \mathscr{H}$, $a(f)^*$ is indeed the adjoint of $a(f)$ with respect to $\langle , \rangle_{q,t}$.

Lemma 6. *For all $f \in \mathscr{H}$, $\xi, \eta \in \mathscr{F}$,*

$$\langle a(f)^* \xi, \eta \rangle_{q,t} = \langle \xi, a(f) \eta \rangle_{q,t}.$$

Proof. It suffices to note that, directly from the previous definitions,

$$\begin{aligned}
&\langle a(f)^* g_1 \otimes \ldots \otimes g_n, h_1 \otimes \ldots \otimes h_{n+1} \rangle_{q,t} = \langle f \otimes g_1 \otimes \ldots \otimes g_n, h_1 \otimes \ldots \otimes h_{n+1} \rangle_{q,t} \\
&= \sum_{k=1}^{n+1} q^{k-1} t^{n+1-k} \langle f, h_k \rangle_{\mathscr{H}} \langle g_1 \otimes \ldots \otimes g_n, h_1 \otimes \ldots \otimes \check{h}_k \otimes \ldots \otimes h_{n+1} \rangle_{q,t} \\
&= \langle g_1 \otimes \ldots \otimes g_n, a(f) h_1 \otimes \ldots \otimes h_{n+1} \rangle_{q,t}
\end{aligned}$$

□

Still in line with [BS91], it is convenient define the operator $P_{q,t} : \mathscr{F} \to \mathscr{F}$ allowing one to express the sesquilinear form $\langle , \rangle_{q,t}$ via the usual scalar product $\langle , \rangle_0$ on the full Fock space. Consider the unitary representation $\pi \mapsto U_\pi^{(n)}$ of the symmetric group S_n on $\mathscr{H}^{\otimes n}$, given as

$$U_\pi^{(n)} h_1 \otimes \ldots \otimes h_n = h_{\pi(1)} \otimes \ldots \otimes h_{\pi(n)}.$$

Recalling the permutation statistics given by inversions and co-inversions, defined in the previous section (cf. Definition 8), let

$$P_{q,t} = \bigoplus_{n=0}^{\infty} P_{q,t}^{(n)} \qquad \text{with} \quad P_{q,t}^{(n)} : \mathscr{H}^{\otimes n} \to \mathscr{H}^{\otimes n},$$

$$P_{q,t}^{(n)} := \sum_{\pi \in S_n} q^{\mathrm{inv}(\pi)} t^{\mathrm{cinv}(\pi)} U_\pi^{(n)}, \tag{4.27}$$

where for the unique $\pi \in S_1$, it is understood that $\mathrm{inv}(\pi) = \mathrm{cinv}(\pi) = 0$. Thus, $P_{q,t}^{(1)} = 1$ and there is no change to the scalar product on the single-particle space. For $n \geq 2$, note that for every $1 \leq i < j \leq n$, the pair $(i, j, \pi(i), \pi(j))$ is either an inversion or a coinversion, and therefore $\mathrm{inv}(\pi) + \mathrm{cinv}(\pi) = \binom{n}{2}$. It follows that for $n \geq 2$ and $t \neq 0$, the operator $P_{q,t}^{(n)}$ can equivalently be expressed as

$$P_{q,t}^{(n)} = t^{\binom{n}{2}} \sum_{\pi \in S_n} \left(\frac{q}{t}\right)^{\mathrm{inv}(\pi)} U_\pi^{(n)} = t^{\binom{n}{2}} P_{q/t}^{(n)}, \tag{4.28}$$

where $P_q^{(n)}$ denotes the analogous operator on the subspace $\mathscr{H}^{\otimes n}$ of the q-Fock space (see [BS91]).

Lemma 7. *For all $\xi, \eta \in \mathscr{F}$,*

$$\langle \eta, \xi \rangle_{q,t} = \langle \xi, P_{q,t} \eta \rangle_0.$$

Proof. It suffices to prove that for all $n \in \mathbb{N}$ and all $g_i, h_j \in \mathscr{H}$,

$$\begin{aligned}\langle g_1 \otimes \ldots \otimes g_n, h_1 \otimes \ldots \otimes h_n \rangle_{q,t} &= \langle g_1 \otimes \ldots \otimes g_n, P_{q,t} h_1 \otimes \ldots \otimes h_n \rangle_0 \\ &= \sum_{\pi \in S_n} q^{\mathrm{inv}(\pi)} t^{\mathrm{cinv}(\pi)} \langle g_1, h_{\pi(1)} \rangle_{\mathscr{H}} \cdots \langle g_n, h_{\pi(n)} \rangle_{\mathscr{H}}\end{aligned}$$

The claim clearly holds for $n = 1$. Proceeding by induction on n, recall that from the definition of $\langle , \rangle_{q,t}$,

$$\begin{aligned}&\langle g_1 \otimes \ldots \otimes g_n, h_1 \otimes \ldots \otimes h_n \rangle_{q,t} \\ &= \sum_{k=1}^{n} q^{k-1} t^{n-k} \langle g_1, h_k \rangle_{\mathscr{H}} \langle g_2 \otimes \ldots \otimes g_n, h_1 \otimes \ldots \otimes \check{h}_k \otimes \ldots \otimes h_n \rangle_{q,t}.\end{aligned}$$

Letting $S_n^{(k)}$ denote all bijections from $\{2, \ldots, n\}$ to $\{1, \ldots, \check{k}, \ldots, n\}$, note that, for all $n \in \mathbb{N}$, every $\pi \in S_n$ can be uniquely decomposed as a pair (k, σ) for some $k \in [n]$ and $\sigma \in S_{n-1}^{(k)}$ and, conversely, that any such pair gives a distinct element of S_n. Specifically, let $\pi(1) = k$ and $\pi(\ell) = \sigma(\ell)$ for $\ell \in \{2, \ldots, n\}$. Then, noting that the natural correspondences $[n-1] \leftrightarrow \{2, \ldots, n\}$ and $[n-1] \leftrightarrow \{1, \ldots, \check{k}, \ldots n\}$ are order-preserving, the inductive hypothesis on $n-1$ can be written as

$$\langle g_2 \otimes \ldots \otimes g_n, h_1 \otimes \ldots \otimes \check{h}_k \otimes \ldots \otimes h_n \rangle_{q,t} = \sum_{\sigma \in S_{n-1}^{(k)}} q^{\mathrm{inv}(\sigma)} t^{\mathrm{cinv}(\sigma)} \langle g_2, h_{\sigma(2)} \rangle_{\mathscr{H}} \cdots \langle g_n, h_{\sigma(n)} \rangle_{\mathscr{H}},$$

where $\mathrm{inv}(\sigma)$ counts all the pairs $(i, j) \in \{2, \ldots, n\} \times \{1, \ldots, \check{k}, \ldots, n\}$ with $i < j$ and $\pi(i) > \pi(j)$ and $\mathrm{cinv}(\sigma)$ is defined analogously. Furthermore, observing that $\mathrm{inv}(\pi) = \mathrm{inv}(\sigma) + k - 1$ and $\mathrm{cinv}(\pi) = \mathrm{cinv}(\sigma) + n - k$, as demonstrated in the caption of Figure 4-4, it follows that

$$\begin{aligned}&\langle g_1 \otimes \ldots \otimes g_n, h_1 \otimes \ldots \otimes h_n \rangle_{q,t} \\ &= \sum_{k=1}^{n} q^{k-1} t^{n-k} \langle g_1, h_k \rangle_{\mathscr{H}} \sum_{\sigma \in S_{n-1}^{(k)}} q^{\mathrm{inv}(\sigma)} t^{\mathrm{cinv}(\sigma)} \langle g_2, h_{\sigma(2)} \rangle_{\mathscr{H}} \cdots \langle g_n, h_{\sigma(n)} \rangle_{\mathscr{H}} \\ &= \sum_{\pi \in S_n} q^{\mathrm{inv}(\pi)} t^{\mathrm{cinv}(\pi)} \langle g_1, h_{\pi(1)} \rangle_{\mathscr{H}} \langle g_2, h_{\pi(2)} \rangle_{\mathscr{H}} \cdots \langle g_n, h_{\pi(n)} \rangle_{\mathscr{H}}\end{aligned}$$

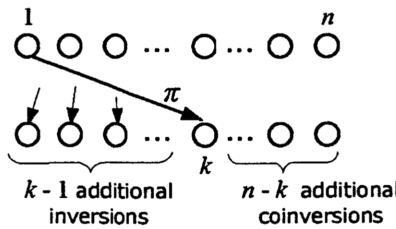

Figure 4-4: If $\pi(1) = k$, then there are exactly $k-1$ elements from the set $\{2,\dots,n\}$ that map under π to an element in $\{1,\dots,k-1\}$ and exactly $n-k$ elements from $\{2,\dots,n\}$ that map to $\{k+1,\dots,n\}$. Thus, there are $k-1$ elements $j > 1$ for which $\pi(j) < \pi(1)$ and $n-k$ elements $\ell > 1$ for which $\pi(\ell) > \pi(1)$. For any other pair $(i,j) \in [n]^2$ with $i \neq 1$, the corresponding inversion or coinversion is accounted for in $\sigma : \{2,\dots,n\} \to \{1,\dots,\check{k},\dots,n\}$. It follows that $\mathrm{inv}(\pi) = k-1+\mathrm{inv}(\sigma)$ and $\mathrm{cinv}(\pi) = n-k+\mathrm{cinv}(\sigma)$.

□

While the following facts, contained in Lemmas 8 through 9, have direct proofs analogous to those in [BS91], it is more convenient to use (4.28) and derive the desired properties from those of the q-Fock space.

Lemma 8. *a) The operator $P_{q,t}$ is positive for all $|q| \leq t$.*
b) The operator $P_{q,t}$ is strictly positive for all $|q| < t$.

Proof. Since $P_{q,t} = \bigoplus_{n=0}^{\infty} P_{q,t}^{(n)}$, it suffices to consider the positivity of $P_{q,t}^{(n)}$. Since $t^{\binom{n}{2}} > 0$, the positivity of $P_{q,t}^{(n)}$ for $|q| \leq t$ follows from (4.28) by the positivity ([BS91]) of $P_{q/t}^{(n)}$ for $|q/t| \leq 1$ and the strict positivity of $P_{q,t}^{(n)}$ for $|q| \leq t$ follows from the strict positivity ([BS91]) of $P_{q/t}^{(n)}$ for $|q/t| < 1$. □

Since $\langle\, ,\, \rangle_{q,t}$ is an inner product on $\mathscr{F}$, the completion of $\mathscr{F}$ yields the desired (q,t)-Fock space.

Definition 9. *The (q,t)-Fock space $\mathscr{F}_{q,t}$ is the completion of $\mathscr{F}$ with respect to $\langle\, ,\, \rangle_{q,t}$.*

Remark 5 For $\dim(\mathscr{H}) \geq 2$, the conditions of Lemma 8 are also necessary. In particular, let e_1, e_2 be two unit vectors in $\mathscr{H}$ with $\langle e_1, e_2\rangle_{q,t} = 0$. Note that

$$\|e_1 \otimes e_2 + e_2 \otimes e_1\|_{q,t}^2 = 2t + 2q, \qquad \text{and} \qquad \|e_1 \otimes e_2 - e_2 \otimes e_1\|_{q,t}^2 = 2t - 2q.$$

Thus, $\langle\, , \,\rangle_{q,t}$ is positive (resp. strictly positive) only if $|q| \leq t$ (resp. $|q| < t$). $\bullet$

Letting $q = t = 1$ recovers the bosonic Fock space. Returning to (4.22) and (4.23), in the bosonic case, the operators $a(f)^*$ and $a(f)$ are unbounded and defined only on the dense domain $\mathscr{F}$. The values of q and t that instead give rise to bounded linear operators on $\mathscr{F}_{q,t}$ are as follows.

Lemma 9. *For the following parameter ranges, $\{a(f) \mid f \in \mathscr{H}\}$ extend to bounded linear operators on $\mathscr{F}_{q,t}$, with norms*

$$\|a(f)\| = \begin{cases} \|f\|_{\mathscr{H}}, & -t \leq q \leq 0 < t \leq 1 \\ \frac{1}{\sqrt{1-q}}\,\|f\|_{\mathscr{H}}, & 0 < q < t = 1 \\ \sqrt{(n_* + 1)\, t^{n_*}}\,\|f\|_{\mathscr{H}}, & 0 < q = t < 1 \\ \sqrt{\frac{t^{\hat{n}+1} - q^{\hat{n}+1}}{t-q}}\,\|f\|_{\mathscr{H}}, & 0 < q < t < 1 \end{cases}$$

for

$$n_* = \lfloor t/(1-t) \rfloor, \qquad \hat{n} = \left\lfloor \frac{\log(1-q) - \log(1-t)}{\log(t) - \log(q)} \right\rfloor.$$

Proof. Let $q \in [-1, 0]$ and $|q| \leq t \leq 1$. For $\xi_n \in \mathscr{H}^{\otimes n}$, by Lemma 5,

$$\begin{aligned} \langle a(f)^*\xi_n, a(f)^*\xi_n\rangle_{q,t} &= \langle \xi_n, a(f)a(f)^*\xi_n\rangle_{q,t} = t^n \langle f, f\rangle_{\mathscr{H}} \langle \xi_n, \xi_n\rangle_{q,t} + q\langle \xi_n, a(f)^*a(f)\xi_n\rangle_{q,t} \\ &= t^n \|f\|_{\mathscr{H}}^2 \|\xi_n\|_{q,t}^2 + q\|a(f)\xi_n\|_{q,t}^2 \leq \|f\|_{\mathscr{H}}^2 \|\xi_n\|_{q,t}^2, \end{aligned}$$

as $q \leq 0$ and $0 < t \leq 1$. Next, for an element $\xi = \sum_{i=0}^{n} \alpha_i \xi_i \in \mathscr{F}$, where $\alpha_i \in \mathbb{C}$ and $\xi_i \in \mathscr{H}^{\otimes i}$, that $a(f)^*$ is linear and $\langle \xi_n, \xi_m\rangle_{q,t} = 0$ whenever $n \neq m$ implies that

$$\langle a(f)^*\xi, a(f)^*\xi\rangle_{q,t} = \sum_{i=0}^{n} |\alpha_i|^2 \langle a(f)^*\xi_i, a(f)^*\xi_i\rangle_{q,t} \leq \sum_{i=0}^{n} |\alpha_i|^2 \|f\|_{\mathscr{H}}^2 \|\xi_i\|_{q,t}^2 = \|f\|_{\mathscr{H}}^2 \|\xi\|_{q,t}^2.$$

Finally, since $\langle a(f)^*\Omega, a(f)^*\Omega\rangle_{q,t} = \|f\|_{\mathscr{H}}^2$, it follows that $\|a(f)\| = \|a(f)^*\| = \|f\|_{\mathscr{H}}$.

For $q \in (0,1)$, analogously to the previous case, it suffices to focus on $\xi \in \mathscr{H}^{\otimes n}$. Again, $\langle a(f)^*\Omega, a(f)^*\Omega\rangle_{q,t} = \|f\|^2_{\mathscr{H}}$ and, for $n \in \mathbb{N}$, (4.28) yields

$$\begin{aligned}\langle a(f)^*\xi, a(f)^*\xi\rangle_{q,t} &= \langle a(f)^*\xi, P^{(n+1)}_{q,t}a(f)^*\xi\rangle_0 = t^{\binom{n+1}{2}}\langle a(f)^*\xi, P^{(n+1)}_{q/t}a(f)^*\xi\rangle_0 \\ &= t^{\binom{n+1}{2}}\langle a(f)^*\xi, a(f)^*\xi\rangle_{q/t}.\end{aligned}$$

Recalling that the creation operators on the (q,t)-Fock and μ-Fock spaces (for $\mu \in [-1,1]$) are defined by the linear extension of the same operator on the dense linear subspace $\mathscr{F}$, it follows that

$$\langle a(f)^*\xi, a(f)^*\xi\rangle_{q/t} = \langle a_{q/t}(f)^*\xi, a_{q/t}(f)^*\xi\rangle_{q/t},$$

where $a^*_{q/t}(f)$ analogously denotes the creation operator on the (q/t)-Fock space. Applying next the familiar bounds on $\|a^*_{q/t}(f)\xi\|_{q/t}$ (e.g. Lemma 4 in [BS91]),

$$\langle a(f)^*\xi, a(f)^*\xi\rangle_{q,t} \le t^{\binom{n+1}{2}}\sum_{k=1}^{n+1}\left(\frac{q}{t}\right)^{k-1}\|f\|^2_{\mathscr{H}}\|\xi\|^2_{q/t} = \begin{cases} t^{\binom{n+1}{2}}\,(n+1)\,\|f\|^2_{\mathscr{H}}\|\xi\|^2_{q/t}, & q = t \\ t^{\binom{n+1}{2}}\,\frac{1-(q/t)^{n+1}}{1-q/t}\,\|f\|^2_{\mathscr{H}}\|\xi\|^2_{q/t}, & q < t \end{cases}$$

But,

$$\|\xi\|^2_{q/t} = \langle \xi, P^{(n)}_{q/t}\xi\rangle_0 = t^{-\binom{n}{2}}\langle \xi, P^{(n)}_{q,t}\xi\rangle_0 = t^{-\binom{n}{2}}\|\xi\|^2_{q,t},$$

and so,

$$\langle a(f)^*\xi, a(f)^*\xi\rangle_{q,t} \le \begin{cases} t^n\,(n+1)\,\|f\|^2_{\mathscr{H}}\|\xi\|^2_{q,t}, & q = t \\ \frac{t^{n+1}-q^{n+1}}{t-q}\|f\|^2_{\mathscr{H}}\|\xi\|^2_{q,t}, & q < t. \end{cases}$$

It follows that $a(f)^*$ is bounded for $0 < q < t \le 1$ and $0 < q = t < 1$. To recover the corresponding expressions for the norm, let $\xi = f \otimes \ldots \otimes f = f^{\otimes n} \in \mathscr{H}^{\otimes n}$. Then,

$$\begin{aligned}\langle a(f)^*f^{\otimes n}, a(f)^*f^{\otimes n}\rangle_{q,t} &= \langle f^{\otimes n+1}, f^{\otimes n+1}\rangle_{q,t} = \sum_{k=1}^{n+1} q^{k-1}t^{n+1-k}\|f\|^2_{\mathscr{H}}\|f^{\otimes n}\|^2_{q,t} \\ &= \begin{cases} t^n\,(n+1)\,\|f\|^2_{\mathscr{H}}\|\xi\|^2_{q,t}, & q = t \\ \frac{t^{n+1}-q^{n+1}}{t-q}\|f\|^2_{\mathscr{H}}\|\xi\|^2_{q,t}, & q < t \end{cases}\end{aligned}$$

For $0 < q < t = 1$,

$$\sup_{n\in\mathbb{N}} \frac{\langle a(f)^* f^{\otimes n}, a(f)^* f^{\otimes n}\rangle_{q,t}}{\|f^{\otimes n}\|_{q,t}^2} = \frac{1}{1-q}\|f\|_{\mathscr{H}}^2.$$

For $0 < q = t < 1$, one is looking for $n \in \mathbb{N} \cup \{0\}$ that maximizes $s_n := t^n(n+1)$. Since $s_n \geq s_{n-1}$ iff $n \leq t/(1-t)$, the desired maximum is achieved at $n = \lfloor t/(1-t) \rfloor$.

Otherwise, for $0 < q < t < 1$, it remains to maximize $t^{n+1} - q^{n+1}$. Let

$$r_n := \frac{t^n - t^{n+1}}{q^n - q^{n+1}}, \qquad n \in \mathbb{N} \cup \{0\},$$

and note that, since $0 < q < t$, both the numerator and the denominator are strictly positive. Furthermore, $\{r_n\}_{n\geq 0}$ forms a strictly increasing sequence as

$$r_n = \frac{t^n\,(1-t)}{q^n\,(1-q)} = \frac{t}{q} r_{n-1} > r_{n-1}.$$

Now, note that $r_n \leq 1$ iff $t^{n+1} - q^{n+1} \geq t^n - q^n$. Therefore, if $r_1 > 1$, then $t^{n+1} - q^{n+1}$ is maximized for $n = 0$; otherwise, the expression is maximized for the greatest non-negative integer n for which $r_n \leq 1$. A straightforward calculation then yields that $t^{n+1} - q^{n+1}$ is maximized for $n = \hat{n}$, where

$$\hat{n} = \left\lfloor \frac{\log(1-q) - \log(1-t)}{\log(t) - \log(q)} \right\rfloor.$$

□

4.4 The (q,t)-Gaussian Processes

The natural starting point to the probabilistic considerations of this section is the $*$-probability space $(\mathscr{G}_{q,t}, \varphi_{q,t})$, discussed next, formed by the $*$-algebra generated by the creation and annihilation operators on $\mathscr{F}_{q,t}$ and the vacuum-state expectation on the algebra. The $*$-probability space $(\mathscr{G}_{q,t}, \varphi_{q,t})$ provides a convenient setting in which the (q,t)-Gaussian family is subsequently introduced and studied.

4.4.1 The $(\mathscr{G}_{q,t}, \varphi_{q,t})$ $*$-probability space

For a clear introduction to non-commutative probability spaces, the reader is referred to the monograph [NS06]. In the present context, the setting of interest is that of the $(\mathscr{G}_{q,t}, \varphi_{q,t})$ $*$-probability space, formed by:

- the unital $*$-algebra $\mathscr{G}_{q,t}$ generated by $\{a(h) \mid h \in \mathscr{H}\}$;
- the unital linear functional $\varphi_{q,t} : \mathscr{G}_{q,t} \to \mathbb{C}$, $b \mapsto \langle \Omega, b\Omega \rangle_{q,t}$ (i.e. the vacuum expectation state on $\mathscr{G}_{q,t}$.).

In this non-commutative setting, *random variables* are understood to be the elements of $\mathscr{G}_{q,t}$ and of particular interest are their joint mixed moments, i.e. expressions $\varphi_{q,t}(b_1^{\epsilon(1)} \dots b_k^{\epsilon(k)})$ for $b_1, \dots, b_k \in \mathscr{G}_{q,t}$, $k \in \mathbb{N}$ and $\epsilon(1), \dots, \epsilon(k) \in \{1, *\}$. Since $\varphi_{q,t}$ is linear, one is allowed to focus on the joint moments $\varphi_{q,t}(a(h_1)^{\epsilon(1)} \dots a(h_k)^{\epsilon(k)})$. These turn out to be intimately connected with two combinatorial objects, discussed next.

For $n \in \mathbb{N}$, consider first the map ψ_n from $\{1, *\}^n$ to the set of all NE/SE paths of length n (cf. Section 4.2), by which every "$*$" maps to a NE step and every "1" maps to a SE step. Clearly, ψ_n is a bijection and of particular interest is the set $\psi_{2n}^{-1}(D_n)$, where D_n denotes the set of Dyck paths of length $2n$. An example of an element in $\{1, *\}^{14}$ mapping into D_7 is shown in Figure 4-5.

Figure 4-5: $\psi_{14}(*, *, *, *, *, 1, 1, *, 1, 1, 1, 1, *, 1)$.

Given $(\epsilon(1), \dots, \epsilon(2n)) \in \psi^{-1}(D_n)$, let each $*$ encode an "opening" and each 1 encode a "closure" and, given some such string of openings and closures, consider all the ways in which the elements of the string can be organized into disjoint pairs so that (1) an opening is always to the left of the corresponding closure and (2) no opening is left unpaired. Clearly, each fixed $(\epsilon(1), \dots, \epsilon(2n)) \in \psi^{-1}(D_n)$ is thus associated with a distinct set of pair-partitions of $[2n]$, that is, a subset of $\mathscr{P}_2(2n)$ (cf. Section 4.2). Conversely, every pair partition naturally corresponds to a unique string of openings/closures belonging to $\psi^{-1}(D_n)$. Thus, writing $\mathscr{V} \sim_p \mathscr{V}'$ for any $\mathscr{V}, \mathscr{V}' \in \mathscr{P}_2(2n)$ for which the corresponding strings of openings and closures both encode the same string in $\psi^{-1}(D_n)$ defines an equivalence relation. Figure 4-6 shows the six pairings in the equivalence class of the string $(*, *, *, 1, 1, 1) \in \psi^{-1}(D_6)$.

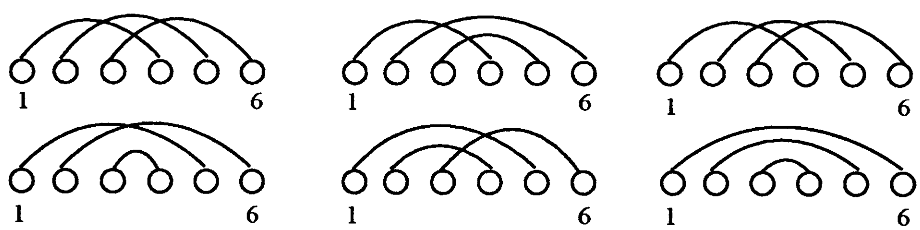

Figure 4-6: The equivalence class of $(*, *, *, 1, 1, 1) \in \psi^{-1}(D_6)$ in $P_2(6)$.

Next, recalling that the purpose is calculating the value of the mixed moment

$$\varphi_{q,t}(a(h_1)^{\epsilon(1)} \dots a(h_{2n})^{\epsilon(2n)}),$$

fix a string $(\epsilon(1), \dots, \epsilon(2n)) \in \psi_{2n}^{-1}(D_n)$ and jointly consider some underlying choice of $h_1, \dots, h_{2n} \in \mathscr{H}$. Each pairing $\mathscr{V} = \{(w_1, z_1), \dots, (w_n, z_n)\}$ in the equivalence class of $(\epsilon(1), \dots, \epsilon(2n))$ is then assigned the weight

$$\operatorname{wt}(\mathscr{V}; h_1, \dots, h_{2n}) := q^{\operatorname{cross}(\mathscr{V})} t^{\operatorname{nest}(\mathscr{V})} \prod_{i=1}^{n} \langle h_{w_i}, h_{z_i} \rangle_{\mathscr{H}},$$

where $\operatorname{cross}(\mathscr{V})$ and $\operatorname{nest}(\mathscr{V})$ correspond, respectively, to the numbers of crossings and nestings in $\mathscr{V}$, as defined in Section 4.2. For example, the top left-most pairing in Figure 4-6 is thus given the weight $q^3 \langle g_1, g_4 \rangle_{\mathscr{H}} \langle g_2, g_5 \rangle_{\mathscr{H}} \langle g_3, g_6 \rangle_{\mathscr{H}}$. For $n \in \mathbb{N}$, let $T_{q,t}(h_1, \dots, h_{2n}; \epsilon(1), \dots, \epsilon(2n))$ denote the generating function of the weighted pair-partitions of $[2n]$, namely

$$\begin{aligned} T_{q,t}(h_1,\ldots,h_{2n};\epsilon(1),\ldots,\epsilon(2n)) &:= \sum_{\substack{\mathscr{V}\in\mathscr{P}_2(2n)\\ \mathscr{V}\sim_p(\epsilon(1),\ldots,\epsilon(2n))}} \mathrm{wt}(\mathscr{V};h_1,\ldots,h_{2n}) \\ &= \sum_{\substack{\mathscr{V}\in\mathscr{P}_2(2n)\\ \mathscr{V}\sim_p(\epsilon(1),\ldots,\epsilon(2n))}} q^{\mathrm{cross}(\mathscr{V})}t^{\mathrm{nest}(\mathscr{V})}\prod_{i=1}^{n}\langle h_{w_i},h_{z_i}\rangle_{\mathscr{H}} \end{aligned} \tag{4.29}$$

where $\mathscr{V}\sim_p(\epsilon(1),\ldots,\epsilon(2n))$ is meant to indicate that $\mathscr{V}$ is in the equivalence class of $(\epsilon(1),\ldots,\epsilon(2n))$ under $\sim_p$ (i.e. $\mathscr{V}$ has $(\epsilon(1),\ldots,\epsilon(2n))$ as its opening/closure string). Writing the weight of a pairing as a product of weights of its pairs, as in the following lemma, is the remaining ingredient in connecting the combinatorial structures at hand to the moments $\varphi_{q,t}(a(h_1)^{\epsilon(1)}\ldots a(h_{2n})^{\epsilon(2n)})$.

Lemma 10. *For $\mathscr{V}=\{(z_1,w_1),\ldots,(z_n,w_n)\}\in\mathscr{P}_2(2n)$ with $w_1<w_2<\ldots<w_n$ (i.e. with pairs indexed in the increasing order of closures) and $h_1,\ldots,h_{2n}\in\mathscr{H}$,*

$$wt(\mathscr{V};h_1,\ldots,h_{2n})=\prod_{i=1}^{n}q^{|\{z_j\,|\,z_i<z_j<w_i,j>i\}|}\;t^{|\{z_j\,|\,1\le z_j<z_i,j>i\}|}\langle h_{z_i},h_{w_i}\rangle_{\mathscr{H}}.$$

Proof. Consider the following procedure for assigning weights to a pairing. Starting with the left-most closure w_1, suppose that it connects to some opening z_1 (where $z_1<w_1$). Since w_1 is indeed the left-most closure, exactly w_1-z_1 pairs will cross the pair (z_1,w_1) and exactly z_1-1 pairs will nest with it. Thus, assign the pair (z_1,w_1) the weight $q^{w_i-z_i}t^{z_i-1}\langle h_{z_i},h_{w_i}\rangle_{\mathscr{H}}$. Consider now the i^{th} closure from the left, w_i, and let $\mathscr{V}'$ be the pairing from which all pairs (z_k,w_k) for $k<i$ have been removed. Then, assigning the pair (z_i,w_i) its weight in $\mathscr{V}'$ according to the previous recipe does not take into account any crossings or nestings that have already been accounted for by the previous pairs. It now suffices to note that, by the end of the procedure, all the crossings and nestings have been taken into account and that the product of the weights of the pairs indeed equals the expression for $\mathrm{wt}(\mathscr{V};h_1,\ldots,h_{2n})$ in (4.29). $\square$

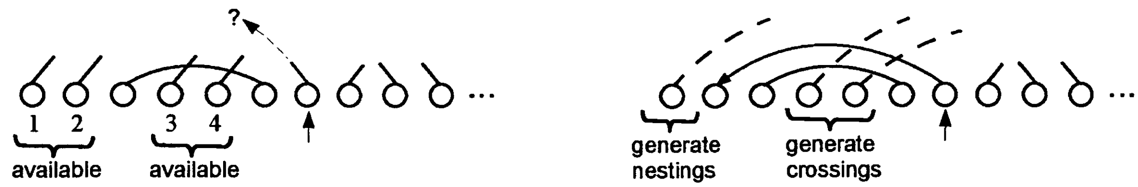

Figure 4-7: The sequential pairing of openings and closures corresponding to the Dyck path of Figure 4-5, with an arrow denoting the currently considered closure. In the left figure, there are four available (i.e. as of yet unpaired) openings. In the right figure, the current closure is paired to the indicated opening, thus incurring two crossings and one nesting. Note that the crossings and nestings incurred by the current closure never include any crossings or nestings already counted in the previous closure.

The sequential procedure for assigning weights to pairings is illustrated in Figure 4-7. We are now ready to calculate the joint mixed moments of interest.

Lemma 11. *For all $n \in \mathbb{N}$ and $\epsilon(1), \ldots, \epsilon(2n) \in \{1, *\}^{2n}$,*

$$\varphi_{q,t}(a(h_1)^{\epsilon(1)} \ldots a(h_{2n-1})^{\epsilon(2n-1)}) = 0$$

$$\varphi_{q,t}(a(h_1)^{\epsilon(1)} \ldots a(h_{2n})^{\epsilon(2n)}) = \begin{cases} 0, & (\epsilon(2n), \ldots, \epsilon(1)) \notin \psi_{2n}^{-1}(D_n) \\ T_{q,t}(h_{2n}, \ldots, h_1; \epsilon(2n), \ldots, \epsilon(1)), & (\epsilon(2n), \ldots, \epsilon(1)) \in \psi_{2n}^{-1}(D_n) \end{cases}$$

$$= \sum_{\mathscr{V} \in \mathscr{P}_2(2n)} \varphi_{q,t}(a(h_{w_1})^{\epsilon(w_1)} a(h_{z_1})^{\epsilon(z_1)}) \ldots \varphi_{q,t}(a(h_{w_n})^{\epsilon(w_n)} a(h_{z_n})^{\epsilon(z_n)}) q^{cross(\mathscr{V})} t^{nest(\mathscr{V})},$$

where each $\mathscr{V}$ is (uniquely) written as a collection of pairs $\{(w_1, z_1), \ldots, (w_n, z_n)\}$ with $w_1 < \ldots < w_n$ and $w_i < z_i$.

Proof. Given the mixed moment $\varphi_{q,t}(a(h_1)^{\epsilon(1)} \ldots a(h_k)^{\epsilon(k)})$, consider the reverse string $\epsilon(k), \epsilon(k-1), \ldots, \epsilon(1)$ and the corresponding NE/SE path $\psi_k(\epsilon(k), \epsilon(k-1), \ldots, \epsilon(1))$. Since $a(h)\Omega = 0$ for all $h \in \mathscr{H}$, by recursively expanding the mixed moment $\varphi_{q,t}(a(h_1)^{\epsilon(1)} \ldots a(h_k)^{\epsilon(k)})$ via (4.22) and (4.23), it immediately follows that the moment is zero if for any $1 \leq i \leq k$, the number of SE steps in $\psi_i(\epsilon(k), \epsilon(k-1), \ldots, \epsilon(i))$ exceeds the corresponding number of NE steps. Moreover, since for all $h_i, g_i \in \mathscr{H}$, $\langle h_1 \otimes \ldots \otimes h_n, g_1 \otimes \ldots \otimes g_m \rangle_{q,t} = 0$ whenever $n \neq m$, it follows that the moment vanishes unless the total number of NE steps in $\psi_k(\epsilon(k), \ldots, \epsilon(1))$ equals the corresponding number of SE steps. This shows that the moment $\varphi_{q,t}(a(h_1)^{\epsilon(1)} \ldots a(h_k)^{\epsilon(k)})$

vanishes if $\psi_k(\epsilon(k),\ldots,\epsilon(1))$ is not a Dyck path. In particular, the mixed moment vanishes if k is odd.

Now let k be even with $\psi(\epsilon(k),\ldots,\epsilon(1)) \in D_n$. For some $m \geq 1$, let $\epsilon(m+1)$ correspond to the first $*$ from the left in the reverse string $(\epsilon(k),\ldots,\epsilon(1))$ (that is, corresponding to the right-most creation operator in $a(h_1)^{\epsilon(1)}\ldots a(h_k)^{\epsilon(k)}$). Then, by (4.22), $a(h_{m+1})$ acts on $h_m\otimes\ldots\otimes h_1$ to produce a weighted sum of $(m-1)$-dimensional products, that is,

$$a(h_{m+1})h_m\otimes\ldots\otimes h_1 = \sum_{i=1}^{m} q^{m-i}t^{i-1}\langle h_{m+1},h_i\rangle_{\mathscr{H}} h_m\otimes\ldots\check{h}_i\otimes\ldots\otimes h_1.$$

At the same time, diagrammatically, $\epsilon(m+1)$ corresponds to the first closure from the left in $\psi(\epsilon(k),\ldots,\epsilon(1))$. Supposing that this closure pairs to the i^{th} opening (from the left), for $1 \leq i \leq m$, the weight of the resulting pair in the sense of Lemma 10 is then $q^{m-i}t^{i-1}\langle h_{m+1},h_i\rangle_{\mathscr{H}}$. Furthermore, the act of removing h_i from the product $h_m\otimes\ldots\otimes h_1$ diagrammatically corresponds to removing the previously completed pairs in the procedure of Lemma 10 and, in both cases, the same iteration is subsequently repeated on the thus reduced object. Now, by definition, summing the weights $\mathrm{wt}(\mathscr{V};h_k,\ldots,h_1)$ over all pairings $\mathscr{V}\sim_p(\epsilon(k),\ldots,\epsilon(1))$ is equivalent to summing the products of the weights of the individual pairs over all the possible ways of matching all closures to openings (and thus, in the above notation, over all choices of i and analogous choices made on the subsequent iterations). Thus,

$$\sum_{\substack{\mathscr{V}\in\mathscr{P}_2(2n)\\ \mathscr{V}\sim_p(\epsilon(k),\ldots,\epsilon(1))}} \mathrm{wt}(\mathscr{V};h_k,\ldots,h_1)$$

is exactly the sum of weights obtained by unfolding the expression $\varphi_{q,t}(a(h_1)^{\epsilon(1)}\ldots a(h_k)^{\epsilon(k)})$ via the recursive definitions (4.22) and (4.23). In other words, we have shown that

$$\varphi_{q,t}(a(h_1)^{\epsilon(1)}\ldots a(h_{2n})^{\epsilon(2n)}) = T_{q,t}(h_{2n},\ldots,h_1;\epsilon(2n),\ldots,\epsilon(1))$$

whenever $(\epsilon(2n), \ldots, \epsilon(1)) \in \psi_{2n}^{-1}(D_n)$.

Finally, that $\varphi_{q,t}(a(h_1)^{\epsilon(1)} \ldots a(h_{2n})^{\epsilon(2n)})$ also equals

$$\sum_{\mathscr{V} \in \mathscr{P}_2(2n)} \varphi_{q,t}(a(h_{w_1})^{\epsilon(w_1)} a(h_{z_1})^{\epsilon(z_1)}) \ldots \varphi_{q,t}(a(h_{w_n})^{\epsilon(w_n)} a(h_{z_n})^{\epsilon(z_n)})\, q^{\mathrm{cross}(\mathscr{V})} t^{\mathrm{nest}(\mathscr{V})}$$

follows immediately from the fact that $\varphi_{q,t}(a(h_{w_j})^{\epsilon(w_j)} a(h_{z_j})^{\epsilon(z_j)}) = 0$ unless $\epsilon(w_j) = 1$ and $\epsilon(z_j) = *$; in other words, unless $\mathscr{V} \sim_p (\epsilon(2n), \ldots, \epsilon(1))$.

□

Much structure will subsequently be derived from the following encoding of the moments of a single element $a(h)$.

Lemma 12. *Given a Dyck path $\psi_n(\epsilon(2n), \ldots, \epsilon(1))$ and $h \in \mathscr{H}$, let $\widetilde{wt}(\epsilon(2n), \ldots, \epsilon(1); h)$ denote the weight of the path taken as the product of the weights of the individual steps, with each NE step assigned unit weight and each SE step falling from height m to height $m-1$ assigned weight*

$$[m]_{q,t} := \sum_{i=1}^{m} q^{m-i} t^{i-1} = \frac{t^m - q^m}{t - q}. \tag{4.30}$$

(See the illustration of Figure 4-8.) Then, $\varphi_{q,t}(a(h)^{\epsilon(1)} \ldots a(h)^{\epsilon(2n)}) = \widetilde{wt}(\epsilon(2n), \ldots, \epsilon(1); h)$.

Proof. Returning to the sequential procedure in the proof of Lemma 10, note that when $h_1 = \ldots = h_{2n} = h$, the choice of an opening for any given closure only affects the weight of the present pair and does not affect that of the subsequently considered pairs. In particular, for any given closure and m available (previously unpaired) openings to the left of it, the generating function of the weight of the current pair is therefore $\sum_{i=1}^{m} q^{m-i} t^{i-1} \|h\|_{\mathscr{H}}^2$. By the same token, the sum of weights of all the pairings $\mathscr{V}$ in the equivalence class of a fixed $(\epsilon(2n), \ldots, \epsilon(1)) \in \psi_n^{-1}(D_n)$ is given by the product of the corresponding generating functions. Finally, note that for any given closure, the "m" is determined by the underlying $(\epsilon(2n), \ldots, \epsilon(1))$; specifically, the reader may verify that m is exactly the height of the Dyck path preceding the

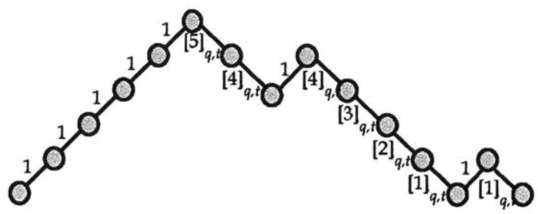

Figure 4-8: Weighted path corresponding to Figure 4-5 for $\|h\|_{\mathscr{H}} = 1$.

corresponding given SE step encoding the given closure. □

Figure 4-8 provides an example of assigning weights to the steps of the Dyck path given by $\psi_{14}(*,*,*,*,*,1,1,*,1,1,1,1,*,1)$ according to the rules defined in the above proof. In particular, in light of Lemma 12, the product of the weights of the individual steps then yields the moment

$$\varphi_{q,t}(a(h)a(h)^*(a(h))^4a(h)^*(a(h))^2(a(h)^*)^5).$$

At this point it is also interesting to note the combinatorial statistics featuring prominently in the previous section, given by inversions and coinversions of permutations, are intimately related to those of crossings and nestings in pair partitions considered presently. In particular, by Lemma 7,

$$\langle g_n \otimes \ldots \otimes g_1, h_1 \otimes \ldots \otimes h_n \rangle_{q,t} = \sum_{\pi \in S_n} q^{\mathrm{inv}(\pi)} t^{\mathrm{cinv}(\pi)} \langle g_n, h_{\pi(1)} \rangle_{\mathscr{H}} \ldots \langle g_1, h_{\pi(n)} \rangle_{\mathscr{H}},$$

while, at the same time,

$$\begin{aligned} \langle g_n \otimes \ldots \otimes g_1, h_1 \otimes \ldots \otimes h_n \rangle_{q,t} &= \langle \Omega, a(g_1) \ldots a(g_n) a^*(h_1) \ldots a(h_n) \Omega \rangle_{q,t} \\ &= \sum_{\substack{\mathscr{V} \in \mathscr{P}_2(2n) \\ \mathscr{V} \sim_p (1,1,\ldots,1,*,*,\ldots,*)}} q^{\mathrm{cross}(\mathscr{V})} t^{\mathrm{nest}(\mathscr{V})} \prod_{i=1}^{n} \langle g_{w_i}, h_{z_i} \rangle_{\mathscr{H}}, \end{aligned}$$

where the second equality follows by Lemma 11. In other words, the value of $\langle g_n \otimes \ldots \otimes g_1, h_1 \otimes \ldots \otimes h_n \rangle_{q,t}$ is computed by summing the weights over pairings $\mathscr{V}$ in the equivalence class of $(1, 1, \ldots, 1, *, *, \ldots, *)$. Writing each such $\mathscr{V}$ in the two-line notation of the previous section, with the $\epsilon(i) = 1$ corresponding to the top row and $\epsilon(j) = *$ to the bottom row, the reader may readily verify that the crossings turn into inversions and nestings into coinversions, and vice-versa.

4.4.2 Joint and marginal statistics of (q, t)-Gaussian elements

Analogously to both classical and free probability and, more generally, to the non-commutative probability constructed over the q-Fock space of [BS91], the so-called "Gaussian elements" will turn out to occupy a fundamental role in the non-commutative probability built over the (q, t)-Fock space. More precisely, for $h \in \mathscr{H}$, the (q, t)-Gaussian element $s_{q,t}(h)$ is defined as the (self-adjoint) element $a(h) + a(h)^*$ of $\mathscr{G}_{q,t}$. In Chapter 5, the (q, t)-Gaussian family will be tied to an extended non-commutative Central Limit Theorem and, in particular, will be identified with the limit of certain normalized sums of random matrix models. Meanwhile, note that the joint moments of the (q, t)-Gaussian elements come in a form that is the (q, t)-analogue of the familiar Wick formula (e.g. [NS06]).

Definition 10. *For $h \in \mathscr{H}$, let the (q, t)-*Gaussian *element $s_{q,t}(h) \in \mathscr{G}_{q,t}$ be given by $s_{q,t}(h) := a(h) + a(h)^*$.*

Lemma 13 (The q, t-Wick Formula). *For all $n \in \mathbb{N}$ and $h_1, \ldots, h_{2n} \in \mathscr{H}$,*

$$\begin{aligned}
\varphi_{q,t}(s_{q,t}(h_1) \ldots s_{q,t}(h_{2n-1})) &= 0 \\
\varphi_{q,t}(s_{q,t}(h_1) \ldots s_{q,t}(h_{2n})) &= \sum_{\mathscr{V}=\{(w_1,z_1),\ldots(w_n,z_n)\}\in\mathscr{P}_2(2n)} q^{cross(\mathscr{V})}\, t^{nest(\mathscr{V})} \prod_{i=1}^{n} \langle h_{w_i}, h_{z_i} \rangle_{\mathscr{H}}.
\end{aligned}$$

Proof. It suffices to note that

$$\varphi_{q,t}(s_{q,t}(h_1) \ldots s_{q,t}(h_n)) = \sum_{\epsilon(1),\ldots,\epsilon(n)\in\{1,*\}} \varphi_{q,t}(a(h_1)^{\epsilon(1)} \ldots a(h_{2n})^{\epsilon(2n)}),$$

from which the result follows directly by Lemma 11 and (4.29). $\square$

Focusing on a single element $s_{q,t}(h)$, the corresponding moments can be expressed in a particularly elegant form. In particular, in light of Lemma 12, the moment $\varphi_{q,t}(s_{q,t}(h)\dots s_{q,t}(h))$ is expressible via generating functions of weighted Dyck paths, which one can then interpret as a continued fraction via a well-known correspondence [Fla80]. Note that the continued-fraction formulation in the following lemma already features in [KZ06], in a more general combinatorial context.

Lemma 14. *The moments of the* (q,t)*-Gaussian element* $s_{q,t}(h)$ *are*

$$\begin{aligned}
\varphi_{q,t}(s_{q,t}(h)^{2n-1}) &= 0\\
\varphi_{q,t}(s_{q,t}(h)^{2n}) &= \|h\|_{\mathscr{H}}^{2n} \sum_{\mathscr{V}\in\mathscr{P}_2(2n)} q^{cross(\mathscr{V})}\, t^{nest(\mathscr{V})} = \|h\|_{\mathscr{H}}^{2n}\,[z^n]\cfrac{1}{1-\cfrac{[1]_{q,t}z}{1-\cfrac{[2]_{q,t}z}{1-\cfrac{[3]_{q,t}z}{\cdots}}}},
\end{aligned}$$

where $[z^n](\cdot)$ *denotes the coefficient of the* z^n *term in the power series expansion of* $(\cdot)$.

Proof.

Expanding the moment and applying Lemma 12,

$$\begin{aligned}
\varphi_{q,t}(s_{q,t}(h)^{2n}) &= \sum_{\epsilon(1),\dots,\epsilon(n)\in\{1,*\}} \mathrm{wt}(\epsilon(2n),\dots,\epsilon(1);h,\dots,h)\\
&= \sum_{(\epsilon(k),\dots,\epsilon(1))\in\psi^{-1}(D_n)} \widetilde{\mathrm{wt}}(\epsilon(k),\dots,\epsilon(1);h).
\end{aligned}$$

The continued fraction expansion of $\sum_{n\geq 0}\varphi_{q,t}(s_{q,t}(h)^{2n})z^n$ is then obtained by the classic encoding of weighted Dyck paths (see [Fla80], also in a more relevant context [Bia97b] and [KZ06]). □

By the previous lemma, the first four even moments of $s_{q,t}(h)$ are thus

$$\begin{aligned}
\varphi_{q,t}(s_{q,t}(h)^0) &= 1 \\
\varphi_{q,t}(s_{q,t}(h)^2) &= \|h\|^2_{\mathscr{H}} \\
\varphi_{q,t}(s_{q,t}(h)^4) &= \|h\|^4_{\mathscr{H}}(1+q+t) \\
\varphi_{q,t}(s_{q,t}(h)^6) &= \|h\|^6_{\mathscr{H}}(1+2q+2t+2qt+q^2+t^2+2q^2t+2qt^2+q^3+t^3),
\end{aligned}$$

which may be verified by enumerating all chord-crossing diagrams with, respectively, between 0 and 3 chords and counting the corresponding crossings and nestings.

Remark 6 For $\|e\|_{\mathscr{H}} = 1$, $\varphi_{q,t}(s_{q,t}(e)^{2n})$ is equal to the generating function of crossings and nestings in $\mathscr{P}_2(2n)$, that is,

$$\varphi_{q,t}(s_{q,t}(e)^{2n}) = \sum_{\mathscr{V}\in\mathscr{P}_2(2n)} q^{\mathrm{cross}(\mathscr{V})}\, t^{\mathrm{nest}(\mathscr{V})},$$

whose continued-fraction encoding, as previously noted, already explicitly features in [KZ06].

Since $[n]_{q,t} = [n]_{t,q}$, the continued-fraction expansion shows that the joint generating function is in fact symmetric in the two variables, that is,

$$\sum_{\mathscr{V}\in\mathscr{P}_2(2n)} q^{\mathrm{cross}(\mathscr{V})}\, t^{\mathrm{nest}(\mathscr{V})} = \sum_{\mathscr{V}\in\mathscr{P}_2(2n)} t^{\mathrm{cross}(\mathscr{V})}\, q^{\mathrm{nest}(\mathscr{V})}.$$

Given the fundamental constraint $|q| < t$, necessary for the positivity of the sesquilinear form $\langle , \rangle_{q,t}$, this symmetry is surprising. Indeed, let

$$T_{q,t}(n) := \begin{cases} \sum_{\mathscr{V}\in\mathscr{P}_2(n)} q^{\mathrm{cross}(\mathscr{V})}\, t^{\mathrm{nest}(\mathscr{V})} & n \text{ even} \\ 0 & n \text{ odd} \end{cases}$$

and note that associating $T_{q,t}(n)$ with the moments of a self-adjoint element (viz. $s_{q,t}(e)$) in a C^* probability space ensures the positive-definiteness of $(T_{q,t}(n))_{n\in\mathbb{N}}$ (as a number sequence) in the range $|q| < t$ [Akh65]. The above symmetry then yields the positive-definiteness of $T_{t,q}(n)$ for $|q| < t$, which, in turn, guarantees the existence of a real measure whose moments are given by $T_{t,q}(n)$ [Akh65]. This observation naturally beckons the construction of spaces that would give rise to such "reflected" Gaussian algebras. •

In the combinatorial theory pioneered by Flajolet and Viennot [Fla80, Vie85], the continued fraction $\frac{1|}{|1}-\frac{\lambda_1 z^2|}{|1}-\frac{\lambda_2 z^2|}{|1}-\frac{\lambda_3 z^2|}{|1}-\ldots$ and the orthogonal polynomial sequence $y_0(z)=1$, $y_1(z)=z$, $zy_n(z)=y_{n+1}(z)+\lambda_n y_{n-1}(z)$ are both encodings of the same, familiar combinatorial object – namely, they both encode the collection of Dyck paths with NE steps carrying unit weight and each SE step of height $n\mapsto n-1$ carrying weight λ_n. Consider therefore the following (further) deformation of the classical and quantum Hermite orthogonal polynomials.

Definition 11. *The (q,t)-Hermite orthogonal polynomial sequence $\{H_n(z;q,t)\}_{n\geq 0}$ is determined by the following three-term recurrence:*

$$zH_n(z;q,t)=H_{n+1}(z;q,t)+[n]_{q,t}H_{n-1}(z;q,t),$$

with $H_0(z;q,t)=1,\ H_1(z;q,t)=z$.

It now follows from either the classical or the combinatorial theory (e.g. [Akh65] or [Vie85]) that the moments of the orthogonalizing linear functional for the (q,t)-Hermite polynomial sequence are those of $s_{q,t}(e)$. In the parameter range for which $s_{q,t}$ is a bounded operator (cf. Lemma 9), the corresponding measure is compactly supported and is therefore uniquely determined by this moment sequence. Thus:

Lemma 15. *The distribution of the (q,t)-Gaussian element $s_{q,t}(e)\in\mathscr{B}(\mathscr{F}_{q,t})$, where $\|e\|_{\mathscr{H}}=1$, is the unique real measure that orthogonalizes the (q,t)-Hermite orthogonal polynomial sequence.*

Remark 7 For $t=1$, the (q,t)-Hermite orthogonal polynomials are a rescaling of the continuous q-Hermite polynomials of Rogers, belonging to the broader Askey-Wilson scheme of q-hypergeometric orthogonal polynomials (e.g. [KLS10]). Considering families built around the more general hypergeometric functions, the

(q, t)-Hermite sequence is readily seen to belong to the *octabasic Laguerre family* introduced by Simion and Stanton [SS96]. In physics, the (q, t)-Hermite appear in the setting of the Chakrabarti-Jagannathan oscillator algebra [CJ91] (see e.g. recurrence (15) in [IB93]).

The continuous q-Hermite have been extensively studied in the past, thereby yielding explicit expressions for the q-Gaussian density [BKS97]. Analogous expressions for the (q, t)-Gaussian measure do not appear to be available at present, beyond the $0 = q < t \leq 1$ case (cf. Lemma 19 of Section 4.5). •

4.4.3 The $(\Gamma_{q,t}, \varphi_{q,t})$ $*$-probability space

While the pair $(\mathscr{G}_{q,t}, \varphi_{q,t})$ provides a natural setting in which one can discuss the mixed moments of the creation operators and field operators, it is not an especially well-behaved non-commutative probability space. In particular, the functional $\varphi_{q,t}$ is neither faithful nor tracial, as $\varphi_{q,t}(a(h)^*a(h)) = 0 \neq \varphi_{q,t}(a(h)a(h)^*) = \|f\|_{\mathscr{H}}$. As in the full Fock space and, more generally, the q-Fock space, one may wish instead to focus on a subalgebra of non-commutative random variables generated by the (q, t)-Gaussian elements of Definition 10. In particular, consider the unital $*$-algebra generated by $\{s_{q,t}(h) \mid h \in \mathscr{H}\}$ and let $\Gamma_{q,t}$ denote its weak closure in $\mathscr{B}(\mathscr{F}_{q,t})$. In other words, $\Gamma_{q,t}$ is the (q, t)-Gaussian von Neumann algebra. While the vacuum expectation state on the corresponding algebra on the q-Fock space, corresponding to the $t = 1$ case in the present setting, is tracial [BS94], the same does not hold in the more general case.

Proposition 4. *For dim($\mathscr{H}$) ≥ 2, $\Gamma_{q,t}$ is tracial if and only if $t = 1$.*

Proof. The forward direction is established in [BS94] in a more general setting. For the converse, it suffices to consider some four vectors $h_1, h_2, h_3, h_4 \in \mathscr{F}_{q,t}$. Using Lemma 13,

$$\begin{aligned}
&\varphi_{q,t}(s_{q,t}(h_1)s_{q,t}(h_2)s_{q,t}(h_3)s_{q,t}(h_4))_{q,t} \\
&= \langle h_1, h_2\rangle_{\mathscr{H}}\langle h_3, h_4\rangle_{\mathscr{H}} + t\langle h_1, h_4\rangle_{\mathscr{H}}\langle h_2, h_3\rangle_{\mathscr{H}} + q\langle h_1, h_3\rangle_{\mathscr{H}}\langle h_2, h_4\rangle_{\mathscr{H}}
\end{aligned}$$

and

$$\begin{aligned}&\varphi_{q,t}(s_{q,t}(h_4)s_{q,t}(h_1)s_{q,t}(h_2)s_{q,t}(h_3))_{q,t}\\&=\langle h_4,h_1\rangle_{\mathscr{H}}\langle h_2,h_3\rangle_{\mathscr{H}}+t\langle h_4,h_3\rangle_{\mathscr{H}}\langle h_1,h_2\rangle_{\mathscr{H}}+q\langle h_4,h_2\rangle_{\mathscr{H}}\langle h_1,h_3\rangle_{\mathscr{H}}.\end{aligned}$$

Since $\mathscr{H}$ is a real Hilbert space, the terms in q are equal in both of the above expressions, but the terms in 1 and t are interchanged. Denoting by $\{e_i\}$ the basis of $\mathscr{H}$ and taking $h_1 = h_2 = e_1$ and $h_3 = h_4 = e_2$, it follows that the two expressions are equal if and only if $t = 1$. $\square$

Remark 8 The fact that the term in q in the previous example remained the same for both $\langle s_{q,t}(h_1)s_{q,t}(h_2)s_{q,t}(h_3)s_{q,t}(h_4)\Omega\rangle_{q,t}$ and $\langle \Omega, s_{q,t}(h_4)s_{q,t}(h_1)s_{q,t}(h_2)s_{q,t}(h_3)\Omega\rangle_{q,t}$ is not coincidental. Indeed, from a combinatorial viewpoint, commuting two products of Gaussian elements is equivalent to rotating (by a fixed number of positions) the chord diagrams corresponding to each of the non-vanishing products of the underlying creation and annihilation operators. The observed equality then follows from the fact that the crossings in a chord diagram are preserved under diagram rotations. The same is however not true of nestings, which is the combinatorial reason for the overall loss of traciality of φ for $t < 1$. •

4.5 Case $0 = q < t$: "t-deformed Free Probability"

The remainder of the paper considers the case $0 = q < t \leq 1$, corresponding to a new single-parameter deformation of the full Boltzmann Fock space of free probability [Voi86, VDN92]. Once again, this deformation will turn out to be particularly natural. In particular, the statistics of the t-deformed semicircular element will be described in an elegant form afforded by the deformed Catalan numbers of Carlitz and Riordan [CR64], the generalized Rogers-Ramanujan continued fraction, and the t-Airy function of Ismail [Ism05]. Moreover, this setting will give rise to a natural counterpart of the result of Wigner [Wig55], as the t-deformed semicircular element will be seen to encode the first-order statistics of correlated Wigner processes. Prior to delving into these properties, we take a moment to show that the von Neumann algebra of bounded linear operators on $\mathscr{F}_{0,t}(\mathscr{H})$ is generated by the annihilation operators $\{a_{0,t}(h)\}_{h\in\mathscr{H}}$.

Lemma 16. *For $q = 0 < t \leq 1$, the von Neumann algebra $\mathscr{W}_{0,t}$ generated by $\{a_{0,t}(h)\}_{h\in\mathscr{H}}$ is $\mathscr{B}(\mathscr{F}_{0,t}(\mathscr{H}))$.*

Proof. The proof follows analogously to the full Boltzmann Fock space setting, namely the $q = 0$, $t = 1$ case. For concreteness, the following sketch adapts the proof of Theorem 5 of [Kem05]. Let $\{e_j\}_{j\in J}$ denote an orthonormal basis for $\mathscr{H}$ and note that $t^N \in \mathscr{W}_{0,t}$, as $a_{0,t}(e_1)a_{0,t}(e_1)^*\Omega = t^0\Omega$ and $a_{0,t}(e_1)a_{0,t}(e_1)^*e_{i_1}\otimes\ldots\otimes e_{i_n} = t^n e_{i_1}\otimes\ldots\otimes e_{i_n}$. Consider now the operator $P_\Omega = t^N - t\sum_{j\in J} a_{0,t}(e_j)^*a_{0,t}(e_j) \in \mathscr{W}_{0,t}$. Since

$$a_{0,t}(e_j)\Omega = 0 \quad \text{and} \quad a_{0,t}(e_j)e_{i_1}\otimes\ldots\otimes e_{i_n} = \begin{cases} 0, & i_1 \neq j \\ t^{n-1}e_{i_2}\otimes\ldots\otimes e_{i_n}, & i_1 = j \end{cases},$$

it follows that $P_\Omega\Omega = \Omega$ and $P_\Omega e_{i_1}\otimes\ldots\otimes e_{i_n} = 0$. Therefore, P_Ω is the projection onto the vacuum. Thus, $\mathscr{W}_{0,t}$ contains the operator $a_{0,t}(e_{i_1})^*\ldots a_{0,t}(e_{i_n})^*P_\Omega\, a_{0,t}(e_{j_1})\ldots a_{0,t}(e_{j_m})$, which is a rank-one operator with image spanned by $e_{i_1}\otimes\ldots\otimes e_{i_n}$ and kernel orthogonal to $e_{j_1}\otimes\ldots\otimes e_{j_m}$. It follows that $\mathscr{W}_{0,t}$ contains all finite-rank operators. Taking closures, $\mathscr{W}_{0,t} \supseteq \mathscr{B}(\mathscr{F}_{0,t}(\mathscr{H}))$ and the result follows. □

4.5.1 The t-semicircular element

In light of its present interpretation in the context of deformed free probability, the *t-semicircular element* is a renaming of the $(0,t)$-Gaussian element $s_{0,t}(h) \in \mathscr{G}_{0,t}$. Specializing accordingly the spectral properties derived in the previous section, the t-semicircular element turns out to encode certain familiar objects in combinatorics and number theory.

Lemma 17. *The moments of the t-semicircular element $s_{0,t}(h)$ are*

$$\begin{aligned} \varphi_{0,t}(s_{0,t}(h)^{2n-1}) &= 0 \\ \varphi_{0,t}(s_{0,t}(h)^{2n}) &= \|h\|_{\mathscr{H}}^{2n} \sum_{\mathscr{V} \in NC_2(2n)} t^{nest(\mathscr{V})} = \|h\|_{\mathscr{H}}^{2n} C_n^{(t)} \end{aligned}$$

where $NC_2(2n)$ denotes the lattice of non-crossing pair-partitions and $C_n^{(t)}$ are referred to as the Carlitz-Riordan t-Catalan (or, rather, q-Catalan[1]) numbers[FH85, CR64], given by the recurrence

$$C_n^{(t)} = \sum_{k=1}^{n} t^{k-1} C_{k-1}^{(t)} C_{n-k}^{(t)}, \tag{4.31}$$

with $C_0^{(t)} = 1$.

Proof. The first two equalities are obtained by Lemma 14, substituting $q = 0$. Next, set $\alpha_n := \sum_{\mathscr{V} \in NC_2(2n)} t^{\mathrm{nest}(\mathscr{V})}$ and note that $\alpha_0 = 1$. For $\pi \in NC_2(2n)$, consider the pair $(1, \beta) \in \pi$ and note that, since π is non-crossing, (1) β is even, and (2) the remaining pairs either belong to the interval $I = \{2, \ldots, \beta - 1\}$ or to $I' = \{\beta + 1, \ldots, n\}$. Thus, given a nesting in π, either both participating pairs belong to I or both belong to I' or one of those pairs is in fact equal to $\{1, \beta\}$. In the latter case, the nesting must be formed by drawing a second pair from I. Summing over all $\pi \in NC_2(2n)$ and conditioning on the choice of $2\beta \in \{1, 2, \ldots, n\}$ recovers the recurrence in (4.31). The corresponding argument is illustrated in Figure 4-9. $\square$

The first few even moments of $s_{0,t}(h)$ are thus $\varphi_{q,t}(s_{q,t}(h)^2) = \|h\|_{\mathscr{H}}^2$, $\varphi_{q,t}(s_{q,t}(h)^4) = \|h\|_{\mathscr{H}}^2(1 + t)$, $\varphi_{q,t}(s_{q,t}(h)^6) = \|h\|_{\mathscr{H}}^2(1 + 2t + t^2 + t^3)$. The deformed Catalan numbers of the above lemma are known to be related to, among other statistics, inversions of Catalan words, Catalan permutations, and the area below lattice paths [FH85].

The generating function of the above moment sequence, particularly its continued-fraction expansion, yields an object of still further interest. Note that for notational

[1] In keeping with the original objects in [CR64], the q-Catalan numbers of Carlitz-Riordan are more frequently written as $q^{-\binom{n}{2}} C_n^{(1/q)}$, i.e. with reversed coefficients compared to the present definition.

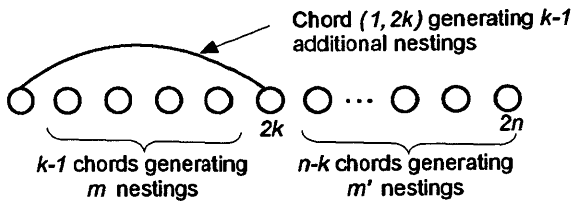

Figure 4-9: Recurrence (4.31): given $\pi \in NC_2(2n)$ with $(1,2k) \in \pi$ such that the chords contained in the interval $I = \{2,\ldots,2k-1\}$ generate m nestings and those contained in $I' = \{2k+1,\ldots,n\}$ generate m' nestings, the total number of nestings in π is then $m+m'+k-1$.

convenience, and without loss of generality, the remainder of the section considers the normalized t-semicirculars $s_{0,t} := s_{0,t}(e)$, where e is a unit vector in $\mathscr{H}$.

Lemma 18. *The moments of the normalized t-semicircular element $s_{0,t}$ are encoded by the generalized Rogers-Ramanujan continued fraction as*

$$\sum_{n\geq 0} \varphi_{0,t}(s_{0,t}(h)^n) z^n = \cfrac{1}{1 - \cfrac{t^0\, z^2}{1 - \cfrac{t^1\, z^2}{1 - \cfrac{t^2\, z^2}{\cdots}}}}$$

The Cauchy transform of the t-semicircular measure $\mu_{0,t}$ associated with $s_{0,t}$ is

$$\int_{\mathbb{R}} \frac{1}{z-\eta}\, d\mu_{0,t}(\eta) = z^{-1} \frac{\sum_{n\geq 0} (-1)^n \frac{t^{n^2}}{(1-t)(1-t^2)\ldots(1-t^n)}\, z^{-2n}}{\sum_{n\geq 0} (-1)^n \frac{t^{n(n-1)}}{(1-t)(1-t^2)\ldots(1-t^n)}\, z^{-2n}}.$$

Proof. The continued fraction follows immediately from Lemma 14 by letting $q = 0$. Since $s_{0,t}(e)$ is bounded, the measure $\mu_{0,t}$ is compactly supported. Therefore, the Cauchy transform of $\mu_{0,t}$ has the power series expansion

$$\int_{\mathbb{R}} \frac{1}{z-\eta}\, d\mu_{0,t}(\eta) = \sum_{n\geq 0} \frac{\varphi_{0,t}(s_{0,t}^n)}{z^{n+1}} = \frac{1}{z} M(1/z),$$

where $M(z) = \sum_{n\geq 0} \varphi_{0,t}(s_{0,t}^n) z^n$ (e.g. [NS06]). It is well known (e.g. [And98]) that the above Rogers-Ramanujan continued-fraction can be written as

$$\frac{\sum_{n\geq 0}(-1)^n \frac{t^{n^2}}{(1-t)(1-t^2)\dots(1-t^n)} z^{2n}}{\sum_{n\geq 0}(-1)^n \frac{t^{n(n-1)}}{(1-t)(1-t^2)\dots(1-t^n)} z^{2n}}.$$

Recalling that the continued fraction encodes $M(z)$ and performing the required change of variable yields the desired Cauchy transform. □

In turn, the functions governing the numerator and denominator of the Cauchy-transform of $\mu_{0,t}$ turn out to have an interpretation as single-parameter deformations of the Airy function. Specifically, let the *t-Airy function* [Ism05] be given by

$$A_t(z) = \sum_{n\geq 0} \frac{t^{n^2}}{(1-t)\dots(1-t^n)}(-z)^n. \tag{4.32}$$

Then:

Corollary 1.

$$\int_{\mathbb{R}} \frac{1}{z-\eta}\, d\mu_{0,t}(\eta) = \frac{1}{z}\frac{A_t(1/z^2)}{A_t(1/(z^2 t))}.$$

Remark 9 While the function $A_q(z) = \sum_{n\geq 0} \frac{q^{n^2}}{(1-q)\dots(1-q^n)}(-z)^n$ features at various points in Ramanujan's work, the nomenclature is more recent. It was proposed by Ismail [Ism05] upon discovering the fact that, analogously to the Airy function in the classical case, the function A_q is involved in the large degree Plancherel-Rotach-type asymptotics for the q-polynomials of the Askey scheme. •

Since the moments of the t-semicircular measure are given by the t-deformed Catalan numbers, the $q = 0$ subfamily of the (q,t)-Hermite orthogonal polynomials of Definition 11 can be viewed as a t-deformed version of the Chebyshev II orthogonal polynomials:

Definition 12. *The t-Chebyshev II orthogonal polynomial sequence $\{U_n(z;t)\}_{n\geq 0}$ is determined by the following three-term recurrence:*

$$zU_n(z;t) = U_{n+1}(z;t) + t^{n-1}U_{n-1}(z;t),$$

with $U_0(z;t) = 1$, $U_1(z;t) = z$.

Remark 10 The orthogonal polynomial sequence encoded by the generalized Rogers-Ramanujan continued fraction was considered by Al-Salam and Ismail in [ASI83], and the polynomial $U_n(z;t)$ is the special case of their polynomial $U_n(z;a,b)$ for $a = 0, b = 1$. Similarly to the previous section, it again follows from the classical theory [Akh65] that the t-semicircular measure $\mu_{0,t}$ is the unique positive measure on $\mathbb{R}$ orthogonalizing the t-Chebyshev II orthogonal polynomial sequence. •

The elegant form of the Cauchy transform of the previous lemma provides means of describing this measure via the zeros of the t-Airy function. This was indeed done in [ASI83] and, adapted to the present setting, is formulated as follows.

Lemma 19 (Corollary 4.5 in [ASI83]). *Let* $\{z_j\}_{j\in\mathbb{N}}$ *denote the sequence of zeros of the rescaled t-Airy function* $A_t(z/t)$. *The measure* $\mu_{0,t}$ *is a discrete probability measure with atoms at*

$$\pm 1/\sqrt{z_j}, \quad j \in \mathbb{N}$$

with corresponding mass

$$-\frac{A_t(z_j)}{2\, z_j\, A_t'(z_j/t)},$$

where $A_t'(z) := \frac{d}{dz}A_t(z)$. *The unique accumulation point of* $\mu_{0,t}$ *is the origin.*

4.5.2 First-order statistics of the Wigner process

For the purpose of this section, consider a *Wigner matrix* $W_N = [w_{i,j}] \in \mathcal{M}_N(L_{\infty-}(\mathbb{R},\mathscr{B},\mathbb{P}))$ to be a self-adjoint random matrix with elements $\{w_{i,j}\}_{1\leq i\leq j\leq N}$ forming a jointly independent collection of centered random variables with unit variance and uniformly bounded[2] moments, i.e. for all $i,j \in [N]$, $\mathbb{E}(w_{i,j}) = 0$, $\mathbb{E}(|w_{i,j}|^2) = 1$ and $\mathbb{E}(|w_{i,j}|^n) \leq c_n \in \mathbb{R}$ for all $n \in \mathbb{N}$. The asymptotics of the distribution of W_N remain an object of extensive study, taking root in the work

[2]The uniform bounds substitute for any assumptions on the identical distribution of the matrix entries. For the (weaker) moment assumptions in the case of identically distributed entries, see e.g. Theorem 2.1.21 in [AGZ10].

of Wigner [Wig55] and further evolving over the following decades. In particular, considering an expectation functional $\varphi_N : \mathcal{M}_N(L_{\infty-}(\mathbb{R}, \mathscr{B}, \mathbb{P})) \to \mathbb{R}$ given by $\varphi_N = \frac{1}{N}\mathrm{Tr} \otimes \mathbb{E}$, the following result is known as Wigner's Semicircle Law.

Theorem (e.g. Theorem 2.1.1 and Lemma 2.1.6 in [AGZ10]). *For W_N a Wigner matrix, the empirical spectral measure of $\tilde{W}_N := W_N/N$ converges in probability to the semicircle distribution, with*

$$\begin{aligned} \lim_{N\to\infty} \varphi_N(\tilde{W}_N^{2n-1}) &= 0 \\ \lim_{N\to\infty} \varphi_N(\tilde{W}_N^{2n}) &= \frac{1}{n+1}\binom{2n}{n} \end{aligned}$$

In particular, W_N converges in moments to the familiar semicircular element $s_{0,1}$. It is natural to expect that by introducing some correlation into Wigner's framework, some deformation of the semicircle law, with an analogous refinement of the Catalan numbers, may be achieved. Indeed, this was obtained by Khorunzhy [Kho01] and further developed by Mazza and Piau [MP02] in relation to the following setup. Let a *Wigner process* refer to the sequence of $\{W_{N,\rho}(k)\}_{k\in\mathbb{N}}$ of Wigner matrices satisfying the following conditions:

- The moments of $W_{N,\rho}(k)$ are uniformly bounded in k, i.e. for all $n \in \mathbb{N}$ and all $i, j \in [N]$, $\mathbb{E}(|w_{i,j}(k)|^n) \leq c_n \in \mathbb{R}$, where $w_{i,j}(k)$ corresponds to the $(i,j)^{\text{th}}$ entry of the matrix $W_{N,\rho}(k)$ and c_n does not depend on k.

- The processes $(w_{i,j}(k))_{k\in\mathbb{N}}$ for $i \leq j$ form a triangular array of independent processes.

- Each process $(w_{i,j}(k))_{k\in\mathbb{N}}$ is ρ-correlated, i.e. for some $|\rho| \leq 1$ and any $1 \leq k \leq m$,
$$\mathbb{E}(w_{i,j}(k)w_{i,j}(m)) = \rho^{m-k}. \tag{4.33}$$

Note that for $\rho = 0$, $\{W_{N,0}(\rho)\}$ is a sequence independent, identically distributed Wigner matrices, whereas for $\rho = 1$, the situation reduces to having copies of the same matrix with $W_{N,1}(1) = W_{N,1}(k)$ for all $k \in \mathbb{N}$. Let

$$B_{n,N} := \varphi_N \left(\frac{W_{N,\rho}(1)}{N} \frac{W_{N,\rho}(2)}{N} \cdots \frac{W_{N,\rho}(n)}{N} \right).$$

The convergence in N of the sequence $B_{n,N}$ was previously established in [Kho01, MP02] and the corresponding limits computed explicitly. The surprise is that the limiting moments are, in fact, those of the t-semicircular element.

Lemma 20. *For $n \in \mathbb{N}$ and $|\rho| \leq 1$,*

$$\lim_{N\to\infty} \varphi_N \left(\frac{W_{N,\rho}(1)}{N} \frac{W_{N,\rho}(2)}{N} \cdots \frac{W_{N,\rho}(2n-1)}{N} \right) = 0 \tag{4.34}$$

$$\lim_{N\to\infty} \varphi_N \left(\frac{W_{N,\rho}(1)}{N} \frac{W_{N,\rho}(2)}{N} \cdots \frac{W_{N,\rho}(2n)}{N} \right) = \rho^n \sum_{\mathscr{V} \in NC_2(2n)} \rho^{2\, nest(\mathscr{V})}. \tag{4.35}$$

In particular, for $\rho \in (0, 1]$,

$$\lim_{N\to\infty} \varphi_N \left(\frac{W_{N,\rho}(1)}{N} \frac{W_{N,\rho}(2)}{N} \cdots \frac{W_{N,\rho}(n)}{N} \right) = \varphi_{0,t}(s_{0,t}(h)^n),$$

for $t = \rho^2$ and $\|h\|_{\mathscr{H}} = \sqrt{\rho}$.

Proof. Expressions (4.34) and (4.35) can be recovered from [Kho01, MP02] (cf. Theorem 1 [MP02]) via Lemma 17. The following self-contained sketch is included for completeness. First note that the general form of both expressions follows analogously to Wigner's proof of the Semicircle Law. In particular, unrolling the normalized trace φ_N,

$$\varphi_N(W_{N,\rho}(1)W_{N,\rho}(2)\ldots W_{N,\rho}(n)) = \frac{1}{N} \sum_{i_1,\ldots,i_n \in [N]} \mathbb{E}(w_{i_1,i_2}(1) w_{i_2,i_3}(2) \ldots w_{i_n,i_1}(n)). \tag{4.36}$$

For a fixed choice of $i_1, \ldots, i_n \in [N]$, the typical argument then proceeds by considering index pairs $\{i_j, i_{j+1}\}$ which repeat and viewing the corresponding pattern as a partition of $[n]$. Since the individual elements $w_{i,j}(k)$ are centered, any partition containing a singleton, i.e. a block formed by a single element, does not contribute to the sum. Next, by counting all the choices of indices $i_1, \ldots, i_n \in [N]$ corresponding to a given partition and taking into account the normalization factors, it can be shown that only the non-crossing pair partitions contribute in the limit. (Note that the counting argument is warranted by the fact that there are uniform bounds, in the time variable n, on the higher moments.) This yields (4.34) and the general form of (4.35).

Considering the left-hand side of (4.36), note that for any index choice $i_1, \ldots, i_n \in [N]$ such that the repetitions in $\{i_1, i_2\}, \{i_2, i_3\}, \ldots, \{i_n, i_1\}$ encode a pair-partition, the expectation $E(w_{i_1,i_2}(1)w_{i_2,i_3}(2) \ldots w_{i_n,i_1}(n))$ factors into second moments. Furthermore, recalling that for all i, j, $\mathbb{E}(w_{i,j}(k)w_{j,i}(m)) = \rho^{m-k}$, one obtains that

$$\lim_{N\to\infty} \varphi_N \left(\frac{W_{N,\rho}(1)}{N} \frac{W_{N,\rho}(2)}{N} \cdots \frac{W_{N,\rho}(2n)}{N} \right) = \sum_{\mathscr{V} \in NC_2(2n)} \rho^{b_1 - a_1} \ldots \rho^{b_n - a_n},$$

where $\mathscr{V} = \{(a_1, b_1), \ldots, (a_n, b_n)\}$ with $1 = a_1 < \ldots < a_n$ and $a_i < b_i$.

It remains to rewrite the above sum in terms of nestings. But, that $\mathscr{V}$ is non-crossing implies that for any $i \in [n]$, $\rho^{b_i - a_i} = \rho^{1+2\times\mathrm{nest}(a_i,b_i;\mathscr{V})}$, where $\mathrm{nest}(a_i, b_i; \mathscr{V})$ denotes the number of nestings that include the pair (a_i, b_i), i.e. $\mathrm{nest}(a_i, b_i; \mathscr{V}) := |\{j \in [n] \mid a_i < a_j < b_j < b_i\}|$. Thus, for any $\mathscr{V} \in NC_2(2n)$, it follows that $\rho^{b_1-a_1} \ldots \rho^{b_n-a_n} = \rho^{n+2\,\mathrm{nest}(\mathscr{V})}$. This yields (4.35) and completes the sketch. □

Remark 11 Taking a process formed from a copy of the same matrix, viz. $W_{N,1}(n) = W_N$ a.e. for all $n \in \mathbb{N}$, yields $\rho = 1 = t$ and recovers the result of Wigner. In particular, denoting $\tilde{W}_N = W_N/N$, the t-Catalan numbers become the (usual) Catalan numbers and

$$\lim_{N\to\infty} \varphi_N \left(\frac{W_{N,1}(1)}{N} \frac{W_{N,1}(2)}{N} \cdots \frac{W_{N,1}(n)}{N} \right) = \lim_{N\to\infty} \varphi_N(\tilde{W}_N{}^n) = \varphi_{0,1}(s_{0,1}{}^n),$$

i.e. $\{\lim_{N\to\infty} \varphi_N(\tilde{W}_N{}^n)\}_{n\in\mathbb{N}}$ is the moment sequence of the semicircular element in free probability. However, for $\rho \in (0, 1]$, it is not obvious that $\{\lim_{N\to\infty} \varphi_N(\tilde{W}_{N,\rho}(1)\,\tilde{W}_{N,\rho}(2) \ldots \tilde{W}_{N,\rho}(n))\}_{n\in\mathbb{N}}$ should still form a moment sequence. Yet, this is indeed the case, and the moments turn out to be those of a t-semicircular element with $t = \rho^2$. A deeper principle underlying this fact is presently unclear. Nevertheless, given their ties to an array of fascinating algebraic and combinatorial objects, their appearance in relation to limits of Wigner processes, and their fundamental role in the generalized non-commutative Central Limit Theorem of Chapter 5, the (q, t)-Gaussians may harbor an additional potential for capturing a broader range of behaviors. $\bullet$

Chapter 5

Two-parameter Non-commutative Central Limit Theorem

In [Spe92], Speicher showed a non-commutative version of the classical Central Limit Theorem (CLT) for mixtures of commuting and anti-commuting elements. Speicher's CLT concerns a sequence of elements $b_1, b_2 \ldots \in \mathcal{A}$ whose terms pair-wise satisfy the deformed commutation relation $b_i b_j = s(j,i) b_j b_i$ with $s(j,i) \in \{-1, 1\}$. It is not a priori clear that the partial sums

$$S_N := \frac{b_1 + \ldots + b_N}{\sqrt{N}} \tag{5.1}$$

should converge in some reasonable sense, nor that the limit should turn out to be a natural refinement of the Wick formula for classical Gaussians, but that indeed turns out to be the case. The following theorem is the "almost sure" version of the Central Limit Theorem of Speicher, presented as an amalgamation of Theorem 1 of [Spe92] and Lemma 1 of [Spe92]. Throughout this paper, $\mathscr{P}_2(2n)$ will denote the collection of pair-partitions of $[2n]$, with each $\mathscr{V} \in \mathscr{P}_2(2n)$ uniquely written as $\mathscr{V} = \{(w_1, z_1), \ldots, (w_n, z_n)\}$ for $w_1 < \ldots < w_n$ and $w_i < z_i$ ($i = 1, \ldots, n$). For further prerequisite definitions, the reader is referred to Section 4.2 of the previous chapter.

Condition 1. *Given a $*$-algebra $\mathscr{A}$ and a state $\varphi : \mathscr{A} \to \mathbb{C}$, consider a sequence $\{b_i\}_{i \in \mathbb{N}}$ of elements of $\mathscr{A}$ satisfying the following:*

i. for all $i \in \mathbb{N}$, $\varphi(b_i) = \varphi(b_i^*) = 0$;

ii. for all for all $i, j \in \mathbb{N}$ with $i < j$ and $\epsilon, \epsilon' \in \{1, *\}$, $\varphi(b_i^{\epsilon} b_i^{\epsilon'}) = \varphi(b_j^{\epsilon} b_j^{\epsilon'})$;

iii. for all $n \in \mathbb{N}$ and all $j(1), \dots, j(n) \in \mathbb{N}$, $\epsilon(1), \dots, \epsilon(n) \in \{1, *\}$, the corresponding mixed moment is uniformly bounded, viz. $|\varphi(\prod_{i=1}^{n} b_{j(i)}^{\epsilon(i)})| \leq \alpha_n$ for some non-negative real α_n;

iv. φ factors over the naturally ordered products in $\{b_i\}_{i \in \mathbb{N}}$, in the sense of Definition 13.

Assume that for all $i \neq j$ and all $\epsilon, \epsilon' \in \{1, *\}$, b_i^{ϵ} and $b_j^{\epsilon'}$ satisfy the commutation relation

$$b_i^{\epsilon} b_j^{\epsilon'} \ = \ s(j,i)\, b_j^{\epsilon'} b_i^{\epsilon}, \qquad s(j,i) \in \{-1, 1\}. \tag{5.2}$$

Theorem 9 (Non-commutative CLT [Spe92]). *Consider a non-commutative probability space $(\mathcal{A}, \varphi)$ and a sequence of elements $\{b_i\}_{i \in \mathbb{N}}$ in $\mathcal{A}$ satisfying Condition 1. Fixing $q \in [-1, 1]$, let the commutation signs $\{s(i,j)\}_{1 \leq i < j}$ be drawn from the collection of independent, identically distributed random variables taking values in $\{-1, 1\}$ with $\mathbb{E}(\mathrm{s}(i,j)) = q$. Then, for almost every sign sequence $\{s(i,j)\}_{1 \leq i < j}$, the following holds: for every $n \in \mathbb{N}$ and all $\epsilon(1), \dots, \epsilon(2n) \in \{1, *\}$,*

$$\lim_{N \to \infty} \varphi(S_N^{\epsilon(1)} \dots S_N^{\epsilon(2n-1)}) = 0, \tag{5.3}$$

$$\lim_{N \to \infty} \varphi(S_N^{\epsilon(1)} \dots S_N^{\epsilon(2n)}) = \sum_{\mathscr{V} \in \mathscr{P}_2(2n)} q^{cross(\mathscr{V})} \prod_{i=1}^{n} \varphi(b^{\epsilon(w_i)} b^{\epsilon(z_i)}), \tag{5.4}$$

with $S_N \in \mathcal{A}$ as given in (5.1), $\mathscr{V} = \{(w_1, z_1), \dots, (w_n, z_n)\}$, and where $cross(\mathscr{V})$ denotes the number of crossings in $\mathscr{V}$ (cf. Definition 7).

The moment expressions in (5.3) and (5.4) may seem familiar. Indeed, this stochastic setting with the average value of the commutation coefficient set to q turns out to be, from the point of view of limiting distributions, equivalent to the setting of bounded linear operators on the the *q-Fock space* $\mathscr{F}_q(\mathscr{H})$ of Bożejko and

Speicher [BS91]. Specifically, given a real, separable Hilbert space $\mathscr{H}$ and two elements $f, g \in \mathscr{H}$, the *creation* and *annihilation* operators on $\mathscr{F}_q(\mathscr{H})$, $a_q(f)^*$ and $a_q(g)$ respectively, satisfy the *q-commutation relation*:

$$a_q(f)a_q(g)^* - q\, a_q(g)^* a_q(f) = \langle f, g\rangle_{\mathscr{H}} 1. \tag{5.5}$$

The mixed moments with respect to the *vacuum expectation state* φ_q of these operators are given by a Wick-type formula which, compared against (5.3) and (5.4), yields that for a unit vector e in $\mathscr{H}$, $\lim_{N\to\infty} \varphi(S_N^{\epsilon(1)} \ldots S_N^{\epsilon(n)}) = \varphi_q(a_q(e)^{\epsilon(1)} \ldots a_q(e)^{\epsilon(n)})$ for all $n \in \mathbb{N}$ and $\epsilon(1), \ldots, \epsilon(2n) \in \{1, *\}$. As described in [Spe92], Theorem 9 can be used to provide a general asymptotic model for operators realizing the relation (5.5), thus providing non-constructive means of settling the conjecture in [FB70].

Finally, any sequence $\{b_i\}_{i\in[n]}$ satisfying Condition 1 has a $*$-representation on $\mathscr{A}_n := \mathcal{M}_2(\mathbb{R})^{\otimes n}$, where $\mathcal{M}_2(\mathbb{R})$ denotes the algebra of 2×2 real matrices. Matricial models for operators satisfying the canonical anti-commutation relation, i.e. the fermionic case corresponding to $q = -1$ in (5.5), are well known and are provided by the so-called Jordan-Wigner transform (see e.g. [CL93] for its appearance in a closely-related context). By extending the transform to the setting where there are both commuting and anti-commuting elements and by applying Theorem 9 , Biane [Bia97a] obtained a random matrix model for operators satisfying the q-commutation relation (5.5). By replacing 2×2 matrices with 4×4 block-diagonal matrices, Kemp [Kem05] similarly obtained models for the corresponding complex family $(a(f) + ia(g))/\sqrt{2}$. To describe the extended Jordan-Wigner model, we make the identification $\mathscr{A}_n \cong \mathcal{M}_{2^n}(\mathbb{R})$ and let the $*$ operation be the conjugate transpose on $\mathcal{M}_{2^n}(\mathbb{R})$. Furthermore, let $\varphi_n : \mathcal{M}_{2^n}(\mathbb{R}) \to \mathbb{C}$ be the positive map $a \mapsto \langle ae_0, e_0\rangle_n$, where $\langle,\rangle_n$ is the usual inner product on $\mathbb{R}^n$ and $e_0 = (1, 0, \ldots, 0)$ an element of the standard basis.

Lemma 21 (Extended Jordan-Wigner Transform [Bia97a]). *Fix $q \in [-1, 1]$ and consider a sequence of commutation coefficients $\{s(i, j)\}_{i\leq j}$ drawn from $\{-1, 1\}$. Con-*

sider the 2×2 *matrices* $\{\sigma_x\}_{x\in\mathbb{R}}, \gamma$ *given as*

$$\sigma_x = \begin{bmatrix} 1 & 0 \\ 0 & x \end{bmatrix}, \qquad \gamma = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

and, for $i = 1, \dots, n$, *let the element* $b_{n,i} \in \mathcal{M}_2(\mathbb{C})^{\otimes n}$ *be given by*

$$b_{n,i} = \sigma_{s(1,i)} \otimes \sigma_{s(2,i)} \otimes \dots \otimes \sigma_{s(i-1,i)} \otimes \gamma \otimes \underbrace{\sigma_1 \otimes \dots \otimes \sigma_1}_{=\sigma_1^{\otimes(n-i)}}. \tag{5.6}$$

Then, for every $n \in \mathbb{N}$, *the non-commutative probablity space* $(\mathcal{A}_n, \varphi_n)$ *and the elements* $b_{n,1}, b_{n,2}, \dots, b_{n,n} \in \mathcal{A}_n$ *satisfy Condition 1.*

5.1 Main Results

This chapter derives a general form of the Non-commutative Central Limit Theorem of [Spe92]. The setting now concerns a sequence $\{b_i\}_{i\in\mathbb{N}}$ of non-commutative random variables satisfying the commutation relation

$$b_i^{\epsilon} b_j^{\epsilon'} \;=\; \mu_{\epsilon',\epsilon}(j,i)\, b_j^{\epsilon'} b_i^{\epsilon} \qquad \text{with} \;\; \epsilon, \epsilon' \in \{1, *\},\; \mu_{\epsilon',\epsilon}(i,j) \in \mathbb{R}, \tag{5.7}$$

for $i \neq j$. The consistency of the above commutation relation is ensured by requiring that for all $i < j$ and $\epsilon, \epsilon' \in \{1, *\}$

$$\mu_{1,1}(i,j) = \frac{1}{\mu_{*,*}(i,j)}, \qquad\qquad \mu_{1,*}(i,j) = \frac{1}{\mu_{*,1}(i,j)}, \tag{5.8}$$

$$\mu_{*,1}(i,j) = t\, \mu_{*,*}(i,j), \qquad\qquad \mu_{\epsilon',\epsilon}(j,i) = \frac{1}{\mu_{\epsilon,\epsilon'}(i,j)}, \tag{5.9}$$

where $t > 0$ is a fixed parameter that will appear explicitly in the limits of interest. The reader is referred to the beginning of Section 5.2 and the Remark 14 of Section 5.3 for a discussion of the relations (5.8)–(5.9). The conditions underlying the extended non-commutative CLT are now the following.

Condition 2. *Given a* $*$*-algebra* $\mathscr{A}$ *and a state* $\varphi : \mathscr{A} \to \mathbb{C}$, *consider a sequence*

$\{b_i\}_{i\in\mathbb{N}}$ *of elements of* $\mathscr{A}$ *satisfying the following:*

i. *for all* $i \in \mathbb{N}$, $\varphi(b_i) = \varphi(b_i^*) = \varphi(b_i b_i) = \varphi(b_i^* b_i^*) = \varphi(b_i^* b_i) = 0$;

ii. *for all* $i, j \in \mathbb{N}$ *and* $\epsilon, \epsilon' \in \{1, *\}$, $\varphi(b_i^{\epsilon} b_i^{\epsilon'}) = \varphi(b_j^{\epsilon} b_j^{\epsilon'})$;

iii. *for all* $n \in \mathbb{N}$ *and all* $j(1), \ldots, j(n) \in \mathbb{N}$, $\epsilon(1), \ldots, \epsilon(n) \in \{1, *\}$, *the corresponding mixed moment is uniformly bounded, i.e.* $|\varphi(\prod_{i=1}^{n} b_{j(i)}^{\epsilon(i)})| \leq \alpha_n$ *for some non-negative real* α_n;

iv. φ *factors over the naturally ordered products in* $\{b_i\}_{i\in\mathbb{N}}$, *in the sense of Definition 13.*

Assume that for all $i \neq j$ *and all* $\epsilon, \epsilon' \in \{1, *\}$, $b_i^{\epsilon(1)}$ *and* $b_j^{\epsilon(2)}$ *satisfy the commutation relation*

$$b_i^{\epsilon} b_j^{\epsilon'} \;=\; \mu_{\epsilon',\epsilon}(j,i)\, b_j^{\epsilon'} b_i^{\epsilon}, \qquad \mu_{\epsilon',\epsilon}(j,i) \in \mathbb{R}\,. \tag{5.10}$$

Theorem 10 (Extended Non-commutative CLT). *Consider a noncommutative probability space* $(\mathcal{A}, \varphi)$ *and a sequence of elements* $\{b_i\}_{i\in\mathbb{N}}$ *in* $\mathcal{A}$ *satisfying Condition 2. Fix* $q \in \mathbb{R}$, $t > 0$ *and let* $\{\mu(i,j)\}_{1\leq i<j}$ *be drawn from a collection of independent, identically distributed, non-vanishing random variables, with*

$$\mathbb{E}(\mu(i,j)) = qt^{-1} \in \mathbb{R}, \qquad \mathbb{E}(\mu(i,j)^2) = 1. \tag{5.11}$$

Letting $\mu_{*,*}(i,j) = \mu(i,j)$ *for* $1 \leq i < j$, *populate the remaining* $\mu_{\epsilon,\epsilon'}(i,j)$, *for* $\epsilon, \epsilon' \in \{1, *\}$ *and* $i \neq j$ $(i, j \in \mathbb{N})$, *by* (5.8) *and* (5.9).

Then, for almost every sequence $\{\mu(i,j)\}_{i\leq j}$, *the following holds: for every* $n \in \mathbb{N}$ *and all* $\epsilon(1), \ldots, \epsilon(2n) \in \{1, *\}$,

$$\lim_{N\to\infty} \varphi(S_N^{\epsilon(1)} \ldots S_N^{\epsilon(2n-1)}) = 0, \tag{5.12}$$

$$\lim_{N\to\infty} \varphi(S_N^{\epsilon(1)} \ldots S_N^{\epsilon(2n)}) = \sum_{\mathcal{V}\in\mathscr{P}_2(2n)} q^{cross(\mathcal{V})} t^{nest(\mathcal{V})} \prod_{i=1}^{n} \varphi(b^{\epsilon(w_i)} b^{\epsilon(z_i)}) \tag{5.13}$$

with $S_N \in \mathcal{A}$ *as given in* (5.1), $\mathscr{V} = \{(w_1, z_1), \ldots, (w_n, z_n)\}$, *and where cross*$(\mathscr{V})$ *denotes the number of crossings in* $\mathscr{V}$ *and nest*$(\mathscr{V})$ *the number of nestings in* $\mathscr{V}$ *(cf. Definition 7).*

The generalized commutation structure of Theorem 10, compared to Theorem 9, generates a second combinatorial statistic in the limiting moments – that of *nestings* in pair partitions. This refinement also extends to the Fock-space level, with the above limits realized as moments of creation and annihilation operators on the (q,t)-*Fock space* introduced in the previous chapter. Compared to (5.5), the operators of interest now satisfy the commutation relation

$$a_{q,t}(f)a_{q,t}(g)^* - q\, a_{q,t}(g)^* a_{q,t}(f) = \langle f, g \rangle_{\mathscr{H}} \, t^N, \tag{5.14}$$

where N is the *number operator.* Note that (5.11), together with the fact that $t > 0$, recovers the fundamental constraint that $|q| < t$ in order for the (q,t)-Fock space to be a bona fide Hilbert space.

The extended non-commutative CLT of Theorem 10 requires that $\varphi(b_i b_i) = \varphi(b_i^* b_i^*) = \varphi(b_i^* b_i) = 0$ (cf. Condition 2), not generally needed in Speicher's setting (Condition 1) except in the asymptotic models for the q-commutation relation (5.5). Remark 12 of Section 5.2 and Remark 15 of Section 5.3 discuss the existence of the limits (5.12) and (5.13) and the form they take in the absence of this requirement. Nevertheless, this additional condition is in fact consistent with the natural choice of matrix models – specifically, extending Lemma 21, the following is a generalized Jordan-Wigner construction. The underlying probability space $(\mathcal{A}_n, \varphi_n)$ remains that of the previous section.

Lemma 22 (Two-parameter Jordan-Wigner Transform). *Fix* $q \in \mathbb{R}$, $t > 0$ *and let* $\{\mu_{\epsilon,\epsilon'}(i,j)\}_{i \neq j, \epsilon, \epsilon' \in \{1,*\}}$ *be a sequence of commutation coefficients, i.e. a sequence of non-zero real numbers satisfying* (5.8) *and* (5.9). *Consider the* 2×2 *matrices*

$\{\sigma_x\}_{x\in\mathbb{R}}, \gamma$ *given by*

$$\sigma_x = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{t}\,x \end{bmatrix}, \qquad \gamma = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

For $i = 1, \ldots, n$, *let* $\mu(i,j) := \mu_{*,*}(i,j)$ *and consider the element* $b_{n,i} \in \mathcal{M}_2(\mathbb{R})^{\otimes n}$ *given by*

$$b_{n,i} = \sigma_{\mu(1,i)} \otimes \sigma_{\mu(2,i)} \otimes \ldots \otimes \sigma_{\mu(i-1,i)} \otimes \gamma \otimes \underbrace{\sigma_1 \otimes \ldots \otimes \sigma_1}_{=\sigma_1^{\otimes(n-i)}}. \tag{5.15}$$

Then, for every $n \in \mathbb{N}$, *the non-commutative probablity space* $(\mathcal{A}_n, \varphi_n)$ *and the elements* $b_{n,1}, b_{n,2}, \ldots, b_{n,n} \in \mathcal{A}_n$ *satisfy Condition 2.*

Finally, analogously to [Bia97a], Theorem 10 and Lemma 22 together yield an asymptotic random matrix models for the creation, annihilation, and field operators on the (q,t)-Fock space:

Corollary 2. *Consider a sequence of commutation coefficients drawn according to Theorem 10 and the corresponding matrix construction of Lemma 22. Let*

$$S_{N,k} := \frac{1}{\sqrt{N}} \sum_{i=N(k-1)+1}^{Nk} b_{Nk,i}. \tag{5.16}$$

Then, for any choice of $k, i(1), \ldots, i(k) \in \mathbb{N}$, $\epsilon(1), \ldots, \epsilon(k) \in \{1, *\}$,

$$\lim_{N\to\infty} \varphi_{Nk}(S_{N,i(1)}^{\epsilon(1)} \cdots S_{N,i(k)}^{\epsilon(k)}) = \varphi_{q,t}(a_{q,t}(e_1)^{\epsilon(1)} \ldots a_{q,t}(e_k)^{\epsilon(k)}) \tag{5.17}$$

and

$$\lim_{N\to\infty} \varphi_{Nk}((S_{N,i(1)} + S_{N,i(1)}^*) \cdots (S_{N,i(k)} + S_{N,i(k)}^*)) = \varphi_{q,t}(s_{q,t}(e_1) \ldots s_{q,t}(e_k)) \tag{5.18}$$

where $\varphi_{q,t}$ *is the vacuum expectation state on the* (q,t)*-Fock space* $\mathscr{F}_{q,t}(\mathscr{H})$, *operator* $a_{q,t}(e_i)$ *is the twisted annihilation operator on* $\mathscr{F}_{q,t}(\mathscr{H})$ *associated with the element* e_i *of the orthonormal basis of* $\mathscr{H}$, *and* $s_{q,t}(e_i) := a_{q,t}(e_i) + a_{q,t}(e_i)^*$ *is the corresponding field operator.*

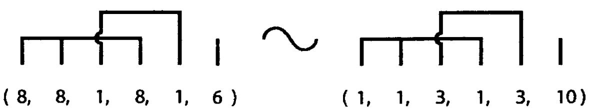

Figure 5-1: Two elements of $[N]^r$ (for $N = 10, r = 6$) that belong to the same equivalence class, where the latter is represented as the corresponding partition $\mathscr{V} \in \mathscr{P}(r)$ given by $\mathscr{V} = \{(1,2,4),(3,5),(6)\}$.

5.1.1 Notation

As previously, denote by $\mathscr{P}(n)$ the collection of partitions of $[n] := \{1, \ldots, n\}$. In this chapter, set partitions will be extensively used to encode equivalence classes of products of random variables, based on the repetition patterns of individual elements. Specifically, any two r-vectors will be declared equivalent if element repetitions occur at same locations in both vectors; i.e. for $(i(1), \ldots, i(r)), (j(1), \ldots, j(r)) \in [N]^r$,

$$(i(1), \ldots, i(r)) \sim (j(1), \ldots, j(r)) \iff \text{for all } 1 \leq k_1 < k_2 \leq r, \quad i(k_1) = i(k_2) \text{ iff } j(k_1) = j(k_2). \tag{5.19}$$

It then immediately follows that the equivalence classes of "$\sim$" can be identified with the set $\mathscr{P}(r)$ of the partitions of $[r]$. An example is shown in Figure 5-1. Note that writing "$(i(1), \ldots, i(r)) \sim \mathscr{V}$" will indicate that $(i(1), \ldots, i(r))$ is in the equivalence class identified with the partition $\mathscr{V} \in \mathscr{P}(r)$.

5.2 Two-parameter Non-commutative Central Limit Theorem

The goal of this section is to extend the "deterministic formulation" of the non-commutative Central Limit Theorem of Speicher [Spe92]. The deterministic result differs from the previously stated Theorem 9 in that the sequence of commutation signs $(s(i,j))_{i,j}$, taking values in $\{-1, 1\}$ and associated with the commutation relations $b_i^{\epsilon} b_j^{\epsilon'} = s(j,i) b_j^{\epsilon'} b_i^{\epsilon}$, is now *fixed*. In [Spe92], an analogous Wick-type formula is nevertheless shown to exist, provided the existence of the following limit:

$$\lim_{N\to\infty} \frac{1}{N^{2n}} \sum_{\substack{i(1),\ldots,i(2n)\in[N] \text{ s.t.} \\ (i(1),\ldots,i(2n))\sim\mathscr{V}}} \prod_{(w_j,w_k,z_j,z_k)\in\text{Cross}(\mathscr{V})} s(i(w_j),i(w_k)) \quad := \lambda_{\mathscr{V}}$$

for each pair partition $\mathscr{V} \in \mathscr{P}(2n)$.

At present, the focus is on a sequence $(b_i)_{i\in\mathbb{N}}$ of non-commutative random variables satisfying a more general type of commutation relations, where for all $i \neq j$ and $\epsilon, \epsilon' \in \{1, *\}$,

$$b_i^{\epsilon} b_j^{\epsilon'} \;=\; \mu_{\epsilon',\epsilon}(j,i)\, b_j^{\epsilon'} b_i^{\epsilon} \qquad \text{for some } \;\mu_{\epsilon',\epsilon}(i,j) \in \mathbb{R}. \tag{5.20}$$

At the outset, the sequence of commutation coefficients $\{\mu_{\epsilon,\epsilon'}(i,j)\}_{i\neq j,\epsilon,\epsilon'\in\{1,*\}}$ must satisfy certain properties. In particular, interchanging the roles of i and j in the commutation relation implies that

$$\mu_{\epsilon,\epsilon'}(i,j) = \frac{1}{\mu_{\epsilon',\epsilon}(j,i)}. \tag{A}$$

Similarly, conjugating (via the $*$ operator) both sides of the commutation relation yields

$$\mu_{*,*}(i,j) = \mu_{1,1}(j,i), \quad \mu_{1,*}(i,j) = \mu_{1,*}(j,i), \quad \mu_{*,1}(i,j) = \mu_{*,1}(j,i).$$

(For example, $b_i b_j = \mu_{1,1}(j,i) b_j b_i$ and therefore $b_j^* b_i^* = (b_i b_j)^* = \mu_{1,1}(j,i) b_i^* b_j^*$, but also $b_j^* b_i^* = \mu_{*,*}(i,j) b_i^* b_j^*$.) Therefore, by (A),

$$\mu_{*,*}(i,j) = \frac{1}{\mu_{1,1}(i,j)}, \qquad \mu_{*,1}(i,j) = \frac{1}{\mu_{1,*}(i,j)}. \tag{B}$$

The second key ingredient in a non-commutative CLT is a moment-factoring assumption. As in [Spe92], the factoring is assumed to follow the underlying partition structure. Drawing on the notation of Section 4.2, viz. the equivalence relation "$\sim$" on the set $[N]^r$ of r-tuples in $[N] := \{1,\ldots,N\}$, the two relevant ways in which the moments may be assumed to factor are defined as follows.

Definition 13. *Consider a sequence $\{b_i\}_{i\in\mathbb{N}}$ of random variables, elements of some non-commutative probability space $(\mathcal{A},\varphi)$. The element $b_{i(1)}^{\epsilon(1)}\dots b_{i(n)}^{\epsilon(n)}$, for $\epsilon(1),\dots,\epsilon(n)\in\{1,*\}$ and $i(1),\dots,i(n)\in\mathbb{N}$, is said to be an* interval-ordered product *if $(i(1),\dots,i(n))\sim\mathscr{V}$ where $\mathscr{V}=\{\{1,\dots,k_1\},\{k_1+1,\dots,k_2\},\dots,\{k_{|\mathscr{V}|-1}+1\dots,k_{|\mathscr{V}|}\}\}$ is an interval partition of $[n]$. The same element is said to be a* naturally ordered product *if, in addition, $i(1)<i(k_1+1)\dots<i(k_{|\mathscr{V}|-1}+1)$.*

The state φ is said to factor over naturally (resp. interval) ordered products *in $\{b_i\}_{i\in\mathbb{N}}$ if*

$$\varphi(b_{i(1)}^{\epsilon(1)}\dots b_{i(n)}^{\epsilon(n)})=\varphi(b_{i(1)}^{\epsilon(1)}\dots b_{i(k_1)}^{\epsilon(k_i)})\dots\varphi(b_{i(k_{|\mathscr{V}|-1}+1)}^{\epsilon(k_{|\mathscr{V}|-1}+1)}\dots b_{i(k_{|\mathscr{V}|})}^{\epsilon(k_{|\mathscr{V}|})})$$

whenever $b_{i(1)}^{\epsilon(1)}\dots b_{i(n)}^{\epsilon(n)}$ is a naturally (resp. interval) ordered product.

The following remark ensures that the commutation relations (5.20) are consistent with the moment factoring assumptions.

Remark 12 In assuming φ factors over naturally ordered products, one must be able to bring a moment $\varphi(b_i^{\epsilon_i}b_i^{\epsilon_i'}b_j^{\epsilon_j}b_j^{\epsilon_j'})$ for $i>j$ into naturally-ordered form. Alternatively, should it be further assumed that φ factors over interval-ordered products of the sequence $\{b_i\}_{i\in\mathbb{N}}$, one must allow that $\varphi(b_i^{\epsilon_i}b_i^{\epsilon_i'}b_j^{\epsilon_j}b_j^{\epsilon_j'})=\varphi(b_j^{\epsilon_j}b_j^{\epsilon_j'}b_i^{\epsilon_i}b_i^{\epsilon_i'})$ for all i,j and $\epsilon,\epsilon'\in\{1,*\}$. When commutation coefficients are constrained to take values in $\{-1,1\}$, it is in fact the case that $b_i^{\epsilon_i}b_i^{\epsilon_i'}$ commutes with $b_j^{\epsilon_j}b_j^{\epsilon_j'}$, and the moment-factoring assumptions are consistent with the commutativity structure. However, this need not be the case for the general setting. In particular,

$$\varphi(b_i^{\epsilon_i}b_i^{\epsilon_i'}b_j^{\epsilon_j}b_j^{\epsilon_j'})=\mu_{\epsilon_i,\epsilon_j'}(j,i)\mu_{\epsilon_i,\epsilon_j}(j,i)\mu_{\epsilon_i',\epsilon_j'}(j,i)\mu_{\epsilon_i',\epsilon_j}(j,i)\,\varphi(b_j^{\epsilon_j}b_j^{\epsilon_j'}b_i^{\epsilon_i}b_i^{\epsilon_i'}).$$

The reader may verify that any sequence of real-valued commutation coefficients for which the above product evaluates to unity regardless of the choice of ϵ,ϵ' must in fact take values in $\{-1,1\}$.

Instead, rather than imposing additional restrictions on the sign sequence, the alternative approach is that of restricting the range of φ when applied to the sequence $\{b_i\}$. In particular, by (A)–(B),

$$\mu_{\epsilon_i,\epsilon'_j}(j,i)\mu_{\epsilon_i,\epsilon_j}(j,i)\mu_{\epsilon'_i,\epsilon'_j}(j,i)\mu_{\epsilon'_i,\epsilon_j}(j,i) = 1$$

whenever $\epsilon \neq \epsilon'$. Thus, by imposing that $\varphi(b_i^* b_i^*) = \varphi(b_i b_i) = 0$ for all $i \in \mathbb{N}$, the assumption on the factoring of naturally-ordered second moments conveniently becomes equivalent to factoring of interval-ordered second moments. Note that factoring an interval-ordered product containing higher moments generally still incurs a product of commutation coefficients. However, as will become apparent shortly, the contribution of such expressions vanishes in the limits of interest. •

The stage is now set for the main result of this section.

Theorem 11 (Extended Non-commutative CLT). *Consider a noncommutative probability space $(\mathcal{A},\varphi)$ and a sequence $\{b_i\}_{i\in\mathbb{N}}$ of elements of $\mathcal{A}$ satisfying Condition 2, with the real-valued commutation coefficients $\{\mu_{\epsilon',\epsilon}(i,j)\}$ satisfying the consistency conditions (A)–(B). For $n \in \mathbb{N}$, fix $\epsilon(1),\ldots,\epsilon(2n) \in \{1,*\}$ and, letting $\mathscr{P}_2(2n)$ denote the collection of pair partitions of $[2n]$, assume that for all $\mathscr{V} = \{(w_1,z_1),\ldots,(w_n,z_n)\} \in \mathscr{P}_2(2n)$ the following limit exists:*

$$\lambda_{\mathscr{V},\epsilon(1),\ldots,\epsilon(2n)} := \lim_{N\to\infty} N^{-n} \sum_{\substack{i(1),\ldots,i(2n)\in[N] \text{ s.t.} \\ (i(1),\ldots,i(2n))\sim\mathscr{V}}} \left(\prod_{\substack{(w_j,w_k,z_j,z_k) \\ \in \mathit{Cross}(\mathscr{V})}} \mu_{\epsilon(z_j),\epsilon(w_k)}(i(z_j),i(w_k)) \right.$$
$$\left. \prod_{\substack{(w_j,w_m,z_m,z_j) \\ \in \mathit{Nest}(\mathscr{V})}} \mu_{\epsilon(z_j),\epsilon(z_m)}(i(z_j),i(z_m))\mu_{\epsilon(z_j),\epsilon(w_m)}(i(z_j),i(w_m)) \right), \tag{5.21}$$

where Cross($\mathscr{V}$) and Nest($\mathscr{V}$) denote, respectively, the sets of crossings and nestings in $\mathscr{V}$ (cf. Definition 7) and where the equivalence relation $\sim$ is given by (5.19).

Then, for every $n \in \mathbb{N}$ and all $\epsilon(1), \ldots, \epsilon(2n) \in \{1, *\}$,

$$\lim_{N\to\infty} \varphi(S_N^{\epsilon(1)} \ldots S_N^{\epsilon(2n-1)}) = 0, \tag{5.22}$$

$$\lim_{N\to\infty} \varphi(S_N^{\epsilon(1)} \ldots S_N^{\epsilon(2n)}) = \sum_{\mathscr{V}\in\mathscr{P}_2(2n)} \lambda_{\mathscr{V},\epsilon(1),\ldots,\epsilon(2n)} \prod_{i=1}^{n} \varphi(b^{\epsilon(w_i)} b^{\epsilon(z_i)}), \tag{5.23}$$

for $S_N \in \mathcal{A}$ as given in (5.1) and with each $\mathscr{V} \in \mathscr{P}_2(2n)$ written as $\mathscr{V} = \{(w_1, z_1), \ldots, (w_n, z_n)\}$ for $w_1 < \ldots < w_n$ and $w_i < z_i$ $(i = 1, \ldots, n)$.

Proof of Theorem 11. The notation and the development follow closely those of [Spe92].

Fix $r \in \mathbb{N}$ and $\epsilon(1), \ldots, \epsilon(r) \in \{1, *\}$ and recall that the focus is the $N \to \infty$ limit of the corresponding mixed moment of S_N. Namely, let

$$M_N := \varphi(S_N^{\epsilon(1)} \ldots S_N^{\epsilon(r)}) = \frac{1}{N^{r/2}} \sum_{i(1),\ldots,i(r)\in[N]} \varphi\left(b_{i(1)}^{\epsilon(1)} \ldots b_{i(r)}^{\epsilon(r)}\right).$$

Making use of the previously-defined equivalence relation, M_N can be rewritten as

$$M_N = \sum_{\mathscr{V}\in\mathscr{P}(r)} \frac{1}{N^{r/2}} \sum_{\substack{i(1),\ldots,i(r)\in[N] \text{ s.t.} \\ (i(1),\ldots,i(r))\sim\mathscr{V}}} \varphi\left(b_{i(1)}^{\epsilon(1)} \ldots b_{i(r)}^{\epsilon(r)}\right) = \sum_{\mathscr{V}\in\mathscr{P}(r)} \frac{1}{N^{r/2}} M_N^{\mathscr{V}},$$

where

$$M_N^{\mathscr{V}} := \sum_{\substack{i(1),\ldots,i(r)\in[N] \text{ s.t.} \\ (i(1),\ldots,i(r))\sim\mathscr{V}}} \varphi\left(b_{i(1)}^{\epsilon(1)} \ldots b_{i(r)}^{\epsilon(r)}\right).$$

Focusing on $M_N^{\mathscr{V}}$, suppose first that $\mathscr{V}$ contains a singleton, i.e. a single-element part $\{k\} \in \mathscr{V}$ for some $k \in [r]$. Via the commutation relation (5.10), $b_{i(1)}^{\epsilon(1)} \ldots b_{i(r)}^{\epsilon(r)}$ can be brought into a naturally ordered form (incurring, in the process, a multiplying factor given by the corresponding product of the commutation coefficients). In turn, by the assumption on the factoring of the naturally ordered products (cf. Definition 13), $\varphi\left(b_{i(1)}^{\epsilon(1)} \ldots b_{i(r)}^{\epsilon(r)}\right)$ factors according to the blocks in $\mathscr{V}$. Since $\varphi(b_k) = \varphi(b_k^*) = 0$, it follows that for all $N \in \mathbb{N}$, $M_N^{\mathscr{V}} = 0$ for all partitions $\mathscr{V}$ containing a singleton block.

Focus next on partitions of $[r]$ containing blocks with two or more elements or, equivalently, partitions $\mathscr{V} \in \mathscr{P}(r)$ with $|\mathscr{V}| \leq \lfloor r/2 \rfloor$, where $|\mathscr{V}|$ denotes the number of blocks in $\mathscr{V}$. Recalling that, by the assumption on the existence of uniform bounds on the moments, we have that for all $\mathscr{V} \in \mathscr{P}(r)$,

$$\left|\varphi\left(b_{i(1)}^{\epsilon(1)} \dots b_{i(r)}^{\epsilon(r)}\right)\right| \leq \alpha_{\mathscr{V}}$$

for some $\alpha_{\mathscr{V}} \in \mathbb{R}$. Thus, for a partition $\mathscr{V}$ with ℓ blocks, summing over all $i(1), \dots, i(r) \in [N]$ with $(i(1), \dots, i(r)) \sim \mathscr{V}$ yields

$$|M_N^{\mathscr{V}}| \leq \binom{N}{\ell} \ell! \, \alpha_{\mathscr{V}},$$

and therefore

$$|M_N| \leq \sum_{\mathscr{V} \in \mathscr{P}(r)} \frac{\binom{N}{\ell} \ell!}{N^{r/2}} \alpha_{\mathscr{V}}.$$

Noting that (1) the above sum is taken over a fixed (finite) index r, (2) that the only N-dependent term in the above expression is the ratio $\binom{N}{\ell}/N^{r/2}$ and (3) that $\binom{N}{\ell}/N^{r/2} \to 0$ as $N \to \infty$ for $\ell < \lceil r/2 \rceil$, it follows that only those partitions $\mathscr{V}$ with $|\mathscr{V}| \geq \lceil r/2 \rceil$ contribute to the $N \to \infty$ limit of M_N. But, since $|\mathscr{V}| \leq \lfloor r/2 \rfloor$, it follows that the only non-vanishing contributions are obtained for r even and partitions with exactly $r/2$ blocks — i.e. pair-partitions, $\mathscr{V} \in \mathscr{P}_2(r)$. Therefore, for r odd,

$$\lim_{N \to \infty} \varphi(S_N^{\epsilon(1)} \dots S_N^{\epsilon(r)}) = 0$$

and, otherwise,

$$\lim_{N \to \infty} |M_N| = \sum_{\mathscr{V} \in \mathscr{P}_2(r)} \lim_{N \to \infty} N^{-\frac{r}{2}} \sum_{\substack{i(1), \dots, i(r) \in [N] \text{ s.t.} \\ (i(1), \dots, i(r)) \sim \mathscr{V}}} \varphi\left(b_{i(1)}^{\epsilon(1)} \dots b_{i(r)}^{\epsilon(r)}\right).$$

Next, fixing $i(1), \dots, i(r) \in [N]$ and recalling that $\mathscr{V}$ is a pair-partition of $[r]$, consider the following algorithm for transforming $b_{i(1)}^{\epsilon(1)} \dots b_{i(r)}^{\epsilon(r)}$, via the commutation relation (5.10), into an interval-ordered product. Starting with $i(1)$ and recalling that $\mathscr{V}$ is a pair-partition of $[r]$, let $1 < k_1 \leq r$ denote the unique index for which $i(k_1) = i(1)$.

Consider element $b_{i(1)}$ to be already in place and commute $b_{i(k_1)}$ with the elements to its left until $b_{i(k_1)}$ is immediately to the right of $b_{i(1)}$, recording all the while the commutation coefficients incurred in each transposition. The next iteration, proceeding in the analogous manner, is carried out on the string of length $r-2$ given by $i(2),\ldots,i(\check{k_1}),\ldots,i(r)$, where $i(\check{k_1})$ indicates that $i(k_1)$ has been suppressed from the string. Continuing in this manner, the algorithm terminates when the remaining string is the empty string. The resulting moment is of the form

$$\varphi\left(b_{i(1)}^{\epsilon(1)}\cdots b_{i(r)}^{\epsilon(r)}\right)=\beta_{i(1),\ldots,i(r)}^{\epsilon(1),\ldots,\epsilon(r)}\ \varphi\left(b_{i(w_1)}^{\epsilon(w_1)}b_{i(z_1)}^{\epsilon(z_1)}\cdots b_{i(w_{r/2})}^{\epsilon(w_{r/2})}b_{i(z_{r/2})}^{\epsilon(z_{r/2})}\right),$$

where $\mathscr{V}=\{(w_1,z_1),\ldots,(w_{r/2},z_{r/2})\}$ with $w_1<\ldots<w_{r/2}$ is the underlying pair-partition and $\beta_{i(1),\ldots,i(r)}^{\epsilon(1),\ldots,\epsilon(r)}$ denotes the product of the commutation coefficients incurred in this transformation. Note that though $i(w_j)=i(z_j)$ for all $j=1,\ldots,r/2$, in general $\epsilon(w_j)\neq\epsilon(z_j)$ and the above expression therefore also (artificially) distinguishes between $i(w_j)$ and $i(z_j)$.

While it need not be the case that $i(w_1)<\ldots<i(w_{r/2})$, and the moment

$$\varphi\left(b_{i(w_1)}^{\epsilon(w_1)}b_{i(z_1)}^{\epsilon(z_1)}\cdots b_{i(w_{r/2})}^{\epsilon(w_{r/2})}b_{i(z_{r/2})}^{\epsilon(z_{r/2})}\right)$$

therefore need not be naturally ordered, φ nevertheless factors over the pairs. Specifically, as $\varphi(b_jb_j)=\varphi(b_j^*b_j^*)=0$, it can be assumed that $\epsilon(w_j)\neq\epsilon(z_j)$ for $j=1,\ldots,r/2$. By Remark 12, it then follows that

$$\varphi\left(b_{i(1)}^{\epsilon(1)}\cdots b_{i(r)}^{\epsilon(r)}\right)=\beta_{i(1),\ldots,i(r)}^{\epsilon(1),\ldots,\epsilon(r)}\ \varphi\left(b_{i(w_1)}^{\epsilon(w_1)}b_{i(z_1)}^{\epsilon(z_1)}\right)\cdots\varphi\left(b_{i(w_{r/2})}^{\epsilon(w_{r/2})}b_{i(z_{r/2})}^{\epsilon(w_{r/2})}\right).$$

Next, $\beta_{i(1),\ldots,i(r)}^{\epsilon(1),\ldots,\epsilon(r)}$ can be expressed combinatorially as follows. Fixing some $j\in[r/2]$ and considering the corresponding pair $(w_j,z_j)\in\mathscr{V}$ (where $w_j<z_j$), note that for every $k\in[r/2]$ for which $w_j<w_k<z_j<z_k$, the above algorithm commutes z_j and w_k. Additionally note that this commutation is performed exactly once, on the j^{th} iteration, as the process does not revisit pairs that were brought into the desired form in one of the previous steps. The corresponding contribution to $\beta_{i(1),\ldots,i(r)}^{\epsilon(1),\ldots,\epsilon(r)}$ is

therefore given by $\mu_{\epsilon(z_j),\epsilon(w_k)}(i(z_j),i(w_k))$. Similarly, for every $m \in [r/2]$ for which $w_j < w_m < z_m < z_j$, the above algorithm commutes z_j and z_m as well as z_j and w_m, and both commutations occur exactly once. The corresponding contribution to $\beta^{\epsilon(1),\ldots,\epsilon(r)}_{i(1),\ldots,i(r)}$ is therefore given by $\mu_{\epsilon(z_j),\epsilon(z_m)}(i(z_j),i(z_m))\mu_{\epsilon(z_j),\epsilon(w_m)}(i(z_j),i(w_m))$. Recall now (cf. Definition 7) that the 4-tuple given by $w_j < w_k < z_j < z_k$ is what is referred to as a *crossing* in $\mathscr{V}$ and encoded by $(w_j,w_k,z_j,z_k) \in \mathrm{Cross}(\mathscr{V})$, whereas the 4-tuple $w_j < w_m < z_m < z_j$ is referred to as a *nesting* in $\mathscr{V}$ and is encoded as $(w_j,w_m,z_m,z_j) \in \mathrm{Nest}(\mathscr{V})$. Finally, realizing that the algorithm performs no other commutations than the two types described, it follows that

$$\beta^{\epsilon(1),\ldots,\epsilon(r)}_{i(1),\ldots,i(r)} = \prod_{\substack{(w_j,w_k,z_j,z_k)\\ \in\mathrm{Cross}(\mathscr{V})}} \mu_{\epsilon(z_j),\epsilon(w_k)}(i(z_j),i(w_k)) \times \prod_{\substack{(w_j,w_m,z_m,z_j)\\ \in\mathrm{Nest}(\mathscr{V})}} \mu_{\epsilon(z_j),\epsilon(z_m)}(i(z_j),i(z_m))\mu_{\epsilon(z_j),\epsilon(w_m)}(i(z_j),i(w_m)).$$

The encoding of $\beta^{\epsilon(1),\ldots,\epsilon(r)}_{i(1),\ldots,i(r)}$ through nestings and crossings of $\mathscr{V}$ is illustrated in Figures 5-2 and 5-3.

Putting it all together,

$$\lim_{N\to\infty}\varphi(S_N^{\epsilon(1)}\ldots S_N^{\epsilon(2n)}) = \sum_{\mathscr{V}\in\mathscr{P}_2(2n)}\lim_{N\to\infty}N^{-\frac{r}{2}}\sum_{\substack{i(1),\ldots,i(r)\in[N]\text{ s.t.}\\ (i(1),\ldots,i(r))\sim\mathscr{V}}}\left(\beta^{\epsilon(1),\ldots,\epsilon(r)}_{i(1),\ldots,i(r)}\varphi\left(b^{\epsilon(w_1)}_{i(w_1)}b^{\epsilon(z_1)}_{i(z_1)}\right)\ldots\varphi\left(b^{\epsilon(w_n)}_{i(w_n)}b^{\epsilon(z_n)}_{i(z_n)}\right)\right) \tag{5.24}$$

By the assumption on the covariances of the b_i's and the existence of the limit in (5.21),

$$\lim_{N\to\infty}\varphi(S_N^{\epsilon(1)}\ldots S_N^{\epsilon(2n)}) = \sum_{\mathscr{V}\in\mathscr{P}_2(2n)}\left(\varphi\left(b^{\epsilon(w_1)}b^{\epsilon(z_1)}\right)\ldots\varphi\left(b^{\epsilon(w_n)}b^{\epsilon(z_n)}\right)\lim_{N\to\infty}N^{-\frac{r}{2}}\sum_{\substack{i(1),\ldots,i(r)\in[N]\text{ s.t.}\\ (i(1),\ldots,i(r))\sim\mathscr{V}}}\beta^{\epsilon(1),\ldots,\epsilon(r)}_{i(1),\ldots,i(r)}\right),$$

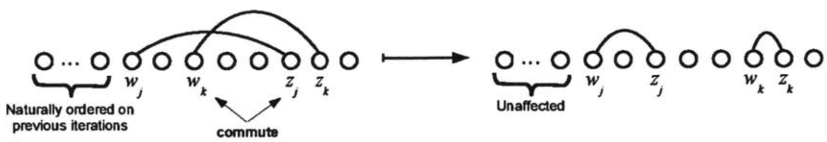

Figure 5-2: The process of bringing a mixed moment into a naturally-ordered form involves commuting all the inversions and all the nestings in each of the underlying pair partitions. In commuting a crossing (w_j, w_k, z_j, z_k), as depicted, the corresponding moment incurs a factor $\mu_{\epsilon(z_j),\epsilon(w_k)}(i(z_j), i(w_k))$.

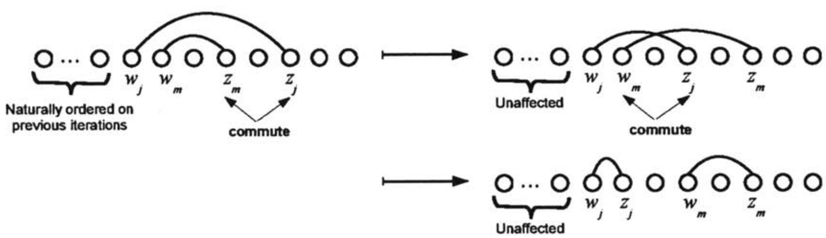

Figure 5-3: The process of bringing a mixed moment into a naturally-ordered form involves commuting all the inversions and all the nestings in each of the underlying pair partitions. In commuting a nesting (w_j, w_m, z_m, z_j), as depicted, the corresponding moment incurs a factor $\mu_{\epsilon(z_j),\epsilon(z_m)}(i(z_j), i(z_m))\mu_{\epsilon(z_j),\epsilon(w_m)}(i(z_j), i(w_m))$.

which yields (5.23) and completes the proof. □

Remark 13 The assumption of Theorem 11 that the covariances of the b_i's are independent of i, namely, that $\varphi(b_i^{\epsilon_1} b_i^{\epsilon_2}) = \varphi(b_j^{\epsilon_1} b_j^{\epsilon_2})$ for all $i, j \in \mathbb{N}$ and $\epsilon_1, \epsilon_2 \in \{1, *\}$, was not used in obtaining (5.22) and (5.24). Provided the existence of the limit in (5.24), the additional assumption is solely used for the purpose of simplifying (5.24) as (5.23). •

The above Theorem 11 differs from Theorem 1 of [Spe92] in the following ways:

i. The more general commutation relation $b_i^{\epsilon} b_j^{\epsilon'} = \mu_{\epsilon',\epsilon}(j, i)\, b_j^{\epsilon'} b_i^{\epsilon}$ with $\mu_{\epsilon,\epsilon'}(i, j) \in \mathbb{R}$ now replaces the commutation relation $b_i^{\epsilon} b_j^{\epsilon'} = s(i, j) b_j^{\epsilon'} b_i^{\epsilon}$ with spins $s(i, j) \in \{-1, 1\}$.

ii. For the purpose of factoring naturally ordered second moments as interval-ordered second moments, it is presently additionally assumed that $\varphi(b_i^* b_i^*) = \varphi(b_i b_i) = 0$. (cf. Remark 12.)

iii. The convergence of the moments now hinges on the existence of a more complicated limit, which is not only a function of the commutation coefficients and of the underlying partition, as was the case in [Spe92], but also on the pattern of adjoints in the mixed moment of interest (i.e. on the string $\epsilon(1), \ldots, \epsilon(n)$).

Note that the assumption on the uniform bounds on the moments is not new, but is instead implicit in [Spe92].

5.3 Stochastic Interpolation

Recall that, analogously to Theorem 1 in [Spe92], the "deterministic version" of the non-commutative CLT hinges on an existence of the limit (5.21), which is determined by the sequence of commutation coefficients $\{\mu_{\epsilon,\epsilon'}(i,j)\}$. Rather than providing more explicit conditions for the existence of the above limit, this section follows the philosophy of [Spe92] and instead considers the scenario where the coefficients "may have been chosen at random". The outcome will be that, starting with a probability law for a single coefficient and extending it to a product measure on the entire coefficient sequence, almost any choice of commutation coefficients will yield a finite and easily describable limit. For this, it is first necessary to define a suitable product measure on the coefficient sequence that is consistent with the dependency structure given by (A)–(B), which is accomplished in Remark 14. In turn, Remark 15 considers the effect on the limit achieved by imposing the vanishing of certain second moments. Finally, Lemma 23 is the remaining ingredient in the "almost sure version" of the non-commutative CLT (viz. the present Theorem 10).

Remark 14 Defining a measure on the sequence of commutation coefficients by focusing on the triangular sequences $\{\mu_{*,*}(i,j)\}_{1\leq i<j}$ and attempting to fix the remaining coefficients via (A)–(B) still leaves one degree of freedom. Namely, $\mu_{*,*}(i,j)$ was not until now explicitly related to $\mu_{*,1}(i,j)$. The need for a third relation governing the sign sequence comes into play when considering positivity requirements. Generally, φ is assumed to be *positive*, that is, if $\varphi(aa^*) \geq 0$ for all $a \in \mathscr{A}$. Then, $\varphi(b_ib_i^*) \geq 0$ and $\varphi(b_ib_jb_j^*b_i^*) \geq 0$ for all $i,j \in \mathbb{N}$. But, by the commutation relations

and the factoring of naturally ordered moments,

$$\varphi(b_i b_j b_j^* b_i^*) = \mu_{*,1}(i,j)\mu_{*,*}(i,j)\varphi(b_i b_i^*)\varphi(b_j b_j^*).$$

If the sequence $b_1, b_2, \ldots$ is such that $\varphi(b_i b_i^*) > 0$ for all i, the commutation signs must therefore also satisfy the following, third, requirement:

$$\frac{\mu_{*,1}(i,j)}{\mid \mu_{*,1}(i,j) \mid} = \frac{\mu_{*,*}(i,j)}{\mid \mu_{*,*}(i,j) \mid}. \tag{C}$$

In the random setting, (C) translates to $\mu_{*,1}(i,j) = \gamma(i,j)\mu_{*,*}(i,j)$ for some random sequence $\{\gamma(i,j)\}$ supported on $(0,\infty)$.

In assuming $\{\gamma(i,j)\}$ to be independent of $\mu_{*,*}(i,j)$ in line with the general philosophy of this section, the reader may soon verify that only the expectation of $\gamma(i,j)$ will matter from the perspective of Lemma 23. Furthermore, since the expectations of $\mu_{*,*}(i,j)$ and $\mu_{*,1}(i,j)$ will be taken to not depend on the index (i,j), one is free to set $t := \mathbb{E}(\gamma(i,j))$. Then, for $i < j$, (C) becomes:

$$\mu_{*,1}(i,j) = t\mu_{*,*}(i,j), \quad t > 0. \tag{C'}$$

•

Remark 15 Beyond the existence of the limit (5.21), the goal of the present section is to develop a probabilistic framework in which this limit takes on a particularly natural form. For this purpose, the basic setting of Theorem 11 will need to fulfill an additional requirement. Specifically, by the assumption of factoring of naturally-ordered moments, $\varphi(b_i b_j b_i^* b_j^*)$ and $\varphi(b_i b_j^* b_i^* b_j)$ for $i < j$ are both brought into their naturally ordered form by performing a single commutation. In the former case, the commutation incurs a factor $\mu_{*,1}(i,j)$ and, in the latter, the factor $\mu_{*,*}(i,j)$. Yet, in the combinatorial formulation, both products are in the equivalence class (in the sense of "$\sim$") of the pair partition $\pi =$ and both are brought into their naturally ordered form by commuting the single crossing in π. Thus, in order for the combinatorial invariant to be preserved, either:

- the expected values of $\mu_{*,1}(i,j)$ and $\mu_{*,*}(i,j)$ must be the same, or,
- one of the two mixed moments vanishes, i.e. $\varphi(b_i b_i^*) = 0$ or $\varphi(b_i^* b_i) = 0$ for all $i \in \mathbb{N}$.

By Remark 14, one may without loss of generality let $\mu_{*,1}(i,j) = t\,\mu_{*,*}(i,j)$. Thus, as the reader may soon be able to verify, opting to make equal the means of $\mu_{*,1}(i,j)$ and $\mu_{*,*}(i,j)$ by letting $t = 1$ reduces the statistics of the desired limit to those of crossings and the outcome is the same as in the case of randomly chosen commutation signs in [Spe92]. The formulation of Lemma 23 instead opts for the second alternative, and the introduction of the second parameter t will give rise to the appearance of a second combinatorial statistic, that of *nestings.*

Note that while φ is assumed to be positive, it is not assumed to be faithful, and there is no contradiction in assuming that $\varphi(b_i^* b_i) = 0$ while $\varphi(b_i b_i^*) \neq 0$. As further discussed in the following section, letting $\varphi(b_i^* b_i) = 0$ and $\varphi(b_i b_i^*) = 1$ will provide an asymptotic model for a family of "twisted" annihilation operators, whereas making the opposite choice would yield the corresponding analogue for the creation operators.

•

Lemma 23. *Fix $0 \leq |q| < t$ and let $\{\mu(i,j)\}_{1 \leq i < j}$ be a collection of independent, identically distributed non-vanishing random variables, with*

$$\mathbb{E}(\mu(i,j)) = qt^{-1} \in \mathbb{R}, \qquad \mathbb{E}(\mu(i,j)^2) = 1.$$

Letting $\mu_{*,*}(i,j) = \mu(i,j)$ for $1 \leq i < j$, populate the remaining $\mu_{\epsilon,\epsilon'}(i,j)$ for $\epsilon, \epsilon' \in \{1, *\}$ by

$$\mu_{1,1}(i,j) = \frac{1}{\mu_{*,*}(i,j)}, \qquad \mu_{1,*}(i,j) = \frac{1}{\mu_{*,1}(i,j)},$$
$$\mu_{*,1}(i,j) = t\,\mu_{*,*}(i,j), \qquad \mu_{\epsilon',\epsilon}(j,i) = \frac{1}{\mu_{\epsilon,\epsilon'}(i,j)},$$

Then, for any $\mathscr{V} \in \mathscr{P}_2(2n)$ and $\epsilon(1), \ldots, \epsilon(2n) \in \{1, *\}$, the limit (5.21) exists a.s. Moreover, if $\mathscr{V}$ is such as to satisfy $(\epsilon(w), \epsilon(z)) = (1, *)$ for all blocks $(w, z) \in \mathscr{V}$, the corresponding limit is given by

$$\lambda_{\mathscr{V},\epsilon(1),\ldots,\epsilon(2n)} = q^{cross(\mathscr{V})}\, t^{nest(\mathscr{V})} \quad a.s.$$

where $cross(\mathscr{V}) = |Cross(\mathscr{V})|$ and $nest(\mathscr{V}) = |Nest(\mathscr{V})|$ denote, respectively, the numbers of crossings and nestings in $\mathscr{V}$ (cf. Definition 7).

Proof. Fix $\mathscr{V} = \{(w_1, z_1), \ldots, (w_n, z_n)\}$ and, recalling that $i(w_m) = i(z_m)$ for all $m \in [n]$, consider the (classical) random variable X_N given by

$$\begin{aligned} X_N := \quad & N^{-n} \sum_{\substack{i(1),\ldots,i(2n)\in[N] \text{ s.t.} \\ (i(1),\ldots,i(2n))\sim\mathscr{V}}} \left(\prod_{\substack{(w_j,w_k,z_j,z_k) \\ \in \mathrm{Cross}(\mathscr{V})}} \mu_{\epsilon(z_j),\epsilon(w_k)}(i(z_j), i(w_k)) \right. \\ & \left. \times \prod_{\substack{(w_\ell,w_m,z_m,z_\ell) \\ \in \mathrm{Nest}(\mathscr{V})}} \mu_{\epsilon(z_\ell),\epsilon(z_m)}(i(z_\ell), i(z_m))\mu_{\epsilon(z_\ell),\epsilon(w_m)}(i(z_\ell), i(w_m)) \right), \end{aligned} \tag{5.25}$$

where the sequence of random variables $\{\mu_{\epsilon,\epsilon'}(i,j)\}_{\epsilon,\epsilon'\in\{1,*\},i,j\in\mathbb{N},i\neq j}$ is obtained by letting $\mu_{*,*}(i,j) = \mu(i,j)$ for $i < j$ and fixing the remaining coefficients as prescribed by (5.8)-(5.9). The first goal is to compute $\mathbb{E}(X_N)$. By the independence assumption, since the overall product includes no repeated terms, the expectation factors over the products. It therefore suffices to evaluate $\mathbb{E}(\mu_{\epsilon(z_j),\epsilon(w_k)}(i(w_j), i(w_k)))$ for each crossing (w_j, w_k, z_j, z_k) and

$$\mathbb{E}(\mu_{\epsilon(z_\ell),\epsilon(z_m)}(i(w_\ell), i(w_m))\mu_{\epsilon(z_\ell),\epsilon(w_m)}(i(w_\ell), i(w_m)))$$

for each nesting $(w_\ell, w_m, z_m, z_\ell)$ of a given pair-partition. At the outset, recall that every pair-partition $\mathscr{V}$ contributing to X_N is such that $(\epsilon(w), \epsilon(z)) = (1, *)$. Then, starting with the crossings and assuming that $i(w_j) = i(z_j) < i(w_k) = i(z_k)$, the corresponding commutation coefficient in (5.25) and its expectation are given as

$$\begin{aligned}&\mu_{*,1}(i(z_j),i(w_k)) = t\,\mu_{*,*}(i(z_j),i(w_k)) = t\,\mu(i(z_j),i(w_k))\\&\overset{\mathbb{E}}{\longmapsto}\ t(q/t) = q.\end{aligned}\tag{5.26}$$

When it is instead the case that $i(w_j) = i(z_j) > i(w_k) = i(z_k)$, it suffices to notice that by (A)–(B), $\mu_{*,1}(i,j) = \mu_{*,1}(j,i)$. The same conclusion then holds and each crossing therefore contributes a factor of q on average. Moving on to nestings, let $(w_\ell, w_m, z_m, z_\ell)$ be a nesting. If $i(w_\ell) = z_\ell < i(w_m) = i(z_m)$, the corresponding commutation coefficient in (5.25) and its expectation are given as

$$\begin{aligned}&\mu_{*,*}(i(z_\ell),i(z_m))\mu_{*,1}(i(z_\ell),i(w_m)) = \mu_{*,*}(i(z_\ell),i(z_m))\,t\,\mu_{*,*}(i(z_\ell),i(w_m))\\&= t(\mu(i(w_\ell),i(w_m)))^2 \overset{\mathbb{E}}{\longmapsto}\ t.\end{aligned}\tag{5.27}$$

If on the other hand $i(w_\ell) = z_\ell > i(w_m) = i(z_m)$, by (A)–(B) the commutation coefficient and its expectation become

$$\begin{aligned}&\mu_{*,*}(i(z_\ell),i(z_m))\mu_{*,1}(i(z_\ell),i(w_m)) = (\mu_{*,*}(i(z_m),i(z_\ell)))^{-1}\mu_{*,1}(i(z_\ell),i(w_m))\\&= (\mu(i(z_m),i(z_\ell)))^{-1}\,t\,\mu(i(z_m),i(z_\ell)) = t\ \overset{\mathbb{E}}{\longmapsto}\ t.\end{aligned}\tag{5.28}$$

Thus, each nesting also contributes a factor of t. It follows that $\mathbb{E}(X_N)$ is given by

$$\mathbb{E}(X_N) = N^{-n} \sum_{\substack{i(1),\dots,i(2n)\in[N]\text{ s.t.}\\(i(1),\dots,i(2n))\sim\mathcal{V}}} q^{\mathrm{cross}(\mathcal{V})}\,t^{\mathrm{nest}(\mathcal{V})} = q^{\mathrm{cross}(\mathcal{V})}\,t^{\mathrm{nest}(\mathcal{V})} N^{-n}\binom{N}{n} n!.\tag{5.29}$$

Thus,

$$\lim_{N\to\infty}\mathbb{E}(X_N) = q^{\mathrm{cross}(\mathcal{V})}\,t^{\mathrm{nest}(\mathcal{V})}.\tag{5.30}$$

It now remains to show that $\lim_{N\to\infty} X_N = \lim_{N\to\infty}\mathbb{E}(X_N)$ a.s., that is, that for every $\eta > 0$, $\mathbb{P}\left(\bigcap_{N\geq 1}\bigcup_{M\geq N}\{|X_M - \mathbb{E}(X_M)| \geq \eta\}\right) = 0$. The calculation is analogous to that in [Spe92]. By the subadditivity of $\mathbb{P}$ and a standard application of Markov inequality,

$$\mathbb{P}\left(\bigcup_{M\geq N}\{|X_M-\mathbb{E}(X_M)|\geq\eta\}\right)\leq\sum_{M\geq N}\mathbb{P}\left(|X_M-\mathbb{E}(X_M)|\geq\eta\right)$$
$$\leq\frac{1}{\eta^2}\sum_{M\geq N}\mathbb{E}(|X_M-\mathbb{E}(X_M)|^2). \tag{5.31}$$

In turn, $\mathbb{E}(|X_M-\mathbb{E}(X_M)|^2)=\mathbb{E}(X_M^2)-\mathbb{E}(X_M)^2$, with

$$\mathbb{E}(X_M)^2=q^{2\,\mathrm{cross}(\mathscr{V})}\,t^{2\,\mathrm{nest}(\mathscr{V})}$$

and

$$\mathbb{E}(X_M^2)=M^{-2n}\sum_{\substack{i(1),\ldots,i(2n)\in[N]\text{ s.t.}\\(i(1),\ldots,i(2n))\sim\mathscr{V},\\j(1),\ldots,j(2n)\in[N]\text{ s.t.}\\(j(1),\ldots,j(2n))\sim\mathscr{V}}}\mathbb{E}\Bigg[\prod_{(w_k,w_\ell)\in\mathrm{Cross}(\mathscr{V})}\Big(\mu_{*,1}(i(w_k),i(w_\ell))\Big)$$
$$\Big(\mu_{*,1}(j(w_k),j(w_\ell))\Big)\prod_{(w_m,w_m)\in\mathrm{Nest}(\mathscr{V})}\Big(\mu_{*,*}(i(w_m),i(w_n))\,\mu_{*,1}(i(w_m),i(w_n))\Big)$$
$$\Big(\mu_{*,*}(j(w_m),j(w_n))\,\mu_{*,1}(j(w_m),j(w_n))\Big)\Bigg] \tag{5.32}$$

where, for convenience of notation, each crossing (w_k,w_ℓ,z_k,z_ℓ) was abbreviated as (w_k,w_ℓ), and similarly for the nestings. Now suppose that for two choices of indices and the corresponding sets (not multisets) $\{i(1),\ldots,i(2n)\}$ and $\{j(1),\ldots,j(2n)\}$, there is at most one index in common, i.e. suppose that $\{i(1),\ldots,i(2n)\}\cap\{j(1),\ldots,j(2n)\}\leq 1$. In that case, $(i(k),i(k'))\neq(j(m),j(m'))$ for all $k,k',m,m'\in[2n]$ with $k\neq k',m\neq m'$. By the independence assumption, the above expectation factors over the product (up to the parenthesized terms) and the contribution of each such $\{i(1),\ldots,i(2n)\}$, $\{j(1),\ldots,j(2n)\}$ is simply $q^{2\mathrm{cross}(\mathscr{V})}\,t^{2\mathrm{nest}(\mathscr{V})}$. Thus, the choices of indices with $\{i(1),\ldots,i(2n)\}\cap\{j(1),\ldots,j(2n)\}\leq 1$ do not contribute to the variance $\mathbb{E}(|X_M-\mathbb{E}(X_M)|^2)$. It now remains to consider the $\Theta(M^{2n-2})$ remaining terms of the sum (5.32).

By the Cauchy-Schwarz inequality, the expectation of the product is bounded, and thus

$$\mathbb{E}(|X_M - \mathbb{E}(X_M)|^2) \leq M^{-2n} M^{2n-2} C = \frac{C}{M^2},$$

where C does not depend on M. Since $\sum_{M \geq 0} M^{-2}$ converges,

$$\lim_{N \to \infty} \sum_{M \geq N} \mathbb{E}(|X_M - \mathbb{E}(X_M)|^2) \to 0,$$

and therefore by (5.31),

$$\mathbb{P}\left(\bigcap_{N \geq 1} \bigcup_{M \geq N} \{|X_M - \mathbb{E}(X_M)| \geq \eta\} \right) = \lim_{N \to \infty} \mathbb{P}\left(\bigcup_{M \geq N} \{|X_M - \mathbb{E}(X_M)| \geq \eta\} \right) = 0.$$

This completes the proof. □

5.4 Random Matrix Models

Considering some prescribed sequence $\{\mu_{\epsilon,\epsilon'}(i,j)\}_{\epsilon,\epsilon' \in \{1,*\}, i,j \in \mathbb{N}, i \neq j}$ of real-valued commmutation coefficients satisfying (A)–(B), Lemma 21 exhibits a set of elements of a matrix algebra that satisfy the corresponding commutativity structure. The construction is analogous to the one given in Lemma 21 and the latter is in fact stated in a form that renders the present generalization natural.

Proof of Lemma 22. To show that $b_j^{\epsilon'} = \mu_{\epsilon',\epsilon}(j,i)\, b_j^{\epsilon'} b_i^{\epsilon}$, it suffices to consider the defns in (5.15) and commute 2×2 matrices. Specifically, let $i < j$ and, by elementary manipulations on tensor products, write

$$\begin{aligned} b_i b_j &= (\sigma_{\mu(1,i)} \sigma_{\mu(1,j)}) \otimes \ldots \otimes (\sigma_{\mu(i-1,i)} \sigma_{\mu(i-1,j)}) \otimes (\gamma \sigma_{\mu(i,j)}) \\ &\quad \otimes (\sigma_1 \sigma_{\mu(i+1,j)})) \otimes \ldots \otimes (\sigma_1 \sigma_{\mu(j-1,j)})) \otimes (\sigma_1 \gamma) \otimes (\sigma_1 \sigma_1)^{\otimes(n-j)} \end{aligned} \tag{5.33}$$

Now note that $\sigma_x \sigma_y = \sigma_y \sigma_x$ for all $x, y \in \mathbb{R}$. Moreover, $\gamma \sigma_x = \sqrt{t x}\, \sigma_x \gamma$. Thus,

$$\gamma \sigma_{\mu(i,j)} = \sqrt{t} \mu(i,j) \sigma_{\mu(i,j)} \gamma \qquad \text{and} \qquad \sigma_1 \gamma = (\sqrt{t})^{-1} \gamma \sigma_1,$$

and, therefore,

$$b_i b_j = \frac{\sqrt{t}\mu(i,j)}{\sqrt{t}} b_j b_i = \mu_{*,*}(i,j) b_j b_i = \mu_{1,1}(j,i) b_j b_i.$$

Next, in commuting b_i^* with b_j, the only non-trivial commutations are that of γ^* with $\sigma_{\mu(i,j)}$ and $\sigma_1^* = \sigma_1$ with γ. Since $\gamma^* \sigma_x = (\sqrt{t}x)^{-1} \sigma_x \gamma^*$, it follows that

$$\begin{aligned} b_i^* b_j &= \frac{1}{\sqrt{t}} \frac{1}{\sqrt{t}\mu(i,j)} b_j b_i^* = \frac{1}{t\,\mu(i,j)} b_j b_i^* = \frac{1}{t\,\mu_{*,*}(i,j)} b_j b_i^* \\ &= \frac{1}{\mu_{*,1}(i,j)} b_j b_i^* = \mu_{1,*}(j,i) b_j b_i^*. \end{aligned}$$

The remaining relations now follow by taking adjoints, and the result is that $b_i^{\epsilon} b_j^{\epsilon'} = \mu_{\epsilon',\epsilon}(j,i)\, b_j^{\epsilon'} b_i^{\epsilon}$.

It remains to show that, in addition to the commutation relation, the resulting matrix sequences also satisfy the assumptions (1)-(4) of Theorem 10. Start by noting that for $a_1, \ldots, a_k \in \mathscr{M}_2$, $\varphi(a_1 \otimes \ldots \otimes a_k) = (a_1)_{11} \ldots (a_k)_{11}$, where $(a)_{11} := \langle e_1 a, e_1 \rangle_2$. It therefore immediately follows that for all $i \in \mathbb{N}$, $\varphi(b_i) = \varphi(b_i^*) = 0$. By the same token, it is also clear that for all $i, j \in \mathbb{N}$, $\varphi(b_i b_i^*) = \varphi(b_j b_j^*) = 1$ and $\varphi(b_i^{\epsilon} b_i^{\epsilon'}) = \varphi(b_i^{\epsilon} b_i^{\epsilon'}) = 0$ for $\epsilon, \epsilon' \in \{1, *\}$ with $(\epsilon, \epsilon') \neq (1, *)$, and, furthermore, $|\varphi(\prod_{i=1}^n b_{j(i)}^{\epsilon(i)})| \leq 1$ for all n and all choices of exponents and indices. The factoring over naturally ordered products also follows immediately, completing the proof. □

Finally, combining Theorem 10 with Lemma 22 immediately yields the desired asymptotic models for the creation, annihilation, and field operators on the (q,t)-Fock space. For instance, the mixed moments of S_N converge to those of the annihilation operator $a(e_1)$, where e_1 is an element of the orthonormal basis of $\mathscr{H}$. More generally, one may consider *systems* of operators, e.g. they specify the *joint mixed moments* of annihilation operators $a(e_1), \ldots, a(e_n)$ associated with basis elements $e_1, \ldots, e_n$. In order to asymptotically realize the joint moments of $a(e_1), \ldots, a(e_n)$ rather than the moments of $a(e_1)$ alone, it suffices to consider a sequence $S_{N,1}, \ldots, S_{N,n}$ of partial sums built from non-intersecting subsets of $\{b_i\}_{i \in \mathbb{N}}$. For instance, the fact that e_i

and e_j are orthogonal for $i \neq j$ and that the moment $\varphi_{q,t}(a(e_i)a(e_j))$ vanishes follows (in this asymptotic setting) from the fact that $\varphi(b_i b_j) = 0$ for $i \neq j$. The general formulation is found in Corollary 2.

Chapter 6

Conclusions and Perspectives

The premise of this thesis are probability theories and matrix limits built around certain types of "commutation structure". From a practical perspective, the promise of such theories is in providing tractable frameworks for dealing with the scaling limits of large random objects. Indeed, as the amounts of available data in experimental science and engineering increase, practictioners are increasingly turning towards random matrix theory as a framework in which to extract the "asymptotic" features from data sets. So far, free probability [Voi86, VDN92] in particular has proven enormously successful in capturing and describing (both algebraically and in a manner yielding efficient computational tools) the asymptotics of a large class of random matrices. As the reader may recall, free probability is a non-commutative probability theory that arises in settings characterized by a lack of commutative structure (in a sense made precise in Chapter 3). As a result, while it describes many matrix limits of interest, free probability cannot describe them all, as certain matrix ensembles still preserve a degree of commutative structure in the limit. Instead, this thesis has focused on the probability theories built around certain *commutation relations* and on the random matrix limits described by the resulting frameworks.

Specifically, this thesis introduces a two-parameter family of Fock spaces – namely, the (q, t)-Fock spaces – and studies the probabilistic aspects of the bounded linear operators on them. The fundamental operators on the (q, t)-Fock space, viz. the creation and annihilation operators, now satisfy the (q, t)-*commutation relation*, known in physics as one of the defining relations of the Chakrabarti-Jagannathan deformed oscillator algebra [CJ91]. This commutation relation unifies a number of the familiar relations found in related contexts, including the q-CR discussed in Chapter 2 and the "physics" q-relations [Bie89, Mac89].

The algebra of bounded linear operators on the (q,t)-Fock space, when endowed with the vacuum expectation state (cf. Chapter 2) gives rise to a rich two-parameter continuum of non-commutative probability theories. Most notably:

- The $q = 0 < t$ case yields a new single-parameter deformation of the full Boltzmann Fock space of free probability [Voi86, VDN92] that is particularly noteworthy for its new and far-reaching ties to a number of well-known objects. In particular, the corresponding deformation of the semicircular measure is encoded, in various forms, via the generalized Rogers-Ramanujan continued fraction (e.g. [And98]), the Rogers-Ramanujan identities (e.g. [And98]), the t-Airy function [Ism05], and the t-Catalan numbers of Carlitz-Riordan [FH85, CR64]. Concerning random matrix limits, the $(0,t)$-Fock space provides a setting in which to realize a generalization of Wigner's semicircle law [Wig55] to time-correlated Wigner processes [MP02]. As discussed shortly, this result hints at new layers of structure that arise from large-dimensional limits of certain matricial stochastic processes.

- The (q,t)-deformed probability naturally arises in the asymptotic setting. Analogously to the q-deformed case [BS91, Spe92], the (q,t)-Gaussian elements can be realized as weak limits of matricial models, via the generalized non-commutative Central Limit Theorem (CLT) theorem developed in Chapter 5. Compared to the original form of the non-commutative CLT introduced in [Spe92], this second-parameter refinement is a consequence of the passage from a commutation structure built around commutation *signs*, taking values in $\{-1,1\}$, to a more general structure based on commutation *coefficients* taking values in $\mathbb{R}$. Beyond the (q,t)-Gaussian statistics, the non-commutative CLT provides a general framework that may be used to derive new statistics along with the corresponding matrix models, thus opening up a number of avenues for further work.

Further on the topic of the matrix limits arising from the present framework, the random matrices obtained by the general Jordan-Wigner transform (cf. Lemma 22 in Chapter 5) yield random matrix models for the creation and annihilation operators on the (q,t)-Fock space (cf. Corollary 2). In this case, the commutative structure does not altogether vanish in the limit of large matrix size, but instead remains present in a relatively simple form, given by the (q,t)-commutation relations. We are therefore dealing with random matrices that are too structured to be asymptotically characterized in the setting of free probability, yet whose limits can nevertheless be described by a well-founded non-commutative probability theory.

Note, however, that the Jordan-Wigner constructions may be taken as prototypes, but not as the ultimate results on the type of matrix limits that fall within the (q,t)-deformed framework. For instance, whereas the corresponding matrices were constructed to satisfy the prescribed commutation relations regardless of their dimension, this requirement only comes into play for sufficiently large dimensions. One may therefore foreseeably consider certain vanishing perturbations of the Jordan-Wigner models, or altogether different models that eventually recover the desired commutation structure. At this point, the reader should also be warned that conjugating any sequence of structured matrices by a fixed unitary matrix will alter the eigenspaces while preserving the commutation relations. The result will be structured matrices with a misleadingly "generic" appearance, whose limits may not fall within the scope of free probability, but instead be described by an asymptotic framework realizing some given type of commutation structure.

6.1 Perspectives

From the interdisciplinary viewpoint, the ultimate goal of non-commutative probability is to provide full and tractable descriptions of the scaling limits of large random objects. However, it is important to note that contrary to the classical intuition, there is no single framework that promises to encompass all of the limits of interest, as different frameworks are built around different types of relations that the limiting objects may satisfy. The frameworks developed in this thesis bring us one step closer to our ultimate goal by providing non-commutative probability theories, built around a two-parameter family of commutation relations, that capture a broader class of random matrix limits. These results, in turn, open up a number of avenues for further work. In particular, one may seek to further develop the probabilistic aspects of the present framework, or one may use the asymptotic tools introduced in this thesis to arrive at new non-commutative probability theories and realize yet a broader class of matrix limits.

As a natural generalization of a well-studied framework [BS91], the (q,t)-Fock space provides a setting in which meaningful two-parameter analogues of known results may be derived. Starting at the combinatorial level, while the statistics of crossings in pair partitions that index the relevant moment formulae in the q-deformed setting have long been well understood [Rio75, Tou52, Tou50a, Tou50b], the combinatorics in the (q,t)-deformed setting have already proved more challenging. Specifically, the combinatorial statistics of crossings in pair partitions are presently replaced by the joint statistics of crossings *and nestings*, whose characterization remains an open problem in combinatorics and is of relevance to broader combinatorial questions [KZ06, Kla06, CDD+07].

The special functions and orthogonal polynomials arising in the q-deformed setting also become their natural two-parameter generalizations in the present framework. In particular, the q-Hermite orthogonal polynomials given by the three-term recurrence $zH_n(z;q,t) = H_{n+1}(z;q,t) + [n]_q H_{n-1}(z;q,t)$ (with $H_0(z;q) = 1$ and $H_1(z;q) = z$) become the *(q,t)-Hermite orthogonal polynomials* $zH_n(z;q,t) = H_{n+1}(z;q,t) + [n]_{q,t} H_{n-1}(z;q,t)$ (with the same initial conditions). Note that the latter were previously considered in the setting of [CJ91], and are also a specialization of the octabasic Laguerre family [SS96]. However, while there is a known expression for the orthogonalizing measure of the q-Hermite polynomials (namely, the q-Gaussian measure, with density given as (3.19) of Chapter 3), no such expression is currently known for that of the (q,t)-Hermite polynomials. One present exception is the $q = 0 < t \leq 1$ case (cf. Definition/Theorem 4 of Chapter 4), where the corresponding (q,t)-Gaussian measure happens to be expressed through a quantum deformation of the Airy function. Characterizing this measure over a broader parameter range may reveal a deeper structure and yet further connections to other fundamental objects.

There are many more combinatorial objects and special functions arising in the q-deformed setting that may admit interesting (q,t)-deformed analogues (see e.g. [EP03]). There is also sophisticated probabilistic structure inherent to the q-deformed setting that one may now hope to extend as well. This will be discussed shortly. But, prior to considering the probabilistic aspects of the (q,t)-Gaussian algebra $\Gamma_{q,t}(\mathscr{H})$, it is natural to ask how much of the operator algebraic structure transfers from the q-deformed setting to the (q,t)-deformed one. As a starting point, it is not presently clear whether the vacuum expectation is faithful on the $\Gamma_{q,t}(\mathscr{H})$, nor whether the vacuum is separating (though it may be readily shown to be cyclic). The vacuum expectation state was shown to not be tracial on $\Gamma_{q,t}(\mathscr{H})$ (cf. Proposition 4 of Chapter 4), but it is unclear whether it is possible to define a non-trivial trace on this algebra. It is also not known whether $\Gamma_{q,t}(\mathscr{H})$ is a factor. The general structure of $\Gamma_{q,t}(\mathscr{H})$ therefore still remains largely hidden, prompting further questions of operator algebraic interest.

6.1.1 Stochastic Integrals

Moving toward more advanced probabilistic aspects of the (q,t)-Gaussian probability space $(\Gamma_{q,t}(\mathscr{H}), \varphi_{q,t})$, one may seek to define Brownian motion and stochastic integrals. As in the q-Gaussian setting, the (q,t)-Brownian motion is readily defined by letting $\mathscr{H} = \mathscr{L}^2_{\mathbb{R}}(\mathbb{R})$ and $X_\tau := s_{q,t}(1_{[0,\tau]})$, where $1_{[0,\tau]}$ is the indicator corresponding to the desired time interval $1_{[0,\tau]}$. In general, the difficulty arises at the level of conditional expectations, as the natural state $\varphi_{q,t}$ on $\Gamma_{q,t}(\mathscr{H})$ is no longer tracial. One may seek to develop the probabilistic aspects around some other state that does happen to be tracial, but, as discussed in the previous section, the latter need not exist.

Nevertheless, there do exist well-developed notions of non-commutative L^p spaces in a non-tracial setting is well developed (see e.g. [Haa79, Kos84]) and, specifically, of conditional expectations (see e.g. [AC82]).

Note that once the fundamental objects are defined, the more advanced results in q-stochastic calculus (see [BS98, DM03, DNN]) make extensive use of the underlying combinatorial structure. Analogously to [BKS97, DM03], one may then seek to define the stochastic integral $\int dX_{\tau_1} \dots dX_{\tau_n}$ via suitable "(q,t)-Wick products". As the combinatorics arising in the (q,t)-deformed setting are a highly natural refinement of those in the q-deformed setting [EP03], one may expect the "(q,t)-stochastic calculus" to be a natural generalization of the q-stochastic calculus. But, the results may nevertheless be surprising. Specifically, the role played by the number operator in the present setting is the reason for the loss of traciality of the vacuum expectation state. Moreover, at the combinatorial level, the appearance of the number operator in the (q,t)-commutation relation yields an "interacting" term on account of which one must consider the structural properties of an operator in conjunction with the dimension of the vector on which it is acting. Yet while not a straightforward task, developing the stochastic calculus in the present setting is warranted on account of the structural depth of the (q,t)-deformed framework as currently revealed.

6.1.2 Matrix limits via the generalized non-commutative CLT

One of the main contributions of this thesis is a generalized non-commutative Central Limit Theorem (CLT) (Theorem 10 of Chapter 5) by which the stochastic mixtures of elements satisfying deformed commutation relations (namely, commuting with respect to some real-valued commutation coefficients) asymptotically obey the (q,t)-Gaussian statistics. Buried underneath this result is in fact a much more general one, introduced as Theorem 11 of Chapter 5). Specifically, Theorem 11 describes the CLT-type limit in a non-commutative setting characterized by an arbitrary sequence of real-valued commutation coefficients and formulates a condition under which such a limit exists. Then, Theorem 10 (via the probabilistic Lemma 23) states that such a limit exists almost surely in the case where the commutation coefficients are taken to be a sequence of independent and identically distributed random variables.

In the original case of Speicher [Spe92], where the commutation coefficients were signs (taking values ± 1), the passage to the i.i.d. scenario was, in a sense, the most natural one and one that ensured a tractable solution. At present, however, the fact that the commutation coefficients are non-vanishing reals allows one to consider a significantly broader range of probability distributions that may still yield a tractable answer. In particular: *what are the limiting (CLT-type) statistics of a non-commutative ensemble with commutation coefficients described by a jointly Gaussian law? How are these statistics affected by the underlying covariance structure?*

One may expect to see interesting statistics arise for certain choices of the covariance structure underlying the commutation coefficients. In the vein of Chapter 4, such statistics may in turn have meaningful representations on new or existing types of Fock spaces. More immediately, however, asymptotic random matrix models for any such statistics are already in place – they are provided by the extended Jordan-Wigner transform (cf. Lemma 22 of Chapter 5), which is presently formulated for an arbitrary sequence of commutation coefficients satisfying the required consistency conditions (cf. Condition 2). In that sense, the results of Chapter 5 form a concrete framework in which to both *discover new statistics* and *realize these as asymptotic random matrix models.*

6.1.3 Limits of correlated Wigner processes

One of the cornerstones of random matrix theory is Wigner's semicircle law [Wig55] (with modern formulation as e.g. Theorem 2.1.1 and Lemma 2.1.6 in [AGZ10]), according to which the moments of a Wigner matrix $\tilde{W}_N$ (in notation of Chapter 4) converge to the Catalan numbers and the spectral measure of $\tilde{W}_N$ converges almost surely to the semicircle law. With free probability, this fact has been extended to a joint convergence of independent Wigner matrices to a system of free semicircular elements (see e.g. Theorem 22.24 in [NS06]).

Lemma 4.5.2 of Chapter 4 achieves something altogether different. Specifically, it uses the $(0,t)$-Fock space framework in order to provide a Hilbert-space realization of the result of Mazza and Piau [MP02] (see also Khorunzhy [Kho01]) on the convergence of the expectation of a correlated Wigner process. Rather than encoding the spectral measure of a single Wigner matrix, the t-semicircular element of Chapter 4 now realizes the first-order statistics of a Wigner process, with the parameter t set to the square of the correlation coefficient for the process. As discussed in Remark 11 of Chapter 4, it is not clear why these first-order statistics should correspond to moments of some operator (though one may hope, as indeed it turns out to be the case, that should such an operator exist, it will be intimately connected to the setting of free probability). A number of questions immediately arise. For instance: *Are there operator realizations of the higher order statistics of the corelated Wigner process? How are these related to the t-semicircular elements?* A better understanding of the asymptotics of the correlated Wigner process, as a meaningful model for correlated noise, is likely to translate into concrete gains in many areas. Going further, one may also consider replacing the Wigner matrices by covariance matrices and studying the asymptotics of the processes that arise. It generally seems unlikely that the existence of the operator realization for the first-order statistics of the correlated Wigner process is coincidental, suggesting instead the possibility of developing operator algebraic frameworks to describe the limits of a broader class of matricial stochastic processes.

Appendix A

The present appendix provides the reader with a brief overview of the algebraic notions used in this thesis.[1] Given the interdisciplinary nature of the thesis, an effort will be made to remain as grounded as possible. The discussion will therefore be restricted to the relevant specializations of the more abstract constructs. For instance, rather than discussing modules, we will be content to work with vector spaces, despite the fact that, for the discussion at hand, the use of the former, more general concept would not have required much additional expenditure in energy.[2] Furthermore, all sets in this chapter will implicitly be assumed non-empty.

Though we will be focusing on relatively advanced material, the pre-requisite background will turn out to be minimal. Specifically, the reader is assumed to already be familiar with the basic notions of *groups* and *vector spaces*. *Algebras*, which are vector spaces that are also closed under some operation of vector multiplication that is compatible with the addition and scalar multiplication, are further discussed in Appendix C, in the context of algebras of operators on normed vector spaces. The reader is also assumed to be comfortable thinking of a *homomorphism* as a structure-preserving map between algebraic objects (with the exact nature of the structure being preserved implied by the context). Similarly, an *isomorphism* is assumed to

[1] Regarding the sources of material for this chapter, most graduate algebra texts will contain the relevant material (e.g. [Rot10]). More specifically, however, the author has found Keith Conrad's extraordinary expository papers, available on-line at [Con], to be brimming with insight on a number of topics discussed presently.

[2] Note ahead of time that while the vector spaces in the present chapter will be taken with respect to an arbitrary field $\mathbb{F}$, starting with Appendix B, the symbol $\mathbb{F}$ will be taken to be a stand-in for either $\mathbb{C}$ or $\mathbb{R}$.

trigger a reflex of thinking of structural equivalence, being a bijective homomorphism whose inverse is also a homomorphism.

A.1 Generators and Relations

At the basic level, *algebra* is concerned with general structure arising from *operations* and *relations* defined on a set. The present section adopts this point of view in order to arrive at a general method for describing a group or a vector space. Note ahead of time that the multiplication operations will not be assumed to be commutative, unless indicated otherwise. In fact, much attention in the later chapters will be paid to certain types of *commutation relations* and on algebras they generate.

As a review of some elementary notions, recall that a *generating set of a group* is a subset that is not contained in any proper subgroup of the group. If S is a subset of a group G, then the smallest subgroup of G containing S is referred to as the *subgroup generated by* S and the elements of S are referred to as *generators*. These definitions can be modified in the obvious way to apply to vector spaces instead of groups.

Now note that the relations between the elements of the generating set reflect on the group that they generate. For example, the subgroup of the additive group $\mathbb{Z}^2$ generated by $\{(1,0),(0,1)\}$ is very different from the one generated by $\{(1,0),(2,0)\}$; the first is in fact $\mathbb{Z}^2$ while the second can be identified with a copy of $\mathbb{Z}$ inside $\mathbb{Z}^2$. Note that in the case of the second generating set, $(2,0) = (1,0) + (1,0)$, while no such relation exists between the elements of the first set.

When given an arbitrary set S and some group operation one wishes to impose, the group that S generates will depend on the *relations* that the elements of S are assumed to satisfy with respect to the group operation. Returning to the previous example and letting $S = \{s_1, s_2\}$ be an arbitrary set, the generators $\{(1,0),(0,1)\}$ should somehow correspond to the case where there are no relations assumed between s_1 and s_2, while the generators $\{(1,0),(2,0)\}$ should correspond to the case where $s_1^2 = s_2$. The remainder of this section is dedicated to making these observations concrete.

A.1.1 Free objects

Let us address the *absence* of relations first. Though most of the present chapter will in fact be concerned with vector spaces, we start by considering a notion of a *free group*, which will play a prominent role in Chapter 3. For that, consider the following useful construction of a group that is generated by the symbol set S with no relations imposed on the elements of S by the group structure.

Construction 1. *Let S be a set of symbols and let $S^{-1} = \{x^{-1} \mid x \in S\}$, where the elements of S^{-1} are also symbols (which are only subsequently to be* interpreted *as actual inverses). Let T be the set of all* non-commutative words *on the symbol set $S \cup S^{-1}$, i.e. T is the set of words $s_1 \dots s_n$ taken over all $n \in \mathbb{N}$ and $s_1, \dots, s_n \in S \cup S^{-1}$, with the additional inclusion of the* empty word *e. The set T_r of* reduced words *is obtained by removing from T all the words in which symbols x and x^{-1} appear consecutively. Lastly, define the binary operation of* concatenation *that takes two words $s_1 s_2 \dots s_n$ and $s'_1 \dots s'_k$ in T_r, creates the new word $s_1 s_2 \dots s_n s'_1 \dots s'_k \in T$, and iteratively removes from it all pairs of consecutive symbols that are of the form xx^{-1} or $x^{-1}x$ for any $x \in S$. It is clear that the resulting word is an element of T_r and it is also clear that the concatenation defines a group structure on T_r, with the empty word as the identity.*

In the above structure, no relation is assumed between the elements of S. However, as an approach to generating objects from sets in the absence of relations, the construction given is highly specific (in particular, the elements of the group and the multiplication map are prescribed). It turns out that the above construction, is in a certain sense, generic.

Definition/Theorem 5. *Given a set S, there exists a group F_S containing S having the following property: for any function f from S to a group H, there exists a unique homomorphism $\theta_f : F_S \to H$ such that $f = \theta_f \circ \iota$, where $\iota : S \to F_S$ is the inclusion map. The group F_S is unique up to isomorphism and it is termed the* free group *generated by S. The set S is said to be a* basis *of F_S.*

In other words, there exists a group F_S containing S such that for any function f from S to a group H, there exists a unique homomorphism θ_f making the following diagram commute:

$$\begin{tikzcd} S \arrow[r, "\iota"] \arrow[dr, "f"'] & F_S \arrow[d, "\theta_f"] \\ & H \end{tikzcd}$$

The above definition is a typical type of definition by a *universal property.* In loose terms, a universal property is an attribute that can be taken to prescribe the structure of objects belonging to a certain category in that, when shared by two or more objects, describes them up to equivalence. In the case of free groups, their universal property is highly natural, as it amounts to requiring that *every* function defined on a *basis* S of a free group and mapping into an arbitrary group H can be uniquely extended to a group homomorphism.

The key remark at this point is that the group obtained in Construction 1 satisfies the universal property. This is almost obvious; indeed, given a generating set S and a function f from S into some group H, it suffices to extend f by setting its value on any reduced word $x_1x_2\dots x_n$ to be $f(x_1)f(x_2)\dots f(x_n)$. The universal property therefore ensures that any free group over S is isomorphic to that of Construction 1. Another immediate corollary of the universal property is that two free groups are isomorphic if and only if their generating sets have the same cardinality.

Returning to the idea of having a function defined on a *basis* S of a group extending uniquely to group homomorphism, it is clear that having relations present on S further constrains such potential extensions when compared to the case of the free group. In fact, by again comparing Construction 1 with Definition 1, the outcome is that, in the presence of relations, such an extension can no longer be guaranteed.

At the same time as discussing free groups, it is also natural to discuss *free products* of (arbitrary) groups. The free product is defined by its universal property as follows.

Definition/Theorem 6. *Given some index set I and groups $\{G_i\}_{i\in I}$, there exists a group G together with unique homomorphisms $\rho_i : G_i \to G$ ($i \in I$), with the* *property that given any group H and homomorphisms $f_i : G_i \to H$, there exists a unique homomorphism $\theta_f : G \to H$ such that $f_i = \rho_i \circ \theta_f$. The group G is unique up to isomorphism and is denoted $*_{i\in I}G_i$ and referred to as the* group free product of $\{G_i\}_{i\in I}$.

The group free product $*_{i\in I}G_i$ is particularly easy to describe. The reader may verify that, as a set, $*_{i\in I}G_i$ is simply given as

$$*_{i\in I}G_i = \{e\} \cup \{g_1 \ldots g_k \mid k \in \mathbb{N},\, g_j \in G_{i(j)},\, g_j \neq e_j,\, i(1) \neq i(2) \neq \ldots \neq i(k)\}, \quad \text{(A.1)}$$

where e again denotes the empty word and e_i the unit element in G_i. (Note that "$i(1) \neq i(2) \neq \ldots \neq i(k)$" stands for the consecutive indices being different, and does not prohibit any non-consecutive index repetitions). Analogously to Construction 1, the multiplication on $*_{i\in I}G_i$ is defined by word concatenation followed by reduction of neighboring terms that come from the same group.

Finally, analogously to a free group, given some generating set S, we will also need a notion of a *free vector space* on S. But, in the vector space setting, the notion we seek is natural. It is due to the following fact: every vector space has a basis.

Definition 14. *Given a set S, a vector space V is the* free vector space on S *if S is a basis of V.*

Vector spaces can also easily be seen to satisfy a universal property. Namely, given a set S, denote by $\mathbb{F}(S)$ the free vector space on S and let $\iota : S \to \mathbb{F}(S)$ denote the inclusion map; then, given any function f from S to an arbitrary vector space W, there clearly exists a unique (vector-space) homomorphism $\theta_f : \mathbb{F}(S) \to W$ such that $f = \theta_f \circ \iota$. It again follows that two free vector spaces are isomorphic if and only if their generating sets have the same cardinality; in particular, every free vector space on n generators is isomorphic to $\mathbb{R}^n$.

Finally, given a set S, the third relevant construct is that of a *free algebra on S*. It is obtained starting with the free vector space over the set of all words formed by elements of S (including the empty word), and endowing this vector space with the multiplication operation defined on the basis elements by word concatenation. The free algebra also satisfies a universal property and its isomorphism class is also determined by the cardinality of its generating set.

A.1.2 Quotients and relations

Quotient spaces arise in settings where a vector space comes with a given partition into equivalence classes *and* there is need to consider the object obtained by imposing the natural vector space structure on these equivalence classes.

Specifically, let V be a vector space over some field $\mathbb{F}$ and let W be a subspace of V. For $v_1, v_2 \in V$, set $v_1 \overset{W}{=} v_2$ (also written $v_1 = v_2 \bmod W$) if and only if $v_1 - v_2 \in W$. It is left as an (easy) exercise for the reader to verify that $\overset{W}{=}$ defines an equivalence relation. In particular, denoting by $[v]$ the equivalence class of an element $v \in V$, it is the case that $[v] = \{v + w \mid w \in W\}$. The key observation at this point is that the collection of equivalence classes $[V] := \{[v] \mid v \in V\}$ comes with a natural vector space structure, with the addition given by

$$[v_1] + [v_2] = [v_1 + v_2] \tag{A.2}$$

and the multiplication by a scalar $\alpha \in \mathbb{F}$ by

$$\alpha[v] = [\alpha v]. \tag{A.3}$$

The reader is again asked to verify that these operations are well-defined and that they satisfy the required axioms.

Definition 15. *Let $\mathbb{F}$ be a field, V a vector space over $\mathbb{F}$, and W a subspace of V. The* quotient space of V and W, *denoted V/W, is the vector space over $\mathbb{F}$ whose elements are the equivalence classes of V under $\overset{W}{=}$, with the addition and scalar multiplication as prescribed by* (A.2) *and* (A.3).

Quotient spaces will arise seamlessly throughout the rest of this thesis. For example, in the following chapter, the ability to pass to the quotient space will allow us to "collapse to 0" the kernel of a any given seminorm on a vector space, thereby obtaining a bona fide normed space. But, first, it will allow us to conclude the discussion of generators and relations.

Specifically, given some arbitrary set S, it is clear how to construct the free vector space on S. If, however, certain relations are to be imposed on the elements of S, it is instead the quotient space that provides the means for the desired construction. Indeed, starting with the free vector space $\mathbb{F}(S)$, one may use the relations on S to impose (in the obvious way) equivalence relations on the whole of F_S. Taking the quotient of $\mathbb{F}(S)$ by the vector space of the equivalence classes formed by the given relations then yields the vector space generated by S. This approach to generating a vector space in the presence of relations is natural, as starting with the free vector space over S ensures that no additional relations get inadvertently included. A very important example of this type of construction is precisely the construction of the *tensor product*, to be discussed in Section A.3.

Finally, if the vector space at hand happens to be an algebra and if the equivalence relations are also compatible with the multiplication operation, then the above discussion naturally extends to *quotient algebras*. Starting with a free algebra over a set S and some relations on S, one can thus construct an algebra whose generators satisfy the given relations. For a first taste of algebras whose generators obey certain commutation relations (a theme that will recur at several points throughout the upcoming chapters), the reader is encouraged to work through the following example.

Example 7 The goal is to construct a unital algebra over $\mathbb{C}$ with generators $\{x_i\}_{i \in I}$ satisfying

$$x_i x_j - \epsilon(i,j) x_j x_i = 0, \quad i,j \in I, \ i \neq j \tag{A.4}$$

where $\epsilon(i,j) = \epsilon(j,i) \in \{-1,1\}$ and I is a totally ordered finite set. For this, start with the free vector space over all words $x_{i_1} \dots x_{i_p}$ with $\{i_1, \dots, i_p\} \subset I$ (for $i_1 < \dots < i_p$) and $x_\emptyset = 1$, and form on it the algebra $\mathcal{A}$ by defining the multiplication operation as word concatenation. Let $\mathcal{I}$ be the two-sided ideal in $\mathcal{A}$

generated by $\{x_i x_j - \epsilon(i,j) x_j x_i \mid i,j \in I,\ i \neq j\}$. Note that $\mathcal{I}$ is a subalgebra of $\mathcal{A}$ and consider the quotient space $\mathcal{A}/\mathcal{I}$. It is easy to see that (A.4) is satisfied on $\mathcal{A}/\mathcal{I}$ and that the latter is indeed an algebra – in particular, $(a_1+\mathcal{I})(a_2+\mathcal{I}) \equiv a_1 a_2$ for every $a_1, a_2 \in \mathcal{A}$. •

A.2 Direct sum

Given two vector spaces V and W over some field $\mathbb{F}$, the cartesian product $V \times W$ can be given the structure of a vector space over $\mathbb{F}$ by defining the operations *component-wise*. Namely, let the addition and the multiplication by a scalar be given by

$$(v_1, w_1) + (v_2, w_2) = (v_1 + v_2, w_1 + w_2), \tag{A.5}$$

$$\alpha(v, w) = (\alpha v, \alpha w), \tag{A.6}$$

for any $(v_1, w_1), (v_2, w_2) \in V \times W$ and $\alpha \in \mathbb{F}$. This structure turns $V \times W$ into a vector space denoted $V \oplus W$, referred to as *the (algebraic) direct sum of V and W*. The notion of the direct sum is formulated analogously for groups, where, given two groups G and H and their cartesian product $G \times H$, the component-wise multiplication is given by

$$(g_1, h_1)(g_2, h_2) = (g_1 g_2, h_1 h_2)$$

for $g_1, g_2 \in G$ and $h_1, h_2 \in H$. These definitions extend analogously to any finite number of groups or vector spaces, by starting from the n-fold cartesian product and defining the operations component-wise.

Example 8 According to the above definition, the vector space $\mathbb{R}^3$ under the usual component-wise addition and multiplication by a real scalar is exactly the triple direct sum $\mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R}$. Note that $\mathbb{R}^3$ is also isomorphic to $\mathbb{R}^2 \oplus \mathbb{R}$ and $\mathbb{R} \oplus \mathbb{R}^2$.

The complex numbers taken as a vector space over $\mathbb{R}$ are isomorphic to $\mathbb{R} \oplus \mathbb{R}$, with the isomorphism given by $x + iy \leftrightarrow (x, y)$. Indeed, $(x_1 + x_2) + i(y_1 + y_2) = (x_1 + iy_1) + (x_2 + iy_2) \leftrightarrow (x_1, y_1) + (x_2, y_2) = (x_1 + x_2) + (y_1, y_2)$ and $\alpha(x + iy) =$

$\alpha x + i\alpha y \leftrightarrow (\alpha x, \alpha y) = \alpha(x, y)$ for $\alpha, x_1, x_2, y_1, y_2 \in \mathbb{R}$. Of course, the complex numbers taken as a vector space over $\mathbb{C}$ are not isomorphic to $\mathbb{R} \oplus \mathbb{R}$, precisely because if $x + iy \leftrightarrow (x, y)$ and $\alpha \in \mathbb{C}$, then $\alpha(x + iy)$ does generally not map to $(\alpha x, \alpha y)$.

It is a shade more subtle to note that the complex numbers taken as a commutative *algebra* over $\mathbb{R}$ are again not isomorphic to $\mathbb{R}\oplus\mathbb{R}$. Instead, $\mathbb{R}\oplus\mathbb{R}$ as a commutative algebra over the real numbers (with component-wise multiplication) is sometimes referred to as the *split complex numbers* $\hat{\mathbb{C}}$. To explain the terminology, note that an element of $\hat{\mathbb{C}}$ can be represented as $x + jy$ where $x, y \in \mathbb{R}$ and $j^2 = 1$ so that $(x_1 + x_2) + j(y_1 + y_2) = (x_1 + jy_1) + (x_2 + jy_2)$ and $(x_1 + jy_1)(x_2 + jy_2) = (x_1x_2 + y_1y_2) + j(x_1y_2 + y_1x_2)$. Consider the map sending $(x + jy)$ to $(x - y, x + y)$. The map is indeed an algebra isomorphism: it clearly preserves the component-wise addition and multiplication by a (real) scalar and, moreover, $(x_1 + jy_1)(x_2 + jy_2) = (x_1x_2 + y_1y_2) + j(x_1y_2 + y_1x_2)$ maps to $(x_1x_2 + y_1y_2 - x_1y_2 - y_1x_2, x_1x_2 + y_1y_2 + x_1y_2 + y_1x_2) = ((x_1 - x_2)(y_1 - y_2), (x_1 + x_2)(y_1 + y_2)) = (x_1 - x_2, x_1 + x_2)(y_1 - y_2, y_1 + y_2)$.
•

The definition extends analogously to any finite number of vector spaces, which, in turn, is used to define direct sums over arbitrary index sets, as follows.

Definition 16. *Given a set J and a family of vector spaces $\{V_\gamma\}_{\gamma\in J}$ over a field $\mathbb{F}$, consider the set of J-tuples $(x_\gamma)_{\gamma\in J}$ with $x_\gamma = 0$ for all but finitely γ. Then, the vector space generated by all such J-tuples with respect to the operations of component-wise addition and multiplication by a scalar is referred to as the* (algebraic) direct sum of *$\{V_\gamma\}_{\gamma\in J}$ and denoted $\bigoplus_{\gamma\in J} V_\gamma$.*

Note that implicitly referring to the above direct sums as *algebraic* is meant to indicate that if the underlying vector spaces happen to be Hilbert spaces, one should nevertheless ignore the natural inner product structure on the resulting vector space. Hilbert-space direct sums are further discussed in Appendix B.

Revisiting the free vector space $\mathbb{F}(X)$ (where X is a set and $\mathbb{F}$ a field) and comparing Definition 14 to Definition 16 above, one can show that $\mathbb{F}(X)$ is the direct sum over X of copies of $\mathbb{F}$. That is,

$$\mathbb{F}(X) = \oplus_{x \in X} \mathbb{F}. \tag{A.7}$$

A.3 Tensor product

Consider again two vector spaces V and W over some field $\mathbb{F}$. Loosely speaking, the tensor product of V and W, denoted $V \otimes W$ can be seen as arising from the free vector space $\mathbb{F}(V \times W)$ subsequently endowed with bilinear structure, akin to multiplication. There are two equivalent definitions that provide very different types of insight into the nature of this space, and are therefore both addressed in what follows.

A.3.1 $V \otimes W$ as a quotient space

The starting point is again the cartesian product $V \times W$. However, instead of the component-wise addition and multiplication by a scalar, this time we wish to endow $V \times W$ with bilinear structure, given by

$$(\alpha_1 v_1 + \alpha_2 v_2, w) = \alpha_1 (v_1, w) + \alpha_2 (v_2, w), \tag{A.8}$$

$$(v, \beta_1 w_1 + \beta_2 w_2) = \beta_1 (v, w_1) + \beta_2 (v, w_2), \tag{A.9}$$

for all $v, v_1, v_2 \in V$, $w, w_1, w_2 \in W$, $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{F}$.

One natural way to realize the above structure first passes through the free vector space $\mathbb{F}(V \times W)$, ensuring the absence of any relations. Of course, it is impossible to impose the above bilinear structure directly "on" the free vector space; indeed, for any $\alpha \in \mathbb{F}$, both (v, w) and $(\alpha v, w)$ are basis elements of $\mathbb{F}(V \times W)$, and it is therefore impossible to have $(\alpha v, w) = \alpha(v, w)$ as required by (A.8). The solution is to pass to the quotient space of $\mathbb{F}(V \times W)$ induced by the equivalence relations

$$(\alpha_1 v_1 + \alpha_2 v_2, w) \sim \alpha_1 (v_1, w) + \alpha_2 (v_2, w), \tag{A.10}$$

$$\alpha(v, \beta_1 w_1 + \beta_2 w_2) \sim \beta_1(v, w_1) + \beta_2(v, w_2). \tag{A.11}$$

Then, once $V \otimes W$ is constructed, a more explicit description in terms of its generators can also be formulated. Specifically:

Definition/Theorem 7. *If V and W are vector spaces over the field $\mathbb{F}$, their* tensor product *$V \otimes W$, is the quotient space $\mathbb{F}(V \times W)/R$, with the kernel R generated by the equivalence relations* (A.10) *and* (A.11). *As a vector space, $V \otimes W$ is spanned by all words $v \otimes w$, for $v \in V, w \in W$, with the rules of addition and multiplication by a scalar:*

$$(\alpha_1 v_1 + \alpha_2 v_2) \otimes w = \alpha_1(v_1 \otimes w) + \alpha_2(v_2 \otimes w), \tag{A.12}$$

$$v \otimes (\beta_1 w_1 + \beta_2 w_2) = \beta_1(v \otimes w_1) + \beta_2 \otimes (v, w_2) \tag{A.13}$$

for all $v, v_1, v_2 \in V$, $w, w_1, w_2 \in W$, $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{F}$.

The elements $v \otimes w$ are referred to as *pure tensors* or *elementary tensors.* Note that while pure tensors span $V \otimes W$, they do not generally form a basis; in fact, there are typically many relations between them. However, the pure tensors formed by the corresponding basis elements do form a basis of the vector-space tensor product. Specifically:

Proposition 5. *Let $\mathbb{F}(E_1)$ and $\mathbb{F}(E_2)$ be free vector spaces over the sets $E_1 = \{e_i\}_{i \in I}$ and $E_2 = \{\hat{e}_j\}_{i \in J}$. Then, $\{e_i \otimes \hat{e}_j\}_{(i,j) \in I \times J}$ is a basis for $\mathbb{F}(E_1) \otimes \mathbb{F}(E_2)$.*

Example 9 Given a field $\mathbb{F}$, let $\mathbb{F}[X]$ be the vector space of all polynomials in variable X and $\mathbb{F}[Y]$ be that of polynomials in variable Y. Clearly, $\mathbb{F}[X]$ and $\mathbb{F}[Y]$ are freely generated, with bases given by $\{1, X, X^2, X^3, \ldots\}$ and $\{1, Y, Y^2, Y^3, \ldots\}$, respectively. So, by the above Proposition 5, $\mathbb{F}[X] \otimes \mathbb{F}[Y]$ is freely generated with basis $\{X^k \otimes Y^m\}_{k,m \geq 0}$. It follows that $\mathbb{F}[X] \otimes \mathbb{F}[Y]$ isomorphic to $\mathbb{F}[X, Y]$, the vector space of polynomials in variables X and Y over $\mathbb{F}$, with the isomorphism given by $\sum_{k,m} c_{k,m} X^k \otimes Y^m \mapsto \sum_{k,m} c_{k,m} X^k Y^m$. •

For the reader who has all along been trying to develop an intuitive idea of what the tensor product "looks like", the above examples hint at a nearly satisfying description. One remaining issue concerns the question of what really *is* a pure tensor $e_i \otimes e_j$ (or more generally $v \otimes w$) and how it relates to the element $e_i \times e_j$ (resp. $v \times w$) of the cartesian product. The approach to defining the tensor product via a universal property, discussed next, has among its many advantages over the constructive formulation the benefit of rendering the nature of a pure tensor transparent.

A.3.2 $V \otimes W$ via a universal property

The more abstract (and powerful) view of the tensor product is as a space through which one can *linearize bilinear maps.* First, given two vector spaces V, W and X over $\mathbb{F}$, the map $f : V \times W \to X$ is a bilinear (or $\mathbb{F}$*-bilinear*) if

$$f(\alpha_1 v_1 + \alpha_2 v_2, w) = \alpha_1 f(v_1, w) + \alpha_2 f(v_2, w), \tag{A.14}$$

$$f(v, \beta_1 w_1 + \beta_2 w_2) = \beta_1 f(v, w_1) + \beta_2 f(v, w_2), \tag{A.15}$$

for all $v, v_1, v_2 \in V$, $w, w_1, w_2 \in W$, $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{F}$. For example, the multiplicative map $\mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is, of course, $\mathbb{R}$-bilinear, and so is the dot product $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, among many other examples.

Proposition 6. *Given vector spaces V and W over $\mathbb{F}$, there exists is a vector space W together with an $\mathbb{F}$-bilinear map $\tau : V \times W \to V \otimes W$ with the following property: for every vector space X and every bilinear map $B : V \times W \to X$, there exists a unique <u>linear</u> map $L : V \otimes W \to X$ such that $B = L \circ \tau$. Furthermore, any two such vector spaces, W and W', are isomorphic.*

A general approach to defining tensor products proceeds by Proposition 6 and interprets $V \otimes W$ as the above space W, with the understanding that $V \otimes W$ is thus specified up to isomorphism. It is another example of a definition by a universal property, as starting form the cartesian product $V \times W$, the property that there is a unique linear map to some vector space that linearizes the bilinear maps on $V \times W$ is enough to characterize that vector space. The reader may readily verify that our earlier construction of the tensor product (via the quotient space) satisfies the universal property above. Specifically, one should check that for $V \otimes W$ as given by Definition/Theorem 7, one has that for every set X and every bilinear map $B : V \times W \to X$, there exists a unique linear map L making the following diagram commute:

$$\begin{tikzcd} V \times W \arrow[r, "\tau"] \arrow[dr, "B"'] & V \otimes W \arrow[d, "L"] \\ & X \end{tikzcd}$$

Drawing on the universal property of the tensor product, it is clear that $v \otimes w$ is now simply the image of $v \times w$ under the map τ. More generally, given any linear function L defined on $V \otimes W$, to evaluate $L(v \otimes w)$ it suffices to evaluate $B(\tau^{-1}(v \otimes w))$. Though the function $V \times W \to V \otimes W$ that we presently call τ typically appears anonymously in the definition of the tensor product, it is in fact one of its crucial properties and is a key ingredient in almost every proof surrounding general properties of the tensor product.

Example 10 Let $V = \mathbb{R}^n$ with the standard basis $E = \{e_1, \dots, e_n\}$. The reader may verify that for $v_1, v_2 \in V$, the pure tensor $v_1 \otimes v_2$ in $V \otimes V$ can be represented as the outer product $v_1 v_2^{\mathrm{T}}$. Indeed, given a bilinear map $B : V \times W \to X$, the map $L : V \otimes V \to X$ defined on the basis elements by $L(e_i \otimes e_j) := B(e_i, e_j)$ extends uniquely by linearity. On the other hand, by bilinearity of B, every element in the image of B can be written as $B(\sum_i \alpha_i e_i, \sum_j \beta_j e_j) = \sum_{i,j} \alpha_i \beta_j B(e_i, e_j) = \sum_{i,j} \alpha_i \beta_j L(e_i \otimes e_j)$. One may now verify that $\sum_{i,j} \alpha_i \beta_j L(e_i \otimes e_j) = L(\sum_{i,j} \alpha_i \beta_j e_i \otimes e_j) = L(\sum_i \alpha_i e_i \otimes \sum_j \alpha_j e_j)$, where the first equality follows by the linearity of L and the second by the nature of the outer product. •

Example 11 Now let $V = \mathrm{End}(\mathbb{R}^n)$ be the space of real $n \times n$ matrices (also denoted $\mathscr{M}_n(\mathbb{R})$ in Chapter 3). Proceeding as in the previous example, the reader may show that the vector space $V \otimes V$ may be taken as the space of all real $n^2 \times n^2$ matrices with the map $\tau : \mathrm{End}(\mathbb{R}^n) \times \mathrm{End}(\mathbb{R}^n) \to \mathrm{End}(\mathbb{R}^n) \otimes \mathrm{End}(\mathbb{R}^n)$ given by the *Kronecker product*. In that case, setting $n = 2$ and taking two matrices

$A, B \in \mathrm{End}(\mathbb{R}^2)$, the pure tensor $A \otimes B$ is given as:

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix}, B = \begin{bmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \end{bmatrix}, \quad A \otimes B = \begin{bmatrix} a_{1,1}b_{1,1} & a_{1,1}b_{1,2} & a_{1,2}b_{1,1} & a_{1,2}b_{1,2} \\ a_{1,1}b_{2,1} & a_{1,1}b_{2,2} & a_{1,2}b_{2,1} & a_{1,2}b_{2,2} \\ a_{2,1}b_{1,1} & a_{2,1}b_{1,2} & a_{2,2}b_{1,1} & a_{2,2}b_{1,2} \\ a_{2,1}b_{2,1} & a_{2,1}b_{2,2} & a_{2,2}b_{2,1} & a_{2,2}b_{2,2} \end{bmatrix}.$$

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A.3.3 Finite tensor products and tensor powers

The previous sections treated the tensor product of two vector spaces, but the definitions generalize analogously to any finite number of vector spaces. (There will be no need in this thesis to consider infinite tensor products, though such a thing is feasible.) First, an extension of bilinearity is called *multilinearity*, where for vector spaces $V_1, \ldots, V_n, Y$ over a field $\mathbb{F}$ a map $f : V_1 \times \ldots \times V_n \to Y$ is multilinear (or n-multilinear) if f is linear in each coordinate while keeping the other coordinates are fixed. By replacing bilinearity by n-multilinearity directly generalizes both of the previous approaches for defining the (n-fold) tensor product and yields $V_1 \otimes \ldots \otimes V_n$.

An important example of an n-fold tensor products is a tensor power $V^{\otimes k}$, defined as the tensor product of V with itself k times. Revisiting Example 11 with $V = \mathrm{End}(\mathbb{R}^n)$, the pure tensors in $V^{\otimes k}$ can be given as the k-fold Kronecker products of matrices. In general, in addition to identifying $V^{\otimes 1} = V$, it is frequently convenient to set $V^{\otimes 0} = \mathbb{F}$. Thus, given a vector space V, one may consider the sequence of vector spaces $V^{\otimes 0}, V^{\otimes 1}, V^{\otimes 2}, \ldots$.

Example 11 brings up another point: what may become of a linear map on V when one instead considers the tensor power $V^{\otimes k}$? It turns out that given vector spaces V and W, one can naturally "boost" any linear transformation $V \to W$ to a linear transformation $V^{\otimes k} \to W^{\otimes k}$, for any $k \in \mathbb{N}$. The reader may verify that when V and W are finite-dimensional vector spaces, one is fundamentally back to dealing with Kronecker products of matrices. More generally:

Proposition 7. *Given vector spaces $V_1, \ldots, V_n$ and $V_1', \ldots, V_n'$, consider the linear transformations $T_i : V_i \to V_i'$ for $i = 1, \ldots, n$. There exists a unique map $T_1 \otimes \ldots \otimes T_n : V_1 \otimes \ldots \otimes V_n \to V_1' \otimes \ldots \otimes V_n'$ that on elementary tensors evaluates to*

$$T_1 \otimes \ldots \otimes T_n(v_1 \otimes \ldots \otimes v_n) = T_1(v_1) \otimes \ldots \otimes T_n(v_n),$$

for all $v_1, \ldots, v_n \in V$.

The above proposition is a fairly direct consequence of the universal property of the tensor products, and may therefore provide another illustration as to why this somewhat more abstract approach to defining the tensor product is generally superior to the explicit construction.

A.3.4 Tensor algebras (algebraic Fock spaces)

This section concerns the algebraic object that plays a central role in this thesis, namely that of the *algebraic Fock space.* As previously discussed, any vector space V gives rise to the sequence of tensor powers $V^{\otimes 0}, V^{\otimes 1}, V^{\otimes 2}, \ldots$. It is thus natural to consider the setting in which these tensor powers can coexist, that is, the vector space

$$\bigoplus_{n \geq 0} V^{\otimes n}.$$

By the properties developed in the previous sections, the above direct sum is the vector space spanned by the elementary tensors $\{v_1 \otimes \ldots \otimes v_n \mid n \in \mathbb{N}, v_1, \ldots, v_n \in V\}$. This observation also suggests that there may be more than the vector space structure present on the above direct sum; in particular, there is a natural notion of vector multiplication that is defined on the pure tensors by $(v_1 \otimes v_n, v_1' \otimes v_m') \mapsto v_1 \otimes v_n \otimes v_1' \otimes v_m'$. However, there is a potential problem with extending this multiplication to the entire vector space because, given an arbitrary element of $\bigoplus_{n \geq 0} V^{\otimes n}$, its decomposition into elementary tensors is highly non-unique and the outcome may a priori depend on this decomposition. This can be remedied either by a choice of a basis (though one must then also show that the map is invariant under the choice of basis) or (more cleanly) by working with the universal property. Either way, the outcome is that:

Proposition 8. *For any vector space V and positive integers n and m, there is a unique bijective bilinear map $V^{\otimes n} \times V^{\otimes m} \to V^{\otimes(n+m)}$ satisfying $(v_1 \otimes \ldots \otimes v_n, v_1' \otimes \ldots \otimes v_m') \to v_1 \otimes \ldots v_n \otimes v_1' \otimes \ldots v_m$ for all $v_1, \ldots, v_n, v_1', \ldots, v_m \in V$.*

The above multiplication is clearly distributive and compatible with scalar multiplication, and can also be readily be shown to be associative. Once we know how to multiply the tensor powers, the multiplication when taken as distributive over the addition then uniquely extends in the natural manner to the whole of $\bigoplus_{n\geq 0} V^{\otimes n}$. For concreteness, every element $w \in \bigoplus_{n\geq 0} V^{\otimes n}$ can be written as some (finite) sum of elements in $V^{\otimes n_1}, \ldots, V^{\otimes n_k}$, where the choice of k (when taken to be minimal) and of $n_1, \ldots, n_k$ is unique. Taking $w, w' \in \bigoplus_{n\geq 0} V^{\otimes n}$ and writing $w = \sum_i w_i, w' = \sum_j w_j'$, where $w_i \in V^{\otimes n_i}, w_j' \in V^{\otimes m_j}$, the resulting multiplication map on $\bigoplus_{n\geq 0} V^{\otimes n}$ satisfies

$$ww' = \sum_i w_i \sum_j w_j' = \sum_{i,j} w_i w_j',$$

with the multiplication on $V^{\otimes n} \times V^{\otimes m}$ as given in Proposition 8 above. It is then straightforward to verify that this vector multiplication turns the corresponding vector space into an algebra, leading to the following definition.

Definition 17. *Given a vector space V, the* tensor algebra *on V is the vector space $\bigoplus_{n\geq 0} V^{\otimes n}$, with the multiplication law given as the natural distributive extension of the multiplication on $V^{\otimes n} \times V^{\otimes m}$.*

Given quantum field theoretic interpretation, with the n^{th} tensor power $V^{\otimes n}$ representing the possible quantum states of an n-particle system, the algebra $\bigoplus_{n\geq 0} V^{\otimes n}$ is referred to as the *algebraic Fock space* over the vector space V.

A.4 Complexification

Complexification is the procedure for "enlarging" a real vector space to a complex vector space in a natural way; let us point out ahead of time that the complexification of $\mathbb{R}^n$ is isomorphic to $\mathbb{C}^n$, and that the complexification of $\mathbb{R}[X]$ is isomorphic to $\mathbb{C}[X]$.

The need to enlarge a vector space arises in many instances in mathematics, and it also arises throughout this thesis. In fact, it occurs at such a fundamental level that most of the time, it will be easy to forget that one is working with a complex vector space. (That is, the fundamental object will be the enlarged vector space, but we will spend most of our time considering what happens to at the level of the real vector space, thanks to Propositions 9 through 11.)

There are two equivalent ways to define complexification: as a direct sum or as a tensor product.

A.4.1 Complexification by direct sum

The idea of complexifying a vector space V via a direct sum comes from thinking of the pair $(v_1, v_2) \in V \times V$ as the formal sum $v_1 + iv_2$, with a suitably defined multiplication by complex scalars. Concretely:

Definition 18. *Given a real vector space V, the* complexification *of V is the vector space $V_{\mathbb{C}} = V \oplus V$ with multiplication law $(\alpha + i\beta)(v_1, v_2) = (\alpha v_1 - \beta v_2, \beta v_1 + \alpha v_2)$.*

Let us consider a few consequences of the above definition. First, given a real vector space V with basis $\{e_i\}$, it is clear that every element of $V_{\mathbb{C}}$ can be written as some linear combination of elements (e_i, e_j). Thus, the set $\{(e_i, e_j)\}$ spans $V_{\mathbb{C}}$, but it is not a basis! Indeed, by the multiplication rule, $i(0, e_j) = (e_j, 0)$. A basis of $V_{\mathbb{C}}$ can instead be taken as follows:

Proposition 9. *If V is a real vector space with an $\mathbb{R}$-basis $\{e_j\}$, then $V_{\mathbb{C}}$ is a complex vector space with a $\mathbb{C}$-basis $\{(e_j, 0)\}$, with the scalar multiplication as given in Definition 18.*

In light of the above proposition, one can consider V as "living" inside $V_{\mathbb{C}}$, which is formally prescribed by the embedding $v \mapsto (v, 0)$. Furthermore, the reader may now see that the complexification of $\mathbb{R}^n$ is isomorphic to $\mathbb{C}^n$ and that the complexification of $\mathbb{R}[X]$ is isomorphic to $\mathbb{C}[X]$. The next important question to address concerns what becomes of linear transformations in this setting.

Proposition 10. *Given two real vector spaces V, V', every $\mathbb{R}$-linear transformation $L : V \to V'$ extends to a unique $\mathbb{C}$-linear transformation $V_{\mathbb{C}} \to V'_{\mathbb{C}}$. Specifically, there is a unique $\mathbb{C}$-linear map $L_{\mathbb{C}} : V_{\mathbb{C}} \to V'_{\mathbb{C}}$ making the following diagram commute:*

$$\begin{tikzcd} V \arrow[r, "L"] \arrow[d] & V' \arrow[d] \\ V_{\mathbb{C}} \arrow[r, "L_{\mathbb{C}}"] & V'_{\mathbb{C}} \end{tikzcd}$$

where the vertical arrows signify the standard embeddings of V into $V_{\mathbb{C}}$ and V' into $V'_{\mathbb{C}}$ given by $v \mapsto (v, 0)$.

In particular, in the matricial setting:

Proposition 11. *Consider two (non-zero) real finite-dimensional vector spaces V, V' with bases $\{e_i\}$ and $\{e'_i\}$, respectively, and let $L : V \to V'$ be an $\mathbb{R}$-linear transformation. Then, the matrix for L with respect to the $\mathbb{R}$-bases $\{e_i\}$ and $\{e'_i\}$ is equal to the matrix for $L_{\mathbb{C}} : V_{\mathbb{C}} \to V'_{\mathbb{C}}$ with respect to the $\mathbb{C}$-bases $\{(e_i, 0)\}$ and $\{(e'_i, 0)\}$.*

For example, given a 2×2 matrix $M : \mathbb{R}^2 \to \mathbb{R}^2$, the corresponding complexification $M_{\mathbb{C}} : (\mathbb{R}^2)_{\mathbb{C}} \to (\mathbb{R}^2)_{\mathbb{C}}$ is given by component-wise action of M on $(\mathbb{R}^2)_{\mathbb{C}}$, that is, as $(v_1, v_2) \mapsto (Mv_1, Mv_2)$. Overall, the answer is that linear transformations behave just as naturally with respect to complexifications as one may hope.

A.4.2 Complexification by tensor product

An alternative, and to some extent simpler, approach complexification is via the tensor product. Given a real vector space V, this time we will wish to consider the complexification of V to be the vector space $\mathbb{C} \otimes_{\mathbb{R}} V$ (where the subscript on the tensor symbol is to make explicit the fact that the tensor product is taken over $\mathbb{R}$).

In this setting, the reader may verify that there is again a natural identification of V as belonging to $\mathbb{C} \otimes_{\mathbb{R}} V$, by the embedding $v \to 1 \otimes v$. Similarly, any $\mathbb{R}$-basis $\{e_i\}$ of V gives rise to a $\mathbb{C}$-basis $\{1 \otimes e_j\}$. (Note that in later chapters, we will frequently be working with complexifications of real vector spaces, rather than starting out with complex vector spaces, precisely for the benefit of working with a basis formed by real tensors.)

An analogue of Proposition 10 is also available and, given two real vector spaces V and V', an $\mathbb{R}$-linear map $L : V \to V'$ can be identified with the $\mathbb{C}$-linear map $1 \otimes L : \mathbb{C} \otimes_{\mathbb{R}} V \to \mathbb{C} \otimes_{\mathbb{R}} V'$. In the finite-dimensional case, the analogue of Proposition 11 holds and given a linear map $L : V \to V'$, the matrix for L with respect to $\{e_i\}$ is equal to the matrix for $1 \otimes L$ with respect to $\{1 \otimes e_i\}$.

Given the above similarities between the two approaches to complexification, it is little surprise that the analogy is in fact systematic. Namely, given a real vector space V, the structures obtained by the two types of complexification are equivalent (up to unique isomorphism):

Proposition 12. *For every real vector space V and $V_{\mathbb{C}}$ its complexification of Definition 18, there is a unique isomorphism $\tau_V : V_{\mathbb{C}} \to \mathbb{C} \otimes_{\mathbb{R}} V$ of $\mathbb{C}$-vector spaces for which the following diagram commutes:*

$$\begin{tikzcd} V \arrow[r] \arrow[dr] & V_{\mathbb{C}} \arrow[d, "\tau_V"] \\ & \mathbb{C} \otimes_{\mathbb{R}} V \end{tikzcd}$$

where the arrows out of V denote the two familiar types of standard embeddings into the corresponding spaces. Moreover, if V and V' are real vector and if $L : V \to V'$ is an $\mathbb{R}$-linear map with complexification $L_{\mathbb{C}}$ given by Proposition 10, then the following diagram of $\mathbb{C}$-linear maps commutes:

$$\begin{tikzcd} V_{\mathbb{C}} \arrow[r, "L_{\mathbb{C}}"] \arrow[d, "\tau_V"'] & V'_{\mathbb{C}} \arrow[d, "\tau_{V'}"] \\ \mathbb{C} \otimes_R V \arrow[r, "1\otimes L"] & \mathbb{C} \otimes_R V' \end{tikzcd}$$

The interested reader in need of exercise may wish to look for a way to write down the action of τ_V on an element $(v_1, v_2) \in V_{\mathbb{C}}$. The answer may be found below.[3] But, this completes our discussion of the algebraic preliminaries.

[3] Answer: $\tau_V(v_1, v_2) = 1 \otimes v_1 + i \otimes v_2$.

Appendix B

The material in this appendix offers a brief review of the prerequisite notions concerning normed vector spaces, Banach spaces, and Hilbert spaces. Many of the definitions and theorems of the present chapter are drawn from John B. Conway's *Course in Functional Analysis* [Con90] and the introductory chapters of the first volume of the *Fundamentals of the theory of operator algebras* by Kadison and Ringrose [KR97]. Most of the examples are drawn from classical analysis, with frequent references to the $\mathscr{L}^p$ spaces, for which the reader is referred to [LL01]. Readers who may be more inclined to focus on the Hilbert-space structure of the $\mathscr{L}^2$ spaces and the concrete benefits associated with such structure are referred to [SS03].

B.1 Normed vector spaces and Banach spaces

A norm is a general notion of length in vector spaces, perhaps the most familiar example of which is the Euclidean norm on the vector space $\mathbb{R}^n$.

Definition 19. *If $\mathscr{X}$ is a vector space over a field $\mathbb{F}$ (to be taken throughout as $\mathbb{R}$ or $\mathbb{C}$), a* norm *is a function $\| \, \| : \mathscr{X} \to [0, \infty)$ with the properties:*

i. $\|x + y\| \leq \|x\| + \|y\|$ *for all* $x, y \in \mathscr{X}$,

ii. $\|\alpha x\| = |\alpha| \|x\|$ *for all* $\alpha \in \mathbb{F}$ *and* $x \in \mathscr{X}$.

iii. $\|x\| = 0$ *only if* $x = 0$.

A *normed space* is a pair $(\mathscr{X}, \| \, \|)$, where $\mathscr{X}$ is a vector space and $\| \, \|$ a norm on $\mathscr{X}$. Every normed space can easily be verified to be metrizable as the distance $\|x - y\|$ for all $x, y \in \mathscr{X}$ indeed defines a metric. A *Banach space* is a normed space that is complete with respect to the metric defined by the norm.

Example 12 Let X be a Hausdorff space, that is, a space endowed with a topology in which any two points have disjoint neighborhoods, i.e. in which any two points can be separated by open sets. (Note that this form of "separability" is, in a sense, the second most elementary property a topology can carry and, from now on, all spaces will be assumed to be Hausdorff.)

Consider the vector space $\mathscr{C}(X)$ of all continuous functions $f : X \to \mathbb{F}$, where the addition of any two functions f and g is given point-wise, by $(f+g)(x) = f(x)+g(x)$ for all $x \in X$, and multiplication by a scalar α is also defined point-wise, as $(\alpha f)(x) = \alpha f(x)$ for all $x \in X$. The subspace of $\mathscr{C}(X)$ given by the *bounded functions*, i.e. functions $f \in \mathscr{C}(X)$ for which $\sup\{|f(x)| \mid x \in X\} < \infty$, is typically denoted $\mathscr{C}_b(X)$ and carries a natural norm. Namely, the mapping $\| \, \| : \mathscr{C}_b(X) \to \mathbb{F}$ given by $\|f\| = \sup\{|f(x)| \mid x \in X\}$ is indeed a norm, referred to as the *uniform norm*.

In practice (or whenever the need to integrate arises), it is useful to consider two subspaces of $\mathscr{C}_b(X)$. For this, it is necessary to place an additional restriction on X, requiring to be locally compact, that is, carrying a topology in which every point has an open neighborhood whose closure is compact. In this setting, one interesting subspace of $\mathscr{C}_b(X)$ is the vector space $\mathscr{C}_0(X)$ formed by the continuous functions that *vanish at infinity*, i.e. continuous functions $f : X \to \mathbb{F}$ having the property that for all $\epsilon > 0$, $\{x \in X \mid f(x)| \geq \epsilon\}$ is compact. Another is the vector space $\mathscr{C}_{00}(X)$ (also denoted $\mathscr{C}_c(X)$) formed by the compactly supported continuous functions. (The reader may verify that $\mathscr{C}_0(X)$ is the closure of $\mathscr{C}_{00}(X)$ in the topology induced by the uniform norm.) Of course, if X happens to be a compact space, then $\mathscr{C}(X) = \mathscr{C}_b(X) = \mathscr{C}_0(X) = \mathscr{C}_{00}(X)$. •

In discussing both general results and concrete examples, we will make frequent use of the notion of equivalence in normed spaces, which is that of isometric isomorphism:

Definition 20. *If $\mathscr{X}$ and $\mathscr{Y}$ are normed spaces, $\mathscr{X}$ and $\mathscr{Y}$ are said to be* isometrically isomorphic *if there exists a surjective linear isometry from $\mathscr{X}$ onto $\mathscr{Y}$, that is, if there exists a linear map $T : \mathscr{X} \to \mathscr{Y}$ such that $T(\mathscr{X}) = \mathscr{Y}$ and $\|T(x)\| = \|x\|$ for all $x \in \mathscr{X}$.*

The previous discussion focused on the normed spaces, but one is frequently presented with a weaker notion. Namely, a function $\mathscr{X} \to [0, \infty)$ satisfying conditions *i* and *ii* of the Definition 19 is instead said to be a *seminorm.* Passing from a seminorm to a norm corresponds to prohibiting the existence of "non-zero vectors of zero length". In fact, every vector space is a priori endowed with a seminorm, namely the *trivial seminorm,* assigning length zero to every element of the space. In some contexts, the existence of a given (non-trivial) seminorm will in fact be enough to derive sufficient amounts of structure for the application at hand. However, in most situations, there is a distinct advantage to instead passing to *equivalence classes* of elements in order to restrict the analysis to a setting where elements different than 0 have strictly positive length. This is accomplished via the quotient space (cf. Appendix A), as in the following example.

Example 13 Let (X, Ψ, μ) be a measure space consisting of a set X, a σ-algebra Ψ of subsets of X, and a (positive) measure $\mu : \Psi \to [0, \infty]$. Let $\mathscr{X}$ be the class of μ-measurable functions $f : \Psi \to \mathbb{C}$ that are p-summable, that is, for which $\left(\int_\Psi |f(x)|^p \, d\mu(x)\right)^{1/p} < \infty$. It can be easily seen that $\mathscr{X}$ is a vector space under point-wise addition and multiplication by scalars.[1] The natural candidate for the norm on $\mathscr{X}$ will be denoted $\| \, \|_p$ and given by $\|f\|_p = \left(\int_\Psi |f(x)|^p \, d\mu(x)\right)^{1/p}$. The reader may verify that $\| \, \|_p$ is a seminorm, but it is not a norm. Indeed, $\| \, \|_p$ evaluates to zero for any any function $f \in \mathscr{X}$ that vanishes everywhere except on a set of zero measure.

At this point, one can easily show that the vector space obtained by dividing out the kernel of $\| \, \|_p$ is a normed space, typically referred to as $\mathscr{L}^p(X, \Psi, \mu)$ (or,

[1]To see this, the reader may wish to recall that for any $\alpha, \beta \in \mathbb{C}$ and $p \geq 1$, $|\alpha + \beta|^p \leq 2^{p-1}(|\alpha|^p + |\beta|^p)$. It immediately follows that for $f, g \in \mathscr{X}$, one must also have $af + bg \in \mathscr{X}$ for all $a, b \in \mathbb{C}$.

simply $\mathscr{L}^p(\mu)$ when the choice of the underlying measurable space is clear). In other words, letting N be the vector space formed by all functions $f \in \mathscr{X}$ that vanish everywhere except on a set of zero measure, the space $\mathscr{L}^p(\mu)$ is given as the quotient $\mathscr{X}/N$. The elements of $\mathscr{L}^p(\mu)$ are therefore equivalence classes of functions that look the same if one ignores what happens on sets of measure zero.

It is an exercise in elementary analysis to check that $\mathscr{L}^p(\mu)$ as defined above is complete, so it may from now on serve as one of our standard examples of Banach spaces. •

On a related note, one may wonder at what generally becomes of a normed space or a Banach space after taking of a quotient by its subspace. This is addressed next.

B.1.1 Quotient spaces as normed spaces

Let $\mathscr{X}$ be a normed space and consider a subspace $\mathscr{X}_0$ of $\mathscr{X}$, where $\mathscr{X}_0$ is not a priori assumed to be closed.[2] Section A.1.2 defined the vector space $\mathscr{X}/\mathscr{X}_0$ and the present concern is with the norm properties of this quotient. First, the reader may verify that the following defines a seminorm on $\mathscr{X}/\mathscr{X}_0$:

$$\| [x] \| := \inf\{\|x+y\|; y \in \mathscr{X}_0\} = \inf\{\|x-y\|; y \in \mathscr{X}_0\} \tag{B.1}$$

for each $[x] \in \mathscr{X}/\mathscr{X}_0$ (in the notation of Section A.1.2). Note that the second equality in (B.1), which follows from the fact that $\mathscr{X}_0$ is a vector space (in particular, if $y \in \mathscr{X}_0$, then $-y \in \mathscr{X}_0$), lends the corresponding functional $\| \; \|$ a clear geometric interpretation: $\| [x] \|$ is simply the *distance* from x to $\mathscr{X}_0$.

Now note that if $\mathscr{X}_0$ is not closed, the seminorm in (B.1) is *not* a norm. Indeed, there must exist some $x \in \mathscr{X}$ not equivalent to zero (i.e. $[x] \neq [0]$) for which the infimum of $\{\|x+y\|; y \in \mathscr{X}_0\}$ vanishes – simply choose x to be a limit point of $\mathscr{X}_0$ that does not belong to it.

[2] *Closure* refers to closure with respect to the subspace topology, with the topology on $\mathscr{X}$ being the default (i.e. norm) topology – see the following section for a discussion of topologies on normed spaces.

If $\mathscr{X}_0$ is closed, then $\mathscr{X}/\mathscr{X}_0$ will generally behave as one may hope. In particular, we have the following:

Theorem 12. *If $\mathscr{X}$ is a normed space and $\mathscr{X}_0$ a closed subspace of $\mathscr{X}$, then $\| \, \|$ as defined in* (B.1) *is a norm on $\mathscr{X}/\mathscr{X}_0$. If $\mathscr{X}$ is a Banach space, so is $\mathscr{X}/\mathscr{X}_0$. Moreover, the map $x \mapsto [x]$ is bounded, has norm less than one, and is continuous.*

B.1.2 Topologies on $\mathscr{X}$

Normed vector spaces come with a natural topology. Specifically, in the setting of a normed space $(\mathscr{X}, \| \, \|)$, the default topology considered will be the *norm topology*, generated by the sets $\{x \in \mathscr{X}; \|x - x_0\| < \epsilon\}$ for all $x_0 \in \mathscr{X}$ and $\epsilon > 0$.

The fact of having a topology on a vector space $\mathscr{X}$ allows one to consider the continuity of functions defined on $\mathscr{X}$. In turn, having this topology come from a norm ensures that two elementary functions are continuous. Specifically, one may check that if $\mathscr{X}$ is a normed space, then

i. the function $\mathscr{X} \times \mathscr{X} \to \mathscr{X}$ given by $(x, y) \mapsto x + y$ is continuous;

ii. the function $\mathbb{F} \times \mathscr{X} \to \mathscr{X}$ given by $(\alpha, x) \mapsto \alpha x$ is continuous.

More generally, a vector space that may not come with a norm, but comes with a topology in which the above two maps are continuous, is referred to as a *topological vector space.*

If $\mathscr{X}$ is a normed space, a second useful topology on $\mathscr{X}$ is derived from the bounded[3] linear functionals. It is referred to as the *weak topology* and is generated by the family of seminorms on $\mathscr{X}$ given by $x \mapsto |L(x)|$ and taken over all bounded linear functionals $L : \mathscr{X} \to \mathbb{F}$. In particular, the weak topology is generated the sets $\{x \in \mathscr{X} \mid \|L(x - x_0)\| < \epsilon\}$ for all $x_0 \in \mathscr{X}$, L a bounded linear functional on $\mathscr{X}$, and $\epsilon > 0$. The weak topology turns $\mathscr{X}$ into a topological vector space with the additional property that 0 is the only element that is in the kernel of all of the seminorms that

[3]For a definition and further discussion of the bounded linear functionals on $\mathscr{X}$, the reader is referred ahead to Sections B.1.3 and B.1.4.

generate it – i.e. the weak topology turns $\mathscr{X}$ into a *locally convex space.* In that sense, the weak topology retains some of the key properties of the norm topology. Of course, as its name indicates, the weak topology is coarser than the norm topology, as the generating sets of the former are also generators of the latter (but not vice-versa).

Two norms will said to be *equivalent* if they define the same topology. Equivalently:

Definition 21. *Two norms* $\| \|_1$ *and* $\| \|_2$ *on a vector space* $\mathscr{X}$ *are said to be equivalent if there exist positive constants* c_-, c_+ *such that for all* $x \in \mathscr{X}$*,*

$$c_- \|x\|_1 \leq \|x\|_2 \leq c_+ \|x\|_1.$$

The following theorem shows that as far as finite-dimensional vector spaces are concerned, the topological properties of the space are those given by the Euclidean norm. However, the reader should keep in mind that the metric properties are generally affected by the choice of the norm.

Theorem 13. *If* $\mathscr{X}$ *is a finite dimensional vector space over* $\mathbb{F}$*, then any two norms on* $\mathscr{X}$ *are equivalent.*

On a more general note, throughout this chapter, the reader should feel free to specialize the discussion to the finite-dimensional setting, thus returning to the realm of linear algebra. However, the same reader should also be forewarned that the spaces that will interest us in this thesis – namely, the normed spaces built on tensor algebras of Section A.3.4 – will be infinite-dimensional.

B.1.3 Linear Operators

When considering maps between two normed vector spaces, the notions of linearity, continuity, and boundedness are intimately connected. First, a map between two vector spaces is typically referred to as an *operator.* When considering an operator from a normed vector space into another, the underlying notion of distance permits to distinguish a class of operators that are *bounded.*

Definition 22. *Given $\mathscr{X}$ and $\mathscr{Y}$ two normed vector spaces, a map $A : \mathscr{X} \to \mathscr{Y}$ is said to be* bounded *if there is a positive constant c such that $\|Ax\| \leq c\|x\|$ for all $x \in \mathscr{X}$.*

For $\mathscr{X}$ and $\mathscr{Y}$ normed vector spaces, the class of bounded linear operators $\mathscr{X} \to \mathscr{Y}$ is denoted $\mathscr{B}(\mathscr{X}, \mathscr{Y})$. Similarly, the class of bounded linear operators $\mathscr{X} \to \mathscr{X}$ is denoted $\mathscr{B}(\mathscr{X})$. Considering the previous definition, there is a natural norm on $\mathscr{B}(\mathscr{X})$, turning $\mathscr{B}(\mathscr{X})$ itself into a normed vector space. In particular, the reader is invited to verify the equivalences stated in the following definition.

Definition 23. *Let $\mathscr{X}$ be a normed vector space. The* norm *of an operator $A \in \mathscr{B}(X)$, denoted $\|A\|$ is given by*

$$\begin{aligned}
\|A\| &= \sup\{\|Ax\| \mid \|x\| \leq 1\} \\
&= \sup\{\|Ax\| \mid \|x\| = 1\} \\
&= \sup\{\|Ax\|/\|x\| \mid x \neq 0\} \\
&= \inf\{c > 0 \mid \|Ax\| \leq c\|x\| \textit{ for } x \in \mathscr{X}\}.
\end{aligned}$$

It is a feature of normed vector spaces that bounded linear operators are continuous, and that, conversely, continous linear operators are bounded. More concretely:

Proposition 13. *Let $\mathscr{X}$ and $\mathscr{Y}$ be normed vector spaces and $A : \mathscr{X} \to \mathscr{Y}$ a linear transformation. The following statements are equivalent:*

i. $A \in \mathscr{B}(\mathscr{X}, \mathscr{Y})$.

ii. A is continuous at 0.

iii. A is continuous at some point.

Note that in the following chapter, it will be important to recall that the identity will always belong to the bounded linear operators, as the corresponding algebra of operators will be *unital.* Of course, there are certainly many examples of bounded linear operators on the familiar normed spaces and the reader is encouraged to look for some concrete examples. The following two will be of use to us further on.

Example 14 If (X, Ψ, μ) is a σ-finite measure space and $f \in \mathscr{L}^{\infty}(X, \Psi, \mu)$, define $M_f : \mathscr{L}^p(X, \Psi, \mu) \to \mathscr{L}^p(X, \Psi, \mu)$, $1 \leq p \leq \infty$, by $M_f g = fg$ for all g in $\mathscr{L}^p(X, \Psi, \mu)$. Then $M_f \in \mathscr{B}(\mathscr{L}^p(X, \Psi, \mu))$ and $\|M_f\| = \|f\|_{\infty}$. •

Example 15 If X and Y are compact spaces and $\tau : Y \to X$ is a continuous map, define $A : \mathscr{C}(X) \to \mathscr{C}(Y)$ by $(Af)(y) = f(\tau(y))$. Then $A \in \mathscr{B}(\mathscr{C}(X), \mathscr{C}(Y))$ and $\|A\| = 1$. •

When considering bounded linear transformations that also happen to be bijective, it is natural to ask what may become of their inverses in this framework. The fact that this inverse will also be bounded is the content of the Inverse Mapping Theorem, which will be further invoked in Appendix C in the context of the *operator spectra.*

Theorem 14 (The Inverse Mapping Theorem). *If $\mathscr{X}$ and $\mathscr{Y}$ are Banach spaces and $A : \mathscr{X} \to \mathscr{Y}$ is a bounded linear transformation that is bijective, then A^{-1} is bounded.*

As far as the background for this thesis, we will be mostly interested in the properties of bounded linear operators on Hilbert spaces, discussed shortly. As for the general theory of bounded linear operators on Banach spaces, the results of interest will be more at home in the setting of Appendix C, and the remainder of the discussion is postponed until then.

B.1.4 Linear Functionals

Given a normed space $\mathscr{X}$, a linear transformation $L : \mathscr{X} \to \mathbb{F}$ is referred to as a *linear functional.* By Proposition 13, the *continuous linear functionals* are the same as the *bounded linear functionals*, and the norm of a linear functional is given by

$$\|L\| := \sup\{|L(x)|; \|x\| \leq 1\}, \tag{B.2}$$

with the remaining equivalent expressions of Definition 23 specializing analogously.

The change in terminology associated with specializing the image of a bounded linear operator is meant to indicate that the resulting objects have special structure. Given a normed space $\mathscr{X}$, let $\mathscr{X}^\star$ denote[4] the collection of all bounded linear functionals on $\mathscr{X}$, i.e. $\mathscr{X}^\star = \mathscr{B}(\mathscr{X}, \mathbb{F})$. Note that $\mathscr{X}^\star$ is indeed a vector space under point-wise addition and multiplication by a scalar; in particular, for any $\alpha, \beta \in \mathbb{F}$ and $L_1, L_2 \in \mathscr{X}^\star$, $(\alpha L_1 + \beta L_2)(x) = \alpha L_1(x) + \beta L_2(x)$ for all $x \in \mathscr{X}$. The space $\mathscr{X}^\star$ is referred to as the *dual space* of $\mathscr{X}^\star$.

The following fact will provide the starting point for the discussion of the Appendix C.

Proposition 14. *If $\mathscr{X}$ is a normed space, then $\mathscr{X}^\star$ is a Banach space. More generally, if $\mathscr{X}^\star$ is non-trivial (i.e. $\mathscr{X}^\star \neq \{0\}$), then $\mathscr{B}(\mathscr{X}, \mathscr{Y})$ is a Banach space if and only if $\mathscr{Y}$ is a Banach space.*

Returning to the familiar function spaces, the following classical theorems, which are a staple of any graduate course in analysis (e.g. [LL01]), provide an illustration of how these notions arise in a more "concrete" setting.

Example 16 Let (X, Ψ, μ) be a measure space and let $1 < p < \infty$. If $1/p + 1/q = 1$ and $g \in L^q(X, \Psi, \mu)$, define $F_g : \mathscr{L}^p(\mu) \to \mathbb{F}$ by

$$F_g(f) = \int f\, g\, d\mu.$$

Then $F_g \in \mathscr{L}^p(\mu)^\star$ and the map $g \mapsto F_g$ defines an isometric isomorphism of $\mathscr{L}^q(\mu)$ onto $\mathscr{L}^p(\mu)^\star$. The analogous result also holds for $p = 1$ and $q = \infty$. •

Example 17 Let X be a locally compact space and let $M(X)$ denote the space of all $\mathbb{F}$-valued regular Borel measures on X, with the norm on $M(X)$ given by the total variation norm. For $\mu \in M(X)$, define $F_\mu : \mathscr{C}_0(X) \to \mathbb{F}$ by

$$F_\mu(f) = \int f\, d\mu.$$

[4] Actually, the usual notation is $\mathscr{X}^*$, but we prefer to use the modified symbol $\star$ to avoid confusion with the conjugation operation, introduced shortly.

Then $F_\mu \in \mathscr{C}_0(X)^\star$ and the map $\mu \mapsto F_\mu$ defines an isometric isomorphism of $M(X)$ onto $\mathscr{C}_0(X)^\star$. •

As a normed space, the dual space $\mathscr{X}^\star$ can be topologized in a obvious way and the corresponding topology is referred to as the *norm topology* or *strong topology* on $\mathscr{X}^\star$. On the other hand, another natural choice is the topology on $\mathscr{X}^\star$ induced by the family of seminorms $L \mapsto L(x)$ taken over all (fixed) $x \in \mathscr{X}$, referred to as the *weak* topology*. The generators of the weak* topology can be taken to be the sets $\{|L(x) - L_0(x)| < \epsilon;\, L \in \mathscr{X}^\star\}$ taken over all $L_0 \in \mathscr{X}^\star$, $x \in \mathscr{X}$, and $\epsilon > 0$. The weak* topology is in many ways more natural than the norm topology. For instance, whereas one can show that the unit ball $\{x \in \mathscr{X}; \|x\| \leq 1\}$ is compact in the norm topology if and only if $\mathscr{X}$ is finite-dimensional, the Banach-Alaoglu Theorem (below) asserts that no such condition is required if one passes to the weak* topology. For an example application of the weak* topology, the reader is referred to Section C.2.1 of Appendix C.

Theorem 15 (Banach-Alaoglu Theorem). *If $\mathscr{X}$ is a normed space, then $\{x \in \mathscr{X}; \|x\| \leq 1\}$ is weak* compact.*

B.2 Inner product spaces and Hilbert spaces

The notion of an inner product on a vector space generalizes the finite-dimensional vector inner product and transports concepts of lengths and angles to the infinite-dimensional setting.

Definition 24. *If $\mathscr{X}$ is a vector space over $\mathbb{F}$, an* inner product *on $\mathscr{X}$ is a function $\langle\,,\rangle : \mathscr{X} \times \mathscr{X} \to \mathbb{F}$ satisfying, for all α, β in $\mathbb{F}$ and x, y, z in $\mathscr{X}$, the following:*

i. $\langle \alpha x + \beta y, z\rangle = \alpha\langle x, z\rangle + \beta\langle y, z\rangle$.

ii. $\langle x, \alpha y + \beta z\rangle = \overline{\alpha}\langle x, y\rangle + \overline{\beta}\langle x, z\rangle$.

iii. $\langle x, x\rangle > 0$ *whenever* $x \neq 0$.

iv. $\langle x, y\rangle = \overline{\langle y, x\rangle}$.

The corresponding pair $(\mathscr{X}, \langle\ ,\ \rangle)$ *is said to be an* inner product space.

Note that by relaxing the requirement *iii* in the definition above to require only that $\langle x, x\rangle \geq 0$ for all $x \in \mathscr{X}$ instead defines what is referred to as a *semi-inner product.* The following is a crucial property of (semi-)inner products that the reader is certain to have previously encountered, at least in the finite-dimensional setting.

Theorem 16 (Cauchy-Schwarz Inequality). *If* $\langle\cdot, \cdot\rangle$ *is a semi-inner product on* $\mathscr{X}$, *then* $|\langle x, y\rangle|^2 \leq \langle x, x\rangle\langle y, y\rangle$ *for all* x *and* x *in* $\mathscr{X}$. *Moreover, equality occurs if and only if there are scalars* α *and* β*, both not* 0*, such that* $\langle\beta x + \alpha y, \beta x + \alpha y\rangle = 0$.

By revisiting Definition 19, the reader may quickly realize that $x \mapsto \sqrt{\langle x, x\rangle}$ defines a norm on $\mathscr{X}$. Thus, all inner product spaces are normed vector spaces. A *Hilbert space* is an inner product space that is complete relative to the metric induced by the corresponding norm. Of the familiar examples of normed spaces, there is one family of function spaces that comes with natural inner product structure:

Example 18 Let (X, Ψ, μ) be a σ-finite measure space and let $\mathscr{L}^2(\mu) \equiv \mathscr{L}^2(X, \Psi, \mu)$ be the familiar space of square integrable functions (cf. Example 13). If f and $g \in \mathscr{L}^2(\mu)$, then by Hölder's inequality $f\overline{g}$ is integrable (i.e. $f\overline{g} \in \mathscr{L}^1(\mu)$). Thus, one may set

$$\langle f, g\rangle = \int f\overline{g}\, d\mu$$

and the reader may verify that this defines an inner product on $\mathscr{L}^2(\mu)$. It is clear that this inner product induces the "usual" norm on $\mathscr{L}^2(\mu)$, which, referring back to Example 13, implies that $\mathscr{L}^2(\mu)$ is complete and is thus a Hilbert space.

The Cauchy-Schwarz inequality then becomes the statement that

$$\langle f, g\rangle \leq \left(\int |f|^2\, d\mu\right)^{\frac{1}{2}} \left(\int |g|^2\, d\mu\right)^{\frac{1}{2}},$$

which is the $p = q = 2$ version of Hölder's inequality. While the natural inner-product structure on $\mathscr{L}^2$ (not shared by $\mathscr{L}^p$ spaces for other values of p) may not look particularly impressive for now, the benefits of the Hilbert-space structure on $\mathscr{L}^2$ will start becoming apparent in the next section.

As a specialization of the above to the case where μ is a discrete measure and X are the positive integers, consider now the space ℓ^2 of square-summable series. Setting $\langle (x_i)_i, (y_j)_j \rangle := \sum_i x_i \overline{y_j}$ then defines an inner product on ℓ^2. The Cauchy-Schwarz inequality then becomes the statement that

$$\left| \sum_{i=1}^{\infty} x_i \overline{y_i} \right|^2 \leq \sum_{j=1}^{\infty} |x_j|^2 \sum_{j=1}^{\infty} |y_j|^2.$$

•

B.2.1 Orthonormal sets and bases

In addition to the natural notion of length (given by the corresponding norm), one of the key attributes of an inner product space is the underlying notion of orthogonality. In particular, given an inner product space $\mathscr{V}$, vectors $v_1, v_2 \in \mathscr{V}$ are said to be *orthogonal* if $\langle v_1, v_2 \rangle = 0$. An array of concepts and results from linear algebra can thereby be transported into the infinite-dimensional setting, starting with:

Definition 25. *An* orthonormal *subset of an inner product space $\mathscr{V}$ is a subset $\mathscr{E}$ having the properties: (a) for e in $\mathscr{E}$, $\|e\| = 1$; (b) if $e_1, e_2 \in \mathscr{E}$ and $e_1 \neq e_2$, then $e_1 \perp e_2$. A* basis *for $\mathscr{V}$ is a maximal orthonormal set.*

Example 19 Returning to the setting of linear algebra, let $\{e_1, \ldots, e_n\}$ be a finite orthonormal basis for some Hilbert space $\mathscr{H}$ and let $\{\epsilon_1, \ldots, \epsilon_m\}$ be a finite orthonormal basis for a Hilbert space $\mathscr{K}$. It can be shown that every linear transform from $\mathscr{H}$ into $\mathscr{K}$ is bounded. Moreover, setting $a_{ij} := \langle Ae_j, \epsilon_i \rangle$ for all $1 \leq j \leq n$, $1 \leq i \leq m$, the resulting $m \times n$ matrix (α_{ij}) is a representation of the operator A with respect to the basis as given. Conversely, every $m \times n$ matrix represents a bounded linear operator $\mathscr{H} \to \mathscr{K}$.

Next, moving to a setting where the basis is countable rather than finite, consider the ℓ^2 space (cf. Example 18) and let $e_1, e_2, \ldots$ be its usual basis, i.e. $e_i =$

$(0, \ldots, 0, 1, 0, \ldots)$ where the single 1 is at position i. If $A \in \mathscr{B}\left(\ell^2\right)$, consider the infinite matrix obtained by taking its $(i,j)^{\text{th}}$ entry as $\alpha_{ij} = \langle Ae_j, e_i \rangle$. The infinite matrix represents A in this basis, analogously to the finite case. However, this representation is frequently of limited value; for example, it is unknown how to calculate the norm of a general infinite matrix from its entries, while even the sufficiency conditions for the boundedness of such a matrix can become unwieldy.

•

As the reader may recall from linear algebra, every linearly independent set can be turned into an orthonormal one via the so-called *Gram-Schmidt orthogonalization process*, and this is more generally true of finite linearly independent sets in an inner product space. In particular, if $\mathscr{V}$ is an inner product space and $\{v_n : n \in \mathbb{N}\}$ is a linearly independent subset of $\mathscr{V}$, then there is an orthonormal set $\{e_n : n \in \mathbb{N}\}$ such that for every n, the linear span of $\{e_1, \ldots, e_n\}$ equals the linear span of $\{v_1, \ldots, v_n\}$.

The following is a useful collection of equivalences regarding the notion of a basis in a Hilbert space.

Theorem 17. *If $\mathscr{H}$ is a Hilbert space and $\mathscr{E}$ is an orthonormal set in $\mathscr{H}$, then the following statements are equivalent:*

i. $\mathscr{E}$ is a basis for $\mathscr{H}$.

ii. If $h \in \mathscr{H}$ and $h \perp \mathscr{E}$, then $h = 0$.

iii. $\mathscr{H}$ equals the closed linear span of $\mathscr{E}$.

iv. If $h \in \mathscr{H}$, then $h = \sum\{\langle h, e\rangle e : e \in \mathscr{E}\}$.

v. If g and $h \in \mathscr{H}$, then

$$\langle g, h \rangle = \sum\{\langle g, e\rangle\langle e, h\rangle : e \in \mathscr{E}\}.$$

vi. If $h \in \mathscr{H}$, then $\|h\|^2 = \sum\{|\langle h, e\rangle|^2 : e \in \mathscr{E}\}$ (Parseval's Identity)

Furthermore, if $\mathscr{H}$ is a Hilbert space, any two bases for $\mathscr{H}$ have the same cardinality.

Note that $\mathscr{E}$ need not in general be a countable set and the items *iv-vi* above may look worriesome. Fortunately, it is the case that if $\mathscr{H}$ is a Hilbert space and $\mathscr{E}$ is an orthonormal set in $\mathscr{H}$, then for any $h \in \mathscr{H}$, $\langle h, e\rangle \neq 0$ for at most countably many $e \in \mathscr{E}$.

Returning to the example of the function space $\mathscr{L}^2(\mu)$, the existence of a basis and the corresponding decomposition properties of Theorem 17 provide powerful tools for representing functions. Indeed, this is the foundation of *Fourier analysis*.[5].

The last property of the above Theorem 17 suggests that the cardinality of the basis is a meaningful invariant for Hilbert spaces. In linear algebra, this quantity is the dimension of the vector space, and that notion will therefore analogously carry over to the general setting:

Definition 26. *The* dimension *of a Hilbert space is the cardinality of the basis and is denoted by* dim $\mathscr{H}$.

As expected, the finite-dimensional Hilbert spaces bring us back to the realm of linear algebra. Of the infinite-dimensional Hilbert spaces, a particularly important class is composed of those spaces that have a countable basis. One such example is the space $\mathscr{L}^2([0,1])$ (with respect to the Lebesgue measure), whose basis is given by the complex exponentials $\psi_n(x) = \exp(i2\pi n x)$ taken over $n \in \mathbb{N}$.

Definition 27. *A Hilbert space is said to be* separable *if it admits a countable orthonormal basis, that is, if* $\dim \mathscr{H} = \aleph_0$.

B.2.2 Isomorphic Hilbert spaces

The notion of structural equivalence in Hilbert space is again that of vector-space isomorphism, augmented by the requirement that this isomorphism should also preserve the inner product. Specifically:

Definition 28. *If $\mathscr{H}$ and $\mathscr{K}$ are Hilbert spaces, a* (Hilbert space) isomorphism

[5]For a thorough and accessible account of Fourier analysis, the reader is referred to Chapter 3 of [SS03]

between $\mathscr{H}$ and $\mathscr{K}$ is a linear surjection $U : \mathscr{H} \to \mathscr{K}$ satisfying

$$\langle Uh, Ug \rangle = \langle h, g \rangle$$

for all h,g in $\mathscr{H}$.

The reader may readily verify that if $\mathscr{H}$ and $\mathscr{K}$ are isomorphic as Hilbert spaces, they are also isomorphic as Banach space, as the isomorphism of Definition 28 also preserves the norm. More generally, a norm-preserving map is called an *isometry* and it is in fact easy to show that if $V : \mathscr{H} \to \mathscr{K}$ is a linear map between Hilbert spaces, then V is an isometry if and only if $\langle Vh, Vg \rangle = \langle h, g \rangle$ for all h, g in $\mathscr{H}$. So, the Hilbert space isomorphism is simply a vector-space isomorphism that is also an isometry.

By making use of the availability of some basis $\mathscr{E}$ on any Hilbert space $\mathscr{H}$, one can construct an explicit isomorphism $U : \mathscr{H} \to \ell^2(\mathscr{E})$, by which one proves the following theorem:

Theorem 18. *Two Hilbert spaces are isomorphic if and only if they have the same dimension.*

Corollary 3. *All separable infinite dimensional Hilbert spaces are isomorphic.*

In particular, all infinite-dimensional separable Hilbert spaces are therefore isomorphic to ℓ^2. Given that the Hilbert spaces of interest in this thesis will indeed be assumed to be separable, the reader may therefore feel free to specialize the remainder of these introductory chapters to their heart's desire.

B.2.3 $\mathscr{B}(\mathscr{H})$

The bounded linear operators on a Hilbert space $\mathscr{H}$ form an algebra, with multiplication defined as composition, denoted $\mathscr{B}(\mathscr{H})$. The present section considers only a handful of their elementary properties, despite the fact that *much* is known about these (see e.g. Chapter 2 of [Con90]). Nevertheless, note ahead of time that we will return to the more advanced aspects of bounded linear operators on Hilbert spaces, towards the end of Appendix C, but only after we demonstrate that this setting is actually "generic".[6]

We start with an elementary (but crucial) example.

Proposition 15. *Let (X, Ψ, μ) a σ-finite measure space and consider the Hilbert space $\mathscr{H} = \mathscr{L}^2(X, \Psi, \mu) \equiv \mathscr{L}^2(\mu)$. If $f \in \mathscr{L}^\infty(\mu)$, define the* multiplication by f operator *$M_f : \mathscr{L}^2(\mu) \to \mathscr{L}^2(\mu)$ by $M_f g = gf$. Then $M_f \in \mathscr{B}(\mathscr{L}^2(\mu))$ and $\|M_f\| = \|f\|_\infty$.*

Of course, the properties of the bounded linear operators on Banach spaces are inherited by $\mathscr{B}(\mathscr{H})$, though the setting of Banach spaces is almost too broad for this remark to be useful. The main new notion to be introduced at this point is that of an *adjoint* of an operator, which will be shown in Appendix C to lend $\mathscr{B}(\mathscr{H})$ its remarkable algebraic structure. In the following chapter, the action of taking the operator adjoint will be considered in a more abstract setting, where the map $A \mapsto A^*$ will become an arbitrary *involution*.

Definition/Theorem 8. *If $\mathscr{H}$ and $\mathscr{K}$ are Hilbert spaces and A in $\mathscr{B}(\mathscr{H}, \mathscr{K})$, then there exists a unique operator $A^* \in \mathscr{B}(\mathscr{K}, \mathscr{H})$, referred to as the* adjoint of A, *satisfying $\langle Ah, k\rangle = \langle h, A^*k\rangle$ for all $h \in \mathscr{H}$, $k \in \mathscr{K}$.*

Proposition 16. *If $A, B \in \mathscr{B}(\mathscr{H})$ and $\alpha \in \mathbb{F}$, then:*

i. $(\alpha A + B)^* = \overline{\alpha} A^* + B^*$.

ii. $(AB)^* = B^* A^*$.

iii. $A^{**} \equiv (A^*)^* = A$.

iv. If A is invertible in $\mathscr{B}(\mathscr{H})$ and A^{-1} is its inverse, then A^* is invertible and $(A^*)^{-1} = (A^{-1})^*$.

v. $\|A\| = \|A^*\| = \|A^*A\|^{1/2}$

[6]Specifically, Appendix C is focused on the more general setting of the so-called *Banach algebras* and, more narrowly, *C^* algebras*. The latter will turn out to come with a representation theorem, namely the *GNS construction*, that will bring the discussion back into the realm of $\mathscr{B}(\mathscr{H})$.

Recall that an important class of bounded linear operators was encountered in Definition 28. These were the linear surjections $U : \mathscr{H} \to \mathscr{K}$ with the property that $\langle Uh, Ug \rangle = \langle h, g \rangle$ for all h, g in $\mathscr{H}$. Such a map U is typically referred to as a *unitary.* Clearly, unitaries are bounded linear operators, and the reader may also easily verify that U is a unitary if and only if $UU^* = U^*U = I$.

We had also previously mentioned that a norm-preserving map is called an *isometry* and that one could show that if $V : \mathscr{H} \to \mathscr{K}$ is a *linear* map between Hilbert spaces, then V is an isometry if and only if $\langle Vh, Vg \rangle = \langle h, g \rangle$ for all h, g in $\mathscr{H}$. It is important to note that a unitary operator is a *surjective* isometry. As an example of an isometry that is not surjective, one may take the left-shift operator $(h_1, h_2, \ldots) \mapsto (0, h_1, h_2, \ldots)$ on ℓ^2.

Just as unitaries can be described by the relation $UU^* = U^*U = I$, the behavior of the operator under the action of taking adjoints will lead us to distinguish the following classes of operators.

Definition 29. *If* $A \in \mathscr{B}(\mathscr{H})$*, then:* A *is* hermitian *or* self-adjoint *if* $A^* = A$*;* A *is* normal *if* $AA^* = A^*A$*;* A *is* positive *or* positive semidefinite *if there exists some* $B \in \mathscr{B}(\mathscr{H})$ *such that* $A = BB^*$*.*

Note that a positive operator in a finite-dimensional setting is a positive semidefinite matrix. In another familiar setting, the reader may verify that if $\mathcal{A} = \mathscr{C}(X)$, then f is positive in $\mathcal{A}$ if and only if $f(x) \geq 0$ for all $x \in X$. Similarly, if $\mathcal{A} = \mathscr{L}^\infty(\mu)$, f is positive in $\mathcal{A}$ iff $f \geq 0$ a.e. (with respect to μ). The positivity of a bounded linear operator on a Hilbert space can in fact be conveniently recast in the following form.

Proposition 17. *If* $\mathscr{H}$ *is a Hilbert space and* $\mathcal{A} \in \mathscr{B}(\mathscr{H})$*, then* A *is positive if and only if* $\langle Ah, h \rangle \geq 0$ *for all* $h \in \mathscr{H}$*.*

Another important class of operators, also familiar from the finite-dimensional setting, are (orthogonal) projections. In particular, an *idempotent* is a bounded linear operator E on $\mathscr{H}$ satisfying $E^2 = E$ and a *projection* is an idempotent P such that

$\ker P = (\text{ran } P)^\perp$. In this general setting, idempotents and projections nevertheless retain most of their key properties:

Proposition 18. *If E is an idempotent on $\mathscr{H}$ and $E \neq 0$, the following statements are equivalent; (a) E is a projection; (b) E is the orthogonal projection of $\mathscr{H}$ onto ran E; (c) $\|E\| = 1$, (d) $E = E^*$, i.e. E is self-adjoint; (e) $EE^* = E^*E$, i.e. E is normal; (f) $\langle Eh, h\rangle \geq 0$ for all $h \in \mathscr{H}$.*

Proposition 19. *(a) E is an idempotent if and only of $I - E$ is an idempotent. (b) ran $E = \ker(I-E)$, $\ker E =$ ran $(I-E)$, and both ran E and $\ker E$ are closed linear subspaces of $\mathscr{H}$. (c) If $\mathscr{M} =$ ran E and $\mathscr{N} = \ker E$, then $\mathscr{M} \cap \mathscr{N} = \{0\}$ and $\mathscr{M} + \mathscr{N} = \mathscr{H}$.*

B.2.4 Topologies on $\mathscr{B}(\mathscr{H})$

Due to the inner product structure on $\mathscr{H}$, there are a number of natural topologies on $\mathscr{B}(\mathscr{H})$, many of which are targeted to specific applications. In addition to the norm topology, we will at present discuss two more relevant ones.

The *strong operator topology (SOT)* on $\mathscr{B}(\mathscr{H})$ is given by the family of seminorms $A \mapsto \|Ah\|$, taken over all $h \in \mathscr{H}$. In particular, SOT is generated by the sets $\{A \in \mathscr{B}(\mathscr{H}); \|(A - A_0)h\| < \epsilon\}$ for all $A_0 \in \mathscr{B}(\mathscr{H})$, $h \in \mathscr{H}$, and $\epsilon > 0$.

The *weak operator topology (WOT)* on $\mathscr{B}(\mathscr{H})$ is given by the family of seminorms $A \mapsto |\langle Ah, g\rangle|$, taken over all $h, g \in \mathscr{H}$. In particular, WOT is generated by the sets $\{A \in \mathscr{B}(\mathscr{H}); |\langle (A - A_0)h, g\rangle| < \epsilon\}$ for all $A_0 \in \mathscr{B}(\mathscr{H})$, $h, g \in \mathscr{H}$, and $\epsilon > 0$.

Clearly, WOT is coarser than SOT as the generators of the former are also the generators of the latter (but not vice versa). From the point of view of convergence, the reader may verify that the net $\{A_i\}$ in $\mathscr{B}(\mathscr{H})$ converges to $A \in \mathscr{B}(\mathscr{H})$ in SOT if and only if $\|(A_i - A)h\| \to 0$ for all $h \in \mathscr{H}$. In contrast, $\{A_i\}$ converges in WOT if and only if $\langle A_i h, g\rangle \to \langle Ah, g\rangle$ for all $h, g \in \mathscr{H}$, and SOT convergence indeed implies WOT convergence, by the Cauchy-Schwarz inequality. Frequently, it is the weaker form that will be useful, as in the following example.

Example 20 Let (X, Ψ, μ) be a σ-finite measure space and for $f \in \mathscr{L}^\infty(\mu)$, let M_f be the multiplication operator on $\mathscr{L}^2(\mu)$ (cf. Example 15). Then, one may check that a net $\{\int_X \cdot f_i \, d\mu\}$ in $\mathscr{L}^1(\mu)^*$ (cf. Example 16) converges weak* to $\int_X \cdot f \, d\mu$ if and only if $\{M_{f_i}\}$ converges in WOT to M_f. •

Analogously to the Banach-Alaoglu Theorem of Section B.1.4, the closed unit ball of $\mathscr{B}(\mathscr{H})$ can be shown to be WOT compact (but not SOT compact unless $\mathscr{H}$ is finite-dimensional). Yet, in other respects, SOT and WOT share a number of properties. For example, it can be shown that if $\mathscr{H}$ is separable, then both SOT and WOT are metrizable on bounded subsets of $\mathscr{B}(\mathscr{H})$. Moreover, from the point of view of the bounded linear functionals on the Banach space $\mathscr{X} = \mathscr{B}(\mathscr{H})$, the two topologies behave quite similarly:

Proposition 20. *For any linear functional $L : \mathscr{B}(\mathscr{H}) \to \mathbb{C}$, the following are equivalent,*

i. L is SOT continuous.

ii. L is WOT continuous.

iii. There are vectors $g_1, \ldots, g_n$, $h_1, \ldots, h_n$ in $\mathscr{H}$ such that $L(A) = \sum_{i=1}^n \langle Ag_i, h_i \rangle$ for every $A \in \mathscr{B}(\mathscr{H})$.

The topologies on $\mathscr{B}(\mathscr{H})$ will become especially important in Appendix C. For instance, the following example lends meaning to the Definition 40 of Section C.2.3.

Example 21 If $\{E_n\}$ is a sequence of pair-wise orthogonal projections on $\mathscr{H}$, then $\sum_{n=1}^\infty E_n$ converges in SOT to the projection of $\mathscr{H}$ onto the closed linear span of $\{E_n(\mathscr{H}) \mid n \geq 1\}$ (i.e. onto the smallest closed linear subspace of $\mathscr{H}$ containing the subspaces $E_n(\mathscr{H})$ for all $n \geq 1$). •

B.2.5 Linear functionals

As in the more general Banach-space setting, one can specialize the discussion of the bounded linear operators between Hilbert spaces to that of the bounded linear functionals on a Hilbert space $\mathscr{H}$. In particular, one may discuss the properties of the dual space of $\mathscr{H}$, denoted $\mathscr{H}^\star$. The fact that the underlying norm was induced by an inner product gives rise to the following powerful representation theorem.

Theorem 19 (Riesz Representation Theorem). *If $\mathscr{H}$ is a Hilbert space and $L : \mathscr{H} \to \mathbb{F}$ a bounded linear functional, then there is a unique vector h_0 in $\mathscr{H}$ such that $L(h) = \langle h, h_0 \rangle$ for every h in $\mathscr{H}$. Moreover, $\|L\| = \|h_0\|$.*

Note that, the natural map $\mathscr{H} \to \mathscr{H}^\star$ is in fact an isomorphism (or anti-isomorphism, depending on the choice of the base field $\mathbb{F}$) between two normed spaces. But, we will not linger further on that fact. Instead, as a more concrete application of the above theorem, consider the resulting representation theorem in $\mathscr{L}^2$.

Corollary 4. *If (X, Ψ, μ) is a measure space and $L : \mathscr{L}^2(\mu) \to \mathbb{F}$ is a bounded linear functional, then there is a unique h_0 in $\mathscr{L}^2(\mu)$ such that for every h in $\mathscr{L}^2(\mu)$,*

$$L(h) = \int h\, \overline{h_0}\, d\mu.$$

B.3 Direct sums and tensor products of Hilbert spaces

The direct sums and tensor products addressed in Appendix A are considered to be *algebraic*, an attribute principally meant to indicate that one is purposefully ignoring the natural Hilbert space structure on the resulting vector spaces. This will be remedied at present. In particular, the following two sections discuss the typical Hilbert-space structure on these objects, which is of central importance to the remainder of this thesis.

B.3.1 Direct sums of Hilbert spaces

Taking direct sums of Hilbert spaces is straightforward: let $\mathscr{H}_1, \dots, \mathscr{H}_n$ be Hilbert spaces and denoting by $\bigoplus_{i=1}^n \mathscr{H}_i$ their algebraic direct sum, set

$$\langle (x_1, \dots, x_n), (y_1, \dots, y_n) \rangle = \langle x_1, y_1 \rangle_{\mathscr{H}_1} + \dots + \langle x_n, y_n \rangle_{\mathscr{H}_n},$$

$$\|(x_1, \dots, x_n)\| = (\|x_1\|^2_{\mathscr{H}_1} + \dots + \|x_n\|^2_{\mathscr{H}_n})^{1/2},$$

for all $(x_1, \dots, x_n), (y_1, \dots, y_n) \in \bigoplus_{i=1}^n \mathscr{H}_i$. The reader may readily verify that $\langle\, , \,\rangle$ gives indeed an inner product and $\|\ \|$ a norm on $\bigoplus_{i=1}^n \mathscr{H}_i$. In fact, $\bigoplus_{i=1}^n \mathscr{H}_i$ is complete with respect to this norm, and the algebraic direct sum as an inner product space thereby becomes a bona fide Hilbert space.

The analogous definitions can be made for any countable family of Hilbert spaces, as follows.

Proposition 21. *If $\mathscr{H}_1, \mathscr{H}_2, \dots$ are Hilbert spaces, let $\mathscr{H}$ be the set of sequences $(h_n)_{n=1}^\infty$, with $h_n \in \mathscr{H}_n$ for all $n \in \mathbb{N}$, such that $\sum_{n=1}^\infty \|h_n\|^2 < \infty$. For $h = (h_n)$ and $g = (g_n)$ in $\mathscr{H}$, define*

$$\langle h, g \rangle = \sum_{n=1}^\infty \langle h_n, g_n \rangle.$$

Then $\langle\, , \,\rangle$ is an inner product on $\mathscr{H}$ and the norm relative to this inner product is $\|h\| = (\sum_{n=1}^\infty \|h_n\|^2)^{\frac{1}{2}}$. With this inner product $\mathscr{H}$ is a Hilbert space.

If $\mathscr{H}_1, \mathscr{H}_2, \dots$ are Hilbert spaces, the space $\mathscr{H}$ of Proposition 21 is called the *(Hilbert space) direct sum* of $\mathscr{H}_1, \mathscr{H}_2, \dots$ and is denoted by $\mathscr{H} =: \bigoplus_{n=1}^i nfty \mathscr{H}_n$. Despite the ambiguous notation, throughout this thesis, we will be careful to point out whenever the direct sums are meant to be taken as algebraic.

B.3.2 Tensor products of Hilbert spaces

Analogously to the previous discussion, there is also a natural definition of a tensor product of Hilbert spaces. As in the algebraic case, the definition will be centered on a universal property governing the behavior of bilinear maps, with some additional "analytic" structure required. First, the maps of interest will be assumed to be bounded. Specifically, let $\mathscr{H}, \mathscr{H}_1, \ldots, \mathscr{H}_n$ be Hilbert spaces and L be a multilinear mapping from $\mathscr{H}_1 \times \cdots \times \mathscr{H}_n$ into $\mathscr{H}$. Then, L is a *bounded multilinear mapping* if, in addition, there exists some $r > 0$ for which

$$\|L(x_1, \ldots, x_n)\| \leq r\|x_1\| \cdots \|x_n\| \quad \text{for all } x_1 \in \mathscr{H}_1, \ldots, x_n \in \mathscr{H}_n.$$

When the image of a bounded multilinear map is $\mathbb{F}$, we will (predictably) refer to it as a bounded multilinear *functional.* There is a subclass of such functionals that is of particular interest for the topic at hand. Namely, let $\mathscr{H}_1, \ldots, \mathscr{H}_n$ be Hilbert spaces with some orthonormal bases $\mathscr{E}_1, \ldots, \mathscr{E}_n$, respectively. If $\gamma : \mathscr{H}_1 \otimes \ldots \otimes \mathscr{H}_n \to \mathbb{F}$ is a bounded multilinear functional, then it can be shown that the value (in the extended sense) of the expression $\sum_{f_1 \in \mathscr{E}_1} \cdots \sum_{f_n \in \mathscr{E}_n} |\gamma(f_1, \ldots, f_n)|^2$ is not affected by the choice of the bases $\mathscr{E}_1, \ldots, \mathscr{E}_n$. A bounded multilinear functional for which this sum is finite is referred to as a *Hilbert-Schmidt functional* on $\mathscr{H}_1 \otimes \ldots \otimes \mathscr{H}_n$, with norm given as

$$\|L\|_2 = \left(\sum_{f_1 \in \mathscr{E}_1} \cdots \sum_{f_n \in \mathscr{E}_n} |L(f_1, \ldots, f_n)|^2 \right)^{\frac{1}{2}}. \tag{B.3}$$

The focus for the universal property of Hilbert tensor product will not merely be bounded multilinear mappings, but the narrower class of the so-called weak Hilbert-Schmidt mappings, defined as follows.

Definition 30. *A* weak Hilbert-Schmidt mapping *from $\mathscr{H}_1 \times \cdots \times \mathscr{H}_n$ into $\mathscr{H}$ is a bounded multilinear mapping L with the following properties:*

i. For each h in $\mathscr{H}$, the mapping L_h defined by

$$L_h(x_1, \ldots, x_n) = \langle L(x_1, \ldots, x_n), h \rangle$$

is a Hilbert-Schmidt functional on $\mathscr{H}_1 \times \cdots \times \mathscr{H}_n$.

ii. There is a real number d such that $\|L_h\|_2 \leq d\|h\|$ for each $h \in \mathscr{H}$.

We are now ready to define the Hilbert space tensor product. The following theorem formulates the desired univesal property.

Theorem 20. *Let $\mathscr{H}_1, \ldots, \mathscr{H}_n$ be Hilbert spaces. There is a Hilbert space $\mathscr{H}$ and a weak Hilbert-Schmidt mapping $\pi : \mathscr{H} \to \mathscr{H}$ with the following property: given any weak Hilbert-Schmidt mapping L from $\mathscr{H}_1 \times \cdots \times \mathscr{H}_n$ into a Hilbert space $\mathscr{K}$, there is a unique bounded linear mapping T from $\mathscr{H}$ into $\mathscr{K}$, such that $L = T \circ \pi$.*

Moreover, for any two Hilbert space $\mathscr{H}$ and $\mathscr{H}'$ as above, with the corresponding maps π and π', there is a unitary transformation U from $\mathscr{H}$ onto $\mathscr{H}'$ such that $\pi' = U \circ \pi$.

The above Hilbert space $\mathscr{H}$ will be denoted $\mathscr{H}_1 \otimes \ldots \otimes \mathscr{H}_n$ and referred to as the *(Hilbert space) tensor product* of $\mathscr{H}_1, \ldots, \mathscr{H}_n$. The universal property can then be rephrased as given any weak Hilbert-Schmidt mapping L from $\mathscr{H}_1 \times \cdots \times \mathscr{H}_n$ into a Hilbert space $\mathscr{K}$, there is a unique bounded linear mapping T making the following diagram commute:

$$\begin{tikzcd} \mathscr{H}_1 \times \cdots \times \mathscr{H}_n \arrow[r, "\pi"] \arrow[dr, "L"'] & \mathscr{H}_1 \otimes \ldots \otimes \mathscr{H}_n \arrow[d, "T"] \\ & \mathscr{K} \end{tikzcd}$$

The resulting Hilbert space $\mathscr{H}_1 \otimes \ldots \otimes \mathscr{H}_n$ is seen to have remarkably natural structure:

Proposition 22. *Let $\mathscr{H}_1, \ldots, \mathscr{H}_n$ be Hilbert spaces with the corresponding bases $\mathscr{E}_1, \ldots, \mathscr{E}_n$, and let $\mathscr{H}_1 \otimes \ldots \otimes \mathscr{H}_n$ be their tensor product with map $\pi : \mathscr{H}_1 \times \cdots \times \mathscr{H}_n \to \mathscr{H}$ given as in Theorem 20. Then, for all $v_i, w_i \in \mathscr{H}_i$ $(i = 1, \ldots, n)$,*

$$\langle \pi(v_1, \ldots, v_n), \pi(w_1, \ldots, w_n) \rangle = \langle v_1, w_1 \rangle \cdots \langle v_n, w_n \rangle.$$

Furthermore, the set $\{\pi(f_1, \ldots, f_n) : f_1 \in \mathscr{E}_1, \ldots, f_n \in \mathscr{E}_n\}$ is an orthonormal basis of $\mathscr{H}$.

Given the Hilbert spaces $\mathscr{H}_1, \ldots, \mathscr{H}_n$ and the map π associated with their tensor product $\mathscr{H}_1 \otimes \ldots \otimes \mathscr{H}_n$, it is natural to write $f_1 \otimes \ldots \otimes f_n$ for the element $\pi(f_1, \ldots, f_n)$

(where $f_i \in \mathscr{H}_i, i = 1, \ldots, n$)), as we have so far been doing when dealing with the algebraic tensor products. The above proposition then indicates that these pure tensors, when drawn from the bases of the underlying Hilbert space, form a basis of the tensor product. The reader may then verify that the Hilbert-space tensor product is (up to a unique Hilbert-space isomorphism) the completion of the algebraic tensor product, relative to the inner product defined by:

$$\langle v_1 \otimes \ldots \otimes v_n, w_1 \otimes \ldots \otimes w_n \rangle = \langle v_1, w_1 \rangle \cdots \langle v_n, w_n \rangle.$$

Appendix C

We presently review some elementary aspects of the theory of *operator algebras*, which form the underpinning of the probabilistic frameworks considered in this thesis. The focus is on the elementary properties of Banach algebras, C^* algebras, and von Neumann algebras. This appendix is principally based on the later chapters of Conway's *Course in Functional Analysis* [Con90] as well as the first volume of the *Fundamentals of the theory of operator algebras* by Kadison and Ringrose [KR97]. The material on von Neumann algebras also draws on a set of unpublished notes by Vaughan Jones, which at the time of this writing can be found floating around the cyberspace.

C.1 Banach Algebras

We start by recalling that throughout this chapter, $\mathbb{F}$ will denote the underlying field of scalars, taken to be either $\mathbb{R}$ or $\mathbb{C}$. Then, an *algebra* over $\mathbb{F}$ will refer to a vector space $\mathcal{A}$ over $\mathbb{F}$ with multiplication $\mathcal{A} \times \mathcal{A} \to \mathcal{A}$ with respect to which $\mathcal{A}$ is also a ring; in particular, the multiplication must satisfy that for every $\alpha \in \mathbb{F}$ and $a, b \in \mathcal{A}$, $\alpha(ab) = (\alpha a)b = a(\alpha b)$.

The central object of this section is the following:

Definition 31. *A* Banach algebra *is an algebra $\mathcal{A}$ over $\mathbb{F}$ that has a norm $\|\cdot\|$ relative to which $\mathcal{A}$ is a Banach space, with the additional property that for all $a, b \in \mathcal{A}$, $\|ab\| \leq \|a\|\|b\|$. A Banach algebra is said to be* unital *if it contains an identity.*

Note that for a unital Banach algebra $\mathcal{A}$ with identity i_d, the map $\alpha \mapsto \alpha i_d$ is an isomorphism of $\mathbb{F}$ into $\mathcal{A}$ with $\|\alpha i_d\| = |\alpha|$. This identification will from now on be thought as the inclusion $\mathbb{F} \subset \mathcal{A}$ and the identity element will be denoted 1.

The following are some standard settings in which the Banach algebras arise naturally; for additional examples, the reader is referred to [Con90].

Example 22 Let X be a compact Hausdorff space and let $\mathcal{A}$ be the continuous functions on X, i.e. $\mathcal{A} = \mathscr{C}(X)$. If the multiplication $\mathcal{A} \times \mathcal{A} \to \mathcal{A}$ is defined as $(f, g) \to fg$ (pointwise), then A is a *commutative* unital Banach algebra with the constant function 1 as the identity.

If X is instead only locally compact and $\mathcal{A}$ further restricted to those functions that are bounded, i.e. $\mathcal{A} = \mathcal{C}_b(X)$, then $\mathcal{A}$ is again a commutative unital Banach algebra.

However, if X is locally compact and $\mathcal{A}$ is composed of continous functions with compact support, i.e. $\mathcal{A} = \mathcal{C}_0(X)$, then $\mathcal{A}$ is a commutative Banach algebra (for multiplication as previously defined), but $\mathcal{A}$ does not contain an identity. •

Example 23 Let (X, Ψ, μ) be a σ-finite measure space and $\mathcal{A}$ be the algebra of all essentially bounded measurable functions on X, i.e. $\mathcal{A} = L^\infty(X, \Psi, \mu)$. Then $\mathcal{A}$ is an abelian unital Banach algebra with under point-wise addition and multiplication, with norm given by the essential supremum norm. •

And there is of course our prototype of a Banach algebra:

Example 24 Let $\mathscr{X}$ be a Banach space and $\mathcal{A}$ the bounded linear operators on $\mathscr{X}$, i.e. $\mathcal{A} = \mathscr{B}(\mathscr{X})$. Defining the multiplication as composition, then $\mathcal{A}$ is a unital Banach algebra. If dim $\mathscr{X} \geq 2$, $\mathcal{A}$ is not abelian.

If $\mathcal{A}$ is instead composed of the subset of operators that are compact, i.e. $\mathcal{A} = \mathscr{B}_0(\mathscr{X})$, then if dim $\mathscr{X} = \infty$, $\mathcal{A}$ becomes a Banach algebra without the identity.

•

The above examples may serve to illustrate the extent to which Banach algebras arise in familiar settings. The remaining sections of this chapter will, by drawing on the dual nature of Banach algebras, develop machinery unique to these objects.

C.1.1 The Spectrum

The spectrum of a bounded linear operator, or, more generally still, of an element of a Banach algebra, is a generalization of the concept of eigenvalues in matrices. We will begin this discussion with revisiting the finite-dimensional setting and generalizing as we move along.

First, let $\mathscr{X}$ be a finite-dimensional Banach space (e.g. Euclidean space, $\mathbb{R}^n$ or $\mathbb{C}^n$) and let A be a (bounded) linear transformation $\mathscr{X} \to \mathscr{X}$. Recall that $A - \lambda I$ will fail to have an inverse in $\mathscr{B}(X)$ for some $\lambda \in \mathbb{C}$ if and only if $(A - \lambda I)e = 0$ for some unit vector e. Such a λ is referred to as an *eigenvalue* of a, with e as the corresponding *eigenvector.*

For $\mathscr{X}$ an arbitrary, possibly infinite-dimensional Banach space, by the Inverse Mapping Theorem (Theorem 14 of Appendix B), the operator $A - \lambda I$ will fail to have an inverse in $\mathscr{B}(\mathscr{X})$ only if $A - \lambda I$ fails to be bijective. However, the converse implication does not hold, and the set of eigenvalues becomes distinct from the set of real or complex numbers for which $(A - \lambda I)$ is not invertible. The latter is instead referred to as the *spectrum of* A, denoted $\mathrm{sp}(A)$. In the general context of the present chapter, the notion of the spectrum of a bounded linear operator on a Banach space extends readily into the unital Banach algebra setting.

Definition 32. *If $\mathcal{A}$ is a unital Banach algebra and $a \in \mathcal{A}$, the* spectrum of a, *denoted $sp(a)$, is defined by*

$$sp(a) = \{\alpha \in \mathbb{C} \mid a - \alpha 1 \textit{ is not invertible in } \mathcal{A}\}.$$

The left spectrum $sp_l(a)$ and the right spectrum $sp_r(a)$ are obtained by considering only left and right inverses, respectively.

The Inverse Mapping Theorem implies that the set of eigenvalues of an operator on an arbitrary Banach space forms a subset of its spectrum. For operators on infinite-dimensional Banach spaces, however, the notion of eigenvalues becomes too restrictive, as illustrated in the following Example 25. In fact, as will become clear in Sections C.2.1 and C.2.3, it is the spectrum that encodes the key characteristics of an operator.

Example 25 Let $\mathscr{H} = \mathscr{L}^2_{\mathbb{C}}([0,1])$ be the Hilbert space of complex-valued square-integrable functions on $[0,1]$ relative to Lebesgue measure. Let $A \in \mathscr{B}(\mathscr{H})$ be given as the multiplication by the identity function, i.e. $(Af)(x) = xf(x)$ for all $f \in \mathscr{H}, x \in [0,1]$. Then, A has no eigenvalues, for if $Af = \lambda f$ for some non-zero vector $f \in \mathscr{H}$ (by the Inverse Mapping Theorem), then $f = 0$ a.e. in $[0,1]$ – a contradiction. Nevertheless, it can be shown (by constructing an appropriate sequence of "approximate" eigenvectors – cf. Example 3.2.2. in [KR97]) that, in fact, $\mathrm{sp}(a) = [0,1]$. •

An elementary question, which this discussion has so far managed to avoid, concerns the conditions under which the spectrum of an operator on a Banach space is non-empty. For example, if $\mathcal{A}$ is an algebra of 2×2 real matrices, then the matrix $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ has an empty spectrum. Indeed, for any $\alpha \in \mathbb{R}$, $\det(A-\alpha I) = \alpha^2+1 > 0$ and $A - \alpha I$ is therefore invertible for all $\alpha \in \mathbb{R}$. Fortunately, it is the case that every bounded linear operator on a complex Banach space, or more generally every element of a Banach algebra over $\mathbb{C}$, must have a non-empty spectrum. For this reason, the Banach algebras of this section will be assumed to be over $\mathbb{C}$.

Theorem 21. *If $\mathcal{A}$ is a unital Banach algebra over $\mathbb{C}$, then for each a in $\mathcal{A}$, $sp(a)$ is a nonempty compact subset of $\mathbb{C}$. Moreover, if $|\alpha| > \|a\|$, then $\alpha \notin sp(a)$.*

Example 26 Let X is a compact Hausdorff space and $\mathcal{A} = \mathscr{C}(X)$. For every $f \in \mathcal{A}$, we have $\mathrm{sp}(f) = f(X)$. Indeed, for every $x \in X$, $f - f(x)1$ vanishes at x and is therefore not invertible as a function on X. Thus, $f(X) \subseteq \mathrm{sp}(f)$. On the other hand, if $\alpha \in \mathbb{F}$ does not belong to $f(X)$, then $f - \alpha 1$ is a non-vanishing function on X and, since f is continuous, it follows that $(f-\alpha)^{-1} \in \mathscr{C}(X)$. In particular, $f - \alpha 1$ is invertible and $\alpha \notin \mathrm{sp}(f)$. •

A useful attribute of the spectrum of an operator is the *spectral radius.* It is the natural generalization of the maximum eigenvalue of a positive semidefinite matrix, both by the nature of its definition and by its resulting properties (see Proposition 23 below).

Definition 33. *Given a unital Banach algebra $\mathcal{A}$ and $a \in \mathcal{A}$, the* spectral radius $r(a)$ *of a is defined by $r(a) = \sup\{|\alpha|; \alpha \in sp(a)\}$.*

Proposition 23. *If $\mathcal{A}$ is a unital Banach algebra over $\mathbb{C}$, then $\lim_{n\to\infty} \|a^n\|^{1/n}$ exists and*

$$r(a) = \lim_{n\to\infty} \|a^n\|^{1/n}.$$

Moreover, if $a, b \in \mathcal{A}$ are two commuting elements, i.e. $ab = ba$, then,

$$r(ab) \leq r(a)r(b) \qquad \text{and} \qquad r(a+b) \leq r(a) + r(b). \tag{C.1}$$

C.2 C^* Algebras

In the finite-dimensional (matricial) setting, the passage from Banach algebras to C^* algebras occurs when the algebraic and analytic structure is refined by taking into account the conjugate transpose. More generally, given a Hilbert space $\mathscr{H}$ and the Banach algebra of bounded linear operators $\mathscr{B}(\mathscr{H})$, the analogous passage is with respect to the operation of taking adjoints. In the modern treatment, the latter is seen as an instance of an algebraic operation referred to as an *involution*, defined as follows.

Definition 34. *A map $a \mapsto a^*$ of $\mathcal{A}$ into $\mathcal{A}$ is an* involution *if for all $a, b \in \mathcal{A}$ and $\alpha, \beta \in \mathbb{C}$, (i) $(a^*)^* = a$, (ii) $(ab)^* = b^*a^*$, and (iii) $(\alpha a + \beta b)^* = \overline{\alpha} a^* + \overline{\beta} b^*$, with the overline denoting complex conjugation.*

Note that if $\mathcal{A}$ is unital, properties (i) and (ii) imply that $1^* = 1$ and, along with property (iii), yield that for any $\alpha \in \mathbb{C}$, $(\alpha 1)^* = \overline{\alpha} 1$.

In the abstract setting, a Banach algebra $\mathcal{A}$ endowed with an involution $* : \mathcal{A} \to \mathcal{A}$ becomes referred to as a *$*$-algebra*. The $*$-algebras provide the basic setting for the discussion of Chapter 3. The defining norm property of Banach algebras implies that in a $*$-algebra, $\|a^*a\| \leq \|a^*\|\|a\|$. In turn, a C^* algebra is obtained by requiring that the previous statement holds as an equality. Specifically:

Definition 35. *A $*$-algebra is a Banach algebra with an involution $*$. A C^* algebra $\mathcal{A}$ is an $*$-algebra such that for every $a \in \mathcal{A}$,*

$$\|a^*a\| = \|a\|^2. \tag{C.2}$$

Note that for any element a of a C^* algebra $\mathcal{A}$, $\|a^*\| = \|a\|$. Indeed, by (C.2), $\|a\|^2 = \|a^*a\| \leq \|a^*\|\|a\|$, so $\|a\| \leq \|a^*\|$ and $\|a^*\| \leq \|a^{**}\| = \|a\|$. Though the requirement (C.2) may look benign at first, it is remarkably strong. As discussed in the remainder of this section, this additional requirement will impart the C^* algebras with remarkably rich structure.

Returning briefly to the familiar setting of operators on Hilbert spaces, note that if $\mathscr{H}$ is a Hilbert space, then $\mathscr{B}(\mathscr{H})$ is a C^* algebra where for each $A \in \mathscr{B}(\mathscr{H})$, A^* is given as the adjoint of A. Whereas $\mathscr{B}(\mathscr{H})$ is a unital C^* algebra, considering the subalgebra $\mathscr{B}_0(\mathscr{H})$ of compact operators yields a C^* algebra that does not contain the unit if $\mathscr{H}$ is infinite-dimensional.

The reader may also verify that a norm-closed subalgebra of $\mathscr{B}(\mathscr{H})$ that contain the adjoints of each of the elements will also be a C^* algebra. Surprisingly, a certain type converse will turn out to hold! (An impatient reader may wish to look ahead to Section C.2.2, concerning representations of C^* algebras.) The following are some examples of C^* algebras.

Example 27 Let X be a compact Hausdorff space and consider the Banach algebra $\mathscr{C}(X)$ (cf. Example 22). Defining the involution as point-wise complex conjugation, i.e. for $f \in \mathscr{C}(X)$, $f^*(x) = \overline{f(x)}$ for all $x \in X$, turns $\mathscr{C}(X)$ into a commutative C^* algebra.

Analogously to Example 22, if X is locally compact, the Banach algebras $\mathcal{C}_b(X)$ and $\mathcal{C}_0(X)$ are again commutative C^* algebras with respect to point-wise complex conjugation. However, $\mathcal{C}_0(X)$ is a C^* algebra without identity. •

Example 28 Defining the involution as point-wise complex conjugation also turns the Banach algebra $L^\infty(X, \Psi, \mu)$ (cf. Example 23) into a C^* algebra, provided the measure space (X, Ψ, μ) is σ-finite. •

Note that for C^* algebras, the suitable notion of algebraic equivalence are provided by $*$-homomorphisms or $*$-isomorphisms. A *$*$-homomorphism* is a homomorphism between $*$-algebras, that is, a homomorphism between algebras that also preserves the involutive structure, i.e. that is compatible with the involution $*$. Similarly, a *$*$-isomorphism* is again an algebra isomorphism compatible with the involution $*$. Naturally, in the light of the defining C^* algebra property (C.2), the presence of a $*$-homomorphism also reflects on the norms defined on the corresponding algebras. In particular:

Proposition 24. *If $\mathcal{A}, \mathcal{A}'$ are C^* algebras and $\rho : \mathcal{A} \to \mathcal{A}'$ is a $*$-homomorphism, then $\|\rho(a)\| \leq \|a\|$ for all $a \in \mathcal{A}$.*

If it is additionally the case that $\|\rho(a)\| = \|a\|$ for every element a of the algebra $\mathcal{A}$, ρ will be referred to as an isometric $*$-homomorphism.

Classes of elements in a C^* algebra

As in the setting of bounded linear operators on Hilbert spaces, the involution distinguishes between several types of elements in a C^* algebra. An element $a \in \mathcal{A}$ is again said to be *normal* if $a^*a = aa^*$. Normal bounded linear operators on Hilbert spaces are of particular importance as they satisfy the *Spectral Theorem* of Section C.2.3. Similarly, an element $u \in \mathcal{A}$ is said to be *unitary* if $u^*u = uu^* = 1$ and it is straightforward to verify that such elements have unit norm. An element $a \in \mathcal{A}$ is said to be *self-adjoint* or *hermitian* if $a = a^*$. In this case, one may wish to show the following useful decomposition: every $b \in \mathcal{A}$ can be written as $b = x + iy$ where x and y are self-adjoint elements of $\mathcal{A}$. A (self-adjoint) element $a \in \mathcal{A}$ is said to be *positive* if $a = bb^*$ for some $b \in \mathcal{A}$.

Spectral properties in C^* algebras

From now on, it will be assumed that the C^* algebras of interest will be unital so that to every element of the algebra, we will then be able to associate a spectrum (cf. Section C.1.1).

The defining property of C^* algebras, (C.2), determines the relation between the spectrum of an element to that of its adjoint. First, by property ii) of Definition 34 of an involution, an element a of a C^* algebra $\mathcal{A}$ is invertible if and only if a^* is invertible and $(a^*)^{-1} = (a^{-1})^*$. Considering instead the elements $a - \alpha 1 \in \mathcal{A}$ and $(a - \alpha 1)^* = a^* - \overline{\alpha} 1 \in \mathcal{A}$ for $\alpha \in \mathbb{C}$ yields that

$$\mathrm{sp}(A^*) = \{\overline{\alpha} \mid \alpha \in \mathrm{sp}\, A\}. \tag{C.3}$$

It is crucial to realize from the above discussion that self-adjoint elements have a spectrum contained within the real line. What is more, one can also show that the spectrum of a self-adjoint element is very explicitly tied to its norm:

Proposition 25. *If $\mathscr{A}$ is a C^*-algebra and a is a self-adjoint element of $\mathscr{A}$, then $\|a\| = r(a)$.*

Finally, it is not immediately clear from the above discussion how the spectrum of an element of $\mathcal{A}$ may change when an element is seen as belonging instead to a subalgebra of $\mathcal{A}$. Fortunately, the spectrum remains invariant:

Proposition 26. *Let $\mathscr{A}$ be a C^*-algebra and $\mathscr{A}'$ a subalgebra of $\mathcal{A}$ that is also a C^*-algebra with the common identity and norm. If $a \in \mathscr{A}$, then $sp_{\mathscr{A}}(a) = sp_{\mathscr{A}'}(a)$.*

It follows that, when it comes to the spectrum of an element, there will be no concern as to the subalgebra of $\mathcal{A}$ in which an element a is considered to "live". In particular, whenever convenient, one may as well consider the C^* algebra generated by $\{a, 1\}$, which is commutative whenever a is normal.

C.2.1 Functional Calculus in C^* algebras

Given an element a in an algebra $\mathcal{A}$, it is clear that polynomials in two variables evaluated on $\{a, a^*\}$ also belong to $\mathcal{A}$. It is thus natural to ask whether an analogous statement may hold for a more general class of functions: say, continuous functions. The so-called *functional calculus* for normal elements in a C^* algebra is the natural framework that makes concrete the notion of "evaluating a continous function on an operator".

The need for a functional calculus is particularly natural in the non-commutative probabilistic setting where elements of $\mathcal{A}$ are interpreted as non-commutative random variables. Of course, in the setting of classical probability, such calculus is immediate: by representing the (classical) random variables as elements in some $\mathscr{L}^p$ space of a given measure space, the needed mechanism is simply that of functional composition. Somewhat surprisingly perhaps, this intuition ends up being highly relevant to the broader operator algebraic setting. Loosely speaking, given a normal element a_0 of a C^* algebra, we will seek to represent a_0 as a continuous function $\hat{a}_0$ on some convenient compact Hausdorff space and to proceed thereafter by functional composition in order to give meaning to "$f(a_0)$". Subsequently, a_0 will be represented as an element of $\mathrm{sp}(a_0)$ so that for any complex continuous function f on $\mathrm{sp}(a_0)$, one has $\mathrm{sp}(f(a_0)) = f(\mathrm{sp}(a_0))$.

For a normal element a_0 of a unital C^* algebra $\mathcal{A}$, let $C^*(a_0)$ be the unital C^* algebra generated by a_0. Note that $C^*(a_0)$ is abelian (commutative). Let Σ_0 the collection of all non-zero homomorphisms of $C^*(a_0) \to \mathbb{C}$. Note that for $h \in \Sigma_0$, we have that $h(a_0) \in \mathbb{C}$ and $h(a_0 - h(a_0)1) = 0$. Therefore, $a_0 - h(a_0)1 \in \mathcal{A}$ is not invertible and $h(a_0) \in \mathrm{sp}(a_0)$ for every $h \in \Sigma_0$. In fact, one can show that the reverse inclusion holds as well and we have:

Proposition 27. *Let $\Sigma_0(a_0) := \{h(a_0) \mid h \in \Sigma_0\}$. Then $sp(a_0) = \Sigma_0(a_0)$.*

In order to consider the continuous functions on Σ_0, the next step consists of topologizing it. The natural candidate is the relative weak* topology inherited from the dual space of $C^*(a_0)$. With this topology, Σ_0 will be referred to as the *maximal* *ideal space* of $C^*(a_0)$. In that setting Σ_0 can be shown to have the following property

Proposition 28. *Σ_0 is a compact Hausdorff space.*

Since Σ_0 is a compact Hausdorff space, then, by the Stone-Weierstrass Theorem, all elements of $\mathscr{C}(\Sigma_0)$ (the complex-valued continuous functions on Σ) can be approximated by polynomials. Using this fact, one can show that there is an isometric $*$-isomorphism of $C^*(a_0)$ onto $\mathscr{C}(\Sigma_0)$. It is given as follows.

Proposition 29. *For any $a \in C^*(a_0)$, Let $\hat{a}$ denote the* Gelfand transform of *a, i.e. the function $\hat{a} : \Sigma_0 \to \mathbb{C}$ with $h \overset{\hat{a}}{\mapsto} h(a)$. Then, the Gelfand transform on $C^*(a_0)$, namely the map $\gamma : C^*(a_0) \to \mathscr{C}(\Sigma_0)$ with $a \overset{\gamma}{\mapsto} \hat{a}$, is an isometric $*$-isomorphism of $C^*(a_0)$ onto $\mathscr{C}(\Sigma_0)$.*

In particular, for any $a \in C^*(a_0)$, one may think of $\hat{a}$ as a copy of a living in $\mathscr{C}(\Sigma_0)$. Now, recall that $\hat{a}$ is the map $h \mapsto h(a) \in \mathrm{sp}(a_0)$. There is therefore a natural isomorphism between $\mathscr{C}(\Sigma_0)$ and $\mathscr{C}(\Sigma_0(a_0)) = \mathscr{C}(\mathrm{sp}(a_0))$. Namely, considering the map $\hat{a}_0 : \Sigma_0 \to \mathrm{sp}(a_0)$ given by $h \overset{\hat{a}_0}{\mapsto} h(a_0)$, the desired isomorphism is the map $\tau_0 : \mathscr{C}(\mathrm{sp}(a_0)) \to \mathscr{C}(\Sigma)$ given by $f \overset{\tau_0}{\mapsto} f \circ \hat{a}_0$. In fact, it can be shown that τ_0 is an isometric $*$-isomorphism. The two such isomorphisms discussed so far in this section are depicted below.

$$\begin{tikzcd} C^*(a_0) \arrow[r, "\gamma_0"] & \mathscr{C}(\Sigma_0) \\ & \mathscr{C}(\mathrm{sp}(a_0)) \arrow[u, "\tau_0"] \end{tikzcd}$$

Now note that $C^*(a_0)$ can be equivalently expressed as the closure of the polynomials in two variables evaluated on $\{a_0, a_0^*\}$. There is nothing to stop us from evaluating these polynomials on elements of of $\mathrm{sp}(a_0)$. In fact:

Proposition 30. *If $p(z, \overline{z})$ is a polynomial in z and $\overline{z}$ and $\gamma_0 : C^*(a_0) \to \mathscr{C}(\Sigma_0)$ the Gelfand transform of $C^*(a_0)$, then $\gamma(p(a_0, a_0^*)) = p \circ \hat{a}_0$.*

Making the anticipated leap from polynomials to continuous functions, it is now clear that the natural notion of "$f(a_0)$" should be the element $\rho_0(f)$, where ρ_0 is the unique map such that the following diagram commutes.

$$\begin{tikzcd} C^*(a_0) \arrow[r, "\gamma_0"] & \mathscr{C}(\Sigma_0) \\ & \mathscr{C}(\mathrm{sp}(a_0)) \arrow[ul, "\rho_0"] \arrow[u, "\tau_0"'] \end{tikzcd}$$

The reader may verify that the following definition is unambiguous:

Definition 36. *For a unital C^* algebra $\mathcal{A}$ and a normal element $a_0 \in A$, let $\rho_0 : \mathscr{C}(sp(a_0)) \to C^*(a_0)$ be given by $\rho_0 = \gamma_0^{-1} \circ \tau_0$ (see above diagram). Then, for every $f \in \mathscr{C}(sp(a_0))$, define*

$$f(a_0) := \rho_0(f).$$

The map $f \mapsto f(a_0)$ fo $\mathscr{C}(sp(a_0)) \to C^*(a_0)$ is called the **functional calculus for** *a.*

The remaining question concerns the spectrum of the element $f(a_0) \in C^*(a_0)$. The Spectral Mapping Theorem, which completes our discussion of the functional calculus in C^* algebras, states that the spectrum of $f(a)$ is precisely the image, under f, of the spectrum of a. The proof is again left as a relatively brief exercise to the reader.[1]

Theorem 22 (Spectral Mapping Theorem). *If $\mathcal{A}$ is a unital C^* algebra and a is a normal element of $\mathcal{A}$, then for every f in $\mathscr{C}(sp(a))$,*

$$sp(f(a)) = f(sp(a)).$$

C.2.2 Hilbert space representations of C^* algebras

In the previous sections, the algebra of bounded linear operators on a Hilbert spaces was invoked as a natural example of a C^* algebras. In a highly concrete sense, it is *the* example. Specifically, the ultimate goal of this section is to explain how any C^* algebra can be represented as a subalgebra of $\mathscr{B}(\mathscr{H})$, for some Hilbert space $\mathscr{H}$.

[1]It may be helpful to remember that ρ_0 is a $*$-isomorphism and to recall Example 26.

The representation is constructive, and is referred to as the *Gelfand-Naimark-Segal* (GNS) construction.

Definition 37. *A* representation *of a C^* algebra $\mathcal{A}$ is a pair $(\pi, \mathscr{H})$, where $\mathscr{H}$ is a Hilbert space and $\pi : \mathcal{A} \to \mathscr{B}(\mathscr{H})$ is a $*$-homomorphism.*

Taking $\mathscr{H}$ to be a Hilbert space and $\mathcal{A}$ to be any C^* subalgebra of $\mathscr{B}(\mathscr{H})$, it immediately follows that the inclusion map $\mathcal{A} \hookrightarrow \mathscr{B}(\mathscr{H})$ is a representation. Two somewhat more sophisticated examples are as follows.

Example 29 Let (X, Ψ, μ) be a σ-finite measure space. Recall that each $f \in \mathscr{L}^\infty(\mu)$ gives rise to a multiplication operator M_f acting on the Hilbert space $\mathscr{H} = \mathscr{L}^2(\mu)$. Then $\pi : \mathscr{L}^\infty(\mu) \to \mathscr{B}(\mathscr{H})$ defined by $\pi(f) = M_f$ is a representation of the C^* algebra $\mathscr{L}^\infty(\mu)$ on $\mathscr{L}^2(\mu)$. •

Example 30 Similarly to the previous example, if X is a compact Hausdorff space and μ a positive Borel measure on X, then $\pi : \mathscr{C}(X) \to \mathscr{B}(\mathscr{L}^2(\mu))$ defined by $\pi(f) = M_f$ is a representation of the C^* algebra $\mathscr{C}(X)$ on $\mathscr{L}^2(\mu)$. •

It is also important to be able to discuss the equivalence of representations. Specifically, two representations $(\pi_1, \mathscr{H}_1)$ and $(\pi_2, \mathscr{H}_2)$ of a C^* algebra $\mathcal{A}$ will be considered *equivalent* if there is Hilbert-space isomorphism $U : \mathscr{H}_1 \to \mathscr{H}_2$ such that $U\pi_1(a)U^{-1} = \pi_2(a)$ for every $a \in \mathcal{A}$.

Note that in the remainder of this section, every C^* algebra will be assumed to be unital and every representation will be assumed to be unit-preserving, in the sense that for a representation $\pi : \mathcal{A} \to \mathscr{B}(\mathscr{H})$, $\pi(1_{\mathcal{A}}) = 1_{\mathscr{B}(\mathscr{H})}$.

A particularly important class of representations of C^* algebras are those representations which are *cyclic*:

Definition 38. *A representation π of a C^* algebra $\mathcal{A}$ is* cyclic *if there is a vector e in $\mathscr{H}$ such that the norm-closure of $\pi(\mathcal{A})e$ equals $\mathscr{H}$.*

In the above definition, e is said to be a *cyclic vector* for the representation π. It is straightforward to check that if $h \in \mathscr{H}$ is a cyclic vector for a representation, then $h/\|h\|$ is also a cyclic vector, justifying the notation. One may verify that the representations of Examples 29 and 30 are cyclic.

Cyclic representations are important in at least two ways. On the one hand, in many concrete settings, $\{\pi(\mathcal{A})e\}$ provides a convenient dense subspace for $\mathscr{H}$. This is is the case for the q-Fock space and the q-Gaussian algebra Γ_q, as discussed further in this chapter. On the other hand, every representation turns out to be equivalent to a direct sum of cyclic representations. Specifically:

Theorem 23. *Let π be a representation of the C^* algebra $\mathcal{A}$. Then, there is a family of cyclic representations $\{\pi_i\}$ of $\mathcal{A}$ such that π and $\hat{\pi} := \oplus_i \pi$ are equivalent.*

A concept intimately connected to cyclic representations is that of a state, defined as follows.

Definition 39. *Let $\mathcal{A}$ be a C^* algebra. A linear functional $\varphi : \mathcal{A} \to \mathbb{C}$ is said to be* positive *if $\varphi(a) \geq 0$ whenever a is a positive element in $\mathcal{A}$. If $\|\varphi\| = 1$, then φ is said to be* a state.

Example 31 Let X be a compact Hausdorff space and $\mathcal{A} = \mathscr{C}(X)$. By Theorem 17 of Appendix B, the positive linear functionals on $\mathcal{A}$ correspond to positive measures on X and the states correspond to probability measures on X. •

The notion of a state arises naturally in the setting of a cyclic representation of a C^* algebra. Let $\pi : \mathcal{A} \to \mathscr{B}(\mathscr{H})$ be a cyclic representation with a cyclic vector e and define the linear functional $\varphi : \mathcal{A} \to \mathbb{C}$ by $\varphi(a) = \langle \pi(a)e, e \rangle$. Then, φ is positive. Indeed, consider $\varphi(a^*a) = \langle \pi(a^*a)e, e \rangle$ and, recalling that φ is a $*$-homomorphism, note that $\pi(a^*a) = \pi(a)^*\pi(a)$. Thus,

$$\varphi(a^*a) = \langle \pi(a)^*\pi(a)e, e \rangle = \langle \pi(a)e, \pi(a)e \rangle = \|\pi(a)e\|^2 \geq 0.$$

Moreover, φ is a bounded linear functional (in the Banach space norm on $\mathcal{A}$) with unit norm. Specifically, by the Cauchy-Schwarz inequality (Theorem 16),

$$\|\varphi\| = \sup_{a\in\mathcal{A}} \frac{|\varphi(a)|}{\|a\|} \leq \sqrt{\langle e, e\rangle} = \|e\|.$$

Then, since φ is taken to be unit-preserving and $|\varphi(1_{\mathcal{A}})| = 1$, it follows that $\|\varphi\| = \|e\|^2$. Thus, $\|\varphi\| = 1$ when the cyclic vector is taken to be a unit vector.

It follows from the previous discussion that the linear functional φ arising from a cyclic representation in the manner described above is a state. The surprise, however, is that every state (or, more generally, every positive linear functional) gives rise to a cyclic representation. This is the essence of the *Gelfand-Naimark-Segal (GNS) construction*:

Theorem 24 (The GNS Construction). *Let $\mathcal{A}$ be a unital C^* algebra.*

If φ is a positive linear functional on $\mathcal{A}$, then there is a cyclic representation $(\pi_\varphi, \mathscr{H}_\varphi)$ of $\mathcal{A}$ with cyclic vector e such that $\varphi(a) = \langle \pi_\varphi(a)e, e\rangle$ for all $a \in \mathcal{A}$. Furthermore, if $(\pi, \mathscr{H})$ is a cyclic representation of $\mathcal{A}$ with cyclic vector e and $\hat{\varphi}(a) := \langle \pi(a)e, e\rangle$ a linear functional on $\mathcal{A}$, then $(\pi_{\hat{\varphi}}, \mathscr{H}_{\hat{\varphi}})$ and $(\pi, \mathscr{H})$ are equivalent.

The following recapitulative example brings together several key notions introduced so far in this chapter.

Example 32 Recall Example 7 of Appendix A, concerned with a complex unital algebra $\mathcal{A}$ with generators $\{x_i\}_{i\in I}$ taken over some totally ordered finite set I and relations

$$x_i x_j - \epsilon(i,j) x_j x_i = 2\delta_{i,j}, \quad i, j \in I,$$

where $\epsilon(i,j) = \epsilon(j,i) \in \{-1, 1\}$. Recall that a basis of $\mathcal{A}$ consists of the words $x_A = x_{i_1} \ldots x_{i_p}$ taken over all $A = \{i_1, \ldots, i_p\} \subset I$ (for $i_1 < \ldots < i_p$) with $x_\emptyset = 1$. Consider an involution $* : \mathcal{A} \to \mathcal{A}$ with the additional property that $x_i^* = x_i$. Then, $\mathcal{A}$ becomes a $*$-algebra, but there is yet no norm defined on it. Let $\tau^\epsilon : \mathcal{A} \to \mathbb{C}$ be the linear map given by $\tau^\epsilon(x_A) = \delta_{A,\emptyset}$ for all $A \subset I$. The reader may now verify that the map $\mathcal{A} \times \mathcal{A} \to \mathbb{C}$ given by $(a_1, a_2) \mapsto \tau^\epsilon(a_1 a_2^*)$ is an inner product and that, with respect to it, the basis $\{x_A\}_{A \subset I}$ is orthonormal. Letting $\| \, \|$ denote the norm on $\mathcal{A}$ induced by this inner product, it is also immediate that $\|aa^*\| \leq \|a\|^2$, and the completion of $\mathcal{A}$, denoted here by $\overline{\mathcal{A}}$, is therefore a C^* algebra.

Now, let $\mathscr{H}$ denote the Hilbert space obtained by completing $\mathcal{A}$ with respect to $\| \, \|$ (i.e. $\mathscr{H}$ ignores the involutive and multiplicative structure present on $\overline{\mathcal{A}}$). To obtain a representation π of $\overline{\mathcal{A}}$ on $\mathscr{H}$, it now suffices to let $\overline{\mathcal{A}}$ act by left multiplication on $\mathscr{H}$, i.e. $\pi(a)h = ah$ for all $a \in \overline{\mathcal{A}}, h \in \mathscr{H}$.

For a context closely related to the setting of this thesis, and one in which the above objects appear naturally, the reader is referred to [Bia97a]. •

In fact, the representation π of the previous example is precisely the outcome of the GNS construction for the C^* algebra $\overline{\mathcal{A}}$ and the state τ^ϵ. Though it is the properties of the GNS representation rather than the details of its construction that are of present concern, it is nevertheless interesting to at least outline the process. Starting with an $*$-algebra $\mathcal{A}$ and a state $\varphi : \mathcal{A} \to \mathcal{A}$, one may define a (semi-)inner product on $\mathcal{A}$ by $(a_1, a_2) \mapsto \varphi(a_1 a_2^*)$. Taking the completion of $\mathcal{A}$ (or, rather, of the quotient of $\mathcal{A}$ by the kernel of the semi-inner product), yields a Hilbert space. Realizing that $\mathcal{A}$ acts on itself by left multiplication then yields the desired representation π. (The fact that $\mathcal{A}$ is assumed to be a C^* algebra, rather than only a $*$-algebra, comes in ensuring that this observation also holds when $\mathcal{A}$ is made to act on the quotient of $\mathcal{A}$ by the kernel of the semi-inner product.) The following are two important examples:

Example 33 Let G be a discrete group and let $\mathbb{C}G$ denote its *group algebra*, that is, the complex vector space having a basis indexed by the elements of G. In particular, every element $a \in \mathbb{C}G$ can be written as $a = \sum_{g \in G} \alpha_g g$ for finitely many $\alpha_g \neq 0$. One may then naturally turn $\mathbb{C}G$ into a $*$-algebra by setting $a^* := \sum_{g \in G} \overline{\alpha}_g g^{-1}$, with the overline denoting complex conjugation. There is also a natural inner product on $\mathbb{C}G$, given by $\langle a, b \rangle = \sum_{g \in G} \overline{a}_g b_g$. The reader may quickly verify that completing $\mathbb{C}G$ with respect to the norm induced by this inner product yields a C^* algebra, denoted $C^*(G)$.

Next, letting 1 denote the unit in G, define a state τ_G on $C^*(G)$ by $\tau_G(a) := \alpha_1$ for $a = \sum_{g \in G} \alpha_g g$. Note that τ_G has the property that $\tau_G(ab) = \tau_G(ba)$ for all $a, b \in G$, and (for reasons that will become apparent shortly) τ_G is referred to as the *canonical trace on G*. Then, the GNS construction yields the representation of $C^*(G)$ on $\ell^2(G)$, with $C^*(G)$ acting on itself by left multiplication. •

Example 34 Let X be a compact Hausdorff space and $\mathcal{A} = \mathscr{C}(X)$. By Example 17 of Appendix B, if L is a positive linear functional on $\mathcal{A}$, then there is a positive measure μ on X such that $L(f) = \int_X f \, d\mu$ for all $f \in \mathcal{A}$. Then, the representation $(\pi, \mathscr{H})$ of $\mathcal{A}$ obtained by the GNS construction is given by $\mathscr{H} = \mathscr{L}^2(\mu)$ and $\pi(f) = M_f$. In other words, $\mathscr{C}(X)$ is represented as acting by left multiplication on $\mathscr{L}^2(\mu)$. •

The flow of our discussion of operator algebras so far was as follows. After first recalling the basics of Banach spaces and Hilbert spaces, we spent some time overviewing the elementary facts surrounding the algebras of operators on these spaces. We then took a leap and transported the discussion to a fully abstract setting, by considering the properties of Banach algebras and, more narrowly, C^* algebras. A few general examples served to illustrate the extent to which C^* algebras can be found in settings other than as subalgebras of $\mathscr{B}(\mathscr{H})$. Yet, the ultimate realization of Section C.2.2 was in fact that C^* algebras "are" subalgebras of $\mathscr{B}(\mathscr{H})$!

The remainder of this chapter will therefore return to the setting of bounded linear operators on Hilbert spaces. In particular, we next consider the spectral decomposition for normal elements of a C^* algebra by first representing these as normal operators on a Hilbert space.

C.2.3 The Spectral Theorem

In rough terms, the Spectral Theorem states that normal operators can be diagonalized. Starting out in a more familiar setting, if N is a normal bounded linear operator on a Hilbert space of dimension $d < \infty$, with eigenvalues $\alpha_1, \ldots, \alpha_d$ (counted with multiplicities), then the corresponding eigenvectors $e_1, \ldots, e_d$ form an orthonormal basis for $\mathscr{H}$. The resulting decomposition

$$N = \sum_{i=1}^{n} \lambda_k E_k, \tag{C.4}$$

where E_k is the orthogonal projection of $\mathscr{H}$ onto $\ker(N - \lambda_k I)$, can be seen as providing a complete statement about the nature and structure of a normal operator. In order to pass from the finite-dimensional setting to that of an arbitrary (or, at least, separable) Hilbert space, several definitions are in order.

Definition 40. *Let X be a set, Ψ a σ-algebra of subsets of X and $\mathscr{H}$ a Hilbert space. A* spectral measure *for $(X, \Psi, \mathscr{H})$ is a function $E : \Psi \to \mathscr{B}(\mathscr{H})$ with the following properties:*

i. for each $\Delta \in \Psi$, $E(\Delta)$ is a projection;

ii. $E(\emptyset) = 0$ and $E(X) = 1$;

iii. $E(\Delta_1 \cap \Delta_2) = E(\Delta_1)E(\Delta_2)$ for $\Delta_1, \Delta_2 \in \Psi$;

iv. if $\{\Delta_n\}_{n\geq 1}$ is a sequence of pair-wise disjoint elements of Ψ, then

$$\mathbb{E}\left(\cup_{n=1}^{\infty} \Delta_n\right) = \sum_{n=1}^{\infty} E(\Delta_n).$$

Note that item (3) of the above definition implies that if Δ_1 and Δ_2 are disjoint, the corresponding projections $E(\Delta_1)$ and $E(\Delta_2)$ are orthogonal. Then, (4) becomes a statement about a sum of pair-wise orthogonal projections, which is well-defined by Example 21 of the previous chapter.

For example, given a compact Hausdorff space X, the Borel σ-field on X and a measure μ, consider the Hilbert space $\mathscr{L}^2(\mu)$. Then, one may easily show that letting $E(\Delta)$ be given as the characteristic function of the Borel subset Δ defines a spectral measure for the corresponding triple. Note that though the Hilbert space in this example is separable, there may be uncountably many projections involved. In order to arrive at a generalization of (C.4), one must use the following fact.

Proposition 31. *If E is a spectral measure for $(X, \Psi, \mathscr{H})$ and $f : X \to \mathbb{C}$ is a bounded Ψ-measurable function, then there is a unique operator $A \in \mathscr{B}(\mathscr{H})$ such that for all $\epsilon > 0$ and elements $\Delta_1, \ldots, \Delta_n$ in Ψ that form a partition of X with $\sup\{|f(x) - f(x')|; x, x' \in \Delta_k\} < \epsilon$ $(1 \leq k \leq n)$, one has*

$$\left\| A - \sum_{k=1}^{n} f(x_k)E(\Delta_k) \right\| < \epsilon,$$

for any $x_k \in \Delta_k$ and all $1 \leq k \leq n$. The operator A is denoted $\int f\, dE$.

Letting X be the spectrum of a normal operator, Ψ the Borel subsets, and recalling that the spectrum of a bounded operator is bounded, one may take the above function f to be the identity. In fact, this is the relevant choice:

Theorem 25 (The Spectral Theorem). *If N is a normal operator, there is a unique spectral measure E on the Borel subsets of $sp(N)$ such that:*

i. $N = \int z dE(z)$;

ii. if O is a nonempty relatively open subset of $sp(N)$, $E(O) \neq 0$;

iii. if $A \in \mathscr{B}(\mathscr{H})$, then $AN = NA$ and $AN^* = N^*A$ if and only if $AE(\Delta) = E(\Delta)A$ for every Borel subset Δ of $sp(N)$.

Applications of the spectral theorem abound; for instance, the Spectral Theorem yields a convenient and profound description of unitary invariance for normal operators (cf. Theorem 10.1, §IX.10 of [Con90]). Another application is the Theorem 26 below, endowing normal operators with natural probabilistic interpretations.

Theorem 26. *If N is a normal operator on $\mathscr{H}$, then there is a measure space (X, Ψ, μ) and a function ϕ in $L^\infty(X, \Psi, \mu)$ such that N is unitarily equivalent on $\mathscr{L}^2(\mu)$ to $M_\phi : L^2(X, \Psi, \mu) \to L^2(X, \Psi, \mu)$, given by $M_\phi(f) = \phi f$ for all $f \in L^\infty(X, \Psi, \mu)$. If $\mathscr{H}$ is separable, then the measure space (X, Ψ, μ) is σ-finite.*

C.3 $\mathscr{B}(\mathscr{H})$ and von Neumann Algebras

Given a subalgebra $\mathcal{A}$ of $\mathscr{B}(\mathscr{H})$, there are two related subalgebras of $\mathscr{B}(\mathscr{H})$ that will play the central role in this chapter. The first is the *commutant of* $\mathcal{A}$, denoted $\mathcal{A}'$, and given as the subset of $\mathscr{B}(\mathscr{H})$ that commutes with every element in $\mathcal{A}$, i.e. $\mathcal{A}' = \{b \in \mathscr{B}(\mathscr{H}) \mid ab = ba, \forall\, a \in \mathcal{A}\}$. The second is the *bicommutant of* $\mathcal{A}$, denoted $\mathcal{A}''$ and defined as the commutant of $\mathcal{A}'$. Clearly, $\mathcal{A} \subseteq \mathcal{A}''$, but the converse need not hold. For example, if $\mathscr{H}$ is the Euclidean space $\mathbb{R}^2$ and $\mathscr{B}(\mathscr{H})$ are therefore the 2×2 real matrices, let $\mathcal{A}$ be the subalgebra generated by the identity and the matrix $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$; then, $\mathcal{A}'$ is the one-dimensional real vector space spanned by the identity, which in turn yields that $\mathcal{A}''$ is the whole of $\mathscr{B}(\mathscr{H})$. The reverse inclusion however does happen to hold if $\mathcal{A}$ is a *self-adjoint* algebra in the finite-dimensional setting:

Proposition 32. *Let $\mathcal{A}$ be a unital self-adjoint subalgebra of $\mathscr{B}(\mathscr{H})$. Then, if $\dim(\mathscr{H}) < \infty$, then $\mathcal{A} = \mathcal{A}''$.*

For $\mathcal{A}$ a unital C^*-subalgebra of $\mathscr{B}(\mathscr{H})$, the relation between $\mathcal{A}$ and $\mathcal{A}''$ can be shown to be the following:

Theorem 27 (von Neumann's Bicommutant Theorem)**.** *If $\mathcal{A}$ is a unital C^*-subalgebra of $\mathscr{B}(\mathscr{H})$, then the SOT-closure of $\mathcal{A}$ equals the WOT-closure of $\mathcal{A}$, which equals $\mathcal{A}''$.*

Since the norm topology is finer than the strong operator topology, it follows that $\mathcal{A}''$ is a C^* algebra. However, either way one chooses to view it, the property of being SOT closed or being equal to its bicommutant lends $\mathcal{A}$ special structure, and the class of such algebras therefore merits distinction:

Definition 41. *A* von Neumann algebra *$\mathcal{A}$ is a C^* subalgebra of $\mathscr{B}(\mathscr{H})$ such that $\mathcal{A}'' = \mathcal{A}$.*

It follows from the definition that $\mathscr{B}(\mathscr{H})$ and $\mathbb{C}$ are von Neumann algebras. By Proposition 32, any finite-dimensional self-adjoint subalgebra of $\mathscr{B}(\mathscr{H})$ containing the unit is a von Neumann algebra. It takes some more work to verify that if (X, Ψ, μ) is a σ-finite measure space, then $\mathcal{A}_\mu = \{M_f \mid f \in \mathscr{L}^\infty(\mu)\} \subset \mathscr{B}(\mathscr{L}^2(\mu))$ is a commutative von Neumann algebra. Starting with these examples, one may easily generate further ones, by taking direct sums or considering the underlying Hilbert-space isomorphisms:

Proposition 33. *i. If $\mathcal{A}_1, \mathcal{A}_2, \ldots$ are von Neumann algebras, then so is $\mathcal{A}_1 \oplus \mathcal{A}_2 \oplus \ldots$.*

ii. If $\mathcal{A}_1, \mathcal{A}_2$ are von Neumann algebras on $\mathscr{H}_1$ and $\mathscr{H}_2$, respectively, and if $U : \mathscr{H}_1 \to \mathscr{H}_2$ is an isomorphism such that $U\mathcal{A}_1U^{-1} = \mathcal{A}_2$, then $U\mathcal{A}_1'U^{-1} = \mathcal{A}_2'$.

Returning to the question of representations in the setting where $\mathcal{A}$ is a C^*-subalgebra of $\mathscr{B}(\mathscr{H})$, let e be a *cyclic vector for* $\mathcal{A}$ if it is a cyclic vector for the representation $(\hookrightarrow, \mathscr{H})$ of $\mathcal{A}$ (cf. Definition 38), i.e. if the norm closure of $\mathcal{A}e$ equals the whole of $\mathscr{H}$. On the other hand, call $\hat{e}$ a *separating vector* for $\mathcal{A}$ if $\hat{e}$ is in the kernel of no other element of $\mathcal{A}$ but the zero element, i.e. if given any $a \in \mathcal{A}$ such that $a\hat{e} = 0$ implies that $a = 0$.

Let e be cyclic for $\mathcal{A}$ and suppose that $be = 0$ for some $b \in \mathcal{A}'$; then for every $a \in \mathcal{A}$, $bae = abe = 0$ implying that, since the closure of $\{ae \mid a \in \mathcal{A}\}$ is all of $\mathscr{H}$, b must be zero. Therefore: if e is a cyclic vector of $\mathcal{A}$, then e is a separating vector for $\mathcal{A}'$. Of course, if $\mathcal{A}$ is a commutative subalgebra of $\mathscr{B}(\mathscr{H})$, then $\mathcal{A} \subseteq \mathcal{A}'$ and every cyclic vector for $\mathcal{A}$ is also separating for $\mathcal{A}$.

One may check that if $\mathcal{A} = \mathscr{B}(\mathscr{H})$, then no vector in $\mathscr{H}$ is separating for $\mathcal{A}$, but that every non-zero vector in $\mathscr{H}$ is a cyclic vector for $\mathcal{A}$. On the other hand, if $\mathcal{A} = \mathbb{C}1$ and $\dim \mathscr{H} \geq 2$, then every non-zero vector is separating for $\mathcal{A}$, but $\mathcal{A}$ has no cyclic vector. If (X, Ψ, μ) is a σ-finite measure space and $f \in \mathscr{L}^2(\mu)$ such that $\mu(\{x \in X \mid f(x) = 0\}) = 0$, then f is a cyclic and separating vector for the von Neumann algebra $A_\mu = \{M_f \mid f \in \mathscr{L}^\infty(\mu)\}$. This last example is special in that $\mathcal{A}_\mu$ a commutative C^* subalgebra of $\mathscr{B}(\mathscr{H})$, where $\mathscr{H}$ is separable. Under these conditions, more can be said:

Theorem 28. *Let $\mathscr{H}$ be a separable Hilbert space and $\mathcal{A}$ a commutative C^* subalgebra of $\mathscr{B}(\mathscr{H})$. The following statements are equivalent:*

i. $\mathcal{A} = \mathcal{A}'$.

ii. *$\mathcal{A}$ has a cyclic vector, contains the identity* 1*, and is SOT closed.*

iii. *There is a compact metric space X, a positive Borel measure μ with support X, and an isomorphism $U : \mathscr{L}^2(\mu) \to \mathscr{H}$ such that $\mathcal{A} = U\mathcal{A}_\mu U^{-1}$, where $\mathcal{A}_\mu = \{M_f \mid f \in \mathscr{L}^\infty(\mu)\}$.*

Moreover, if $\mathcal{A}$ is a commutative C^* subalgebra of $\mathscr{B}(\mathscr{H})$ for $\mathscr{H}$ separable, then $\mathcal{A}$ has a separating vector.

The above item *iii* should not be taken lightly. It establishes an isomorphism between any maximal commutative von Neumann algebra and the algebra $\mathscr{L}^2(\mu)$, for some measurable space with measure μ, acting by multiplication on $\mathscr{L}^\infty(\mu)$. In this sense, the theory of von Neumann algebras is considered to be a *non-commutative measure theory.* In this vein, one may wonder what becomes of the spectral measure of a normal operator living in a von Neumann algebra; indeed, the developments of the following section are central to the operator algebraic approach to non-commutative probability.

C.3.1 Scalar-valued spectral measures

In the remainder of this section, assume that the underlying Hilbert space $\mathscr{H}$ is separable. Let a be a normal bounded linear operator on $\mathscr{H}$ and let $vN(a)$ be the von Neumann algebra generated by a, i.e. the intersection of all von Neumann algebras containing a. By Theorem 28, $vN(a)$ has some separating vector e. Letting E be the (projection-valued) spectral measure for a, define a measure μ on $\mathrm{sp}(a)$ by

$$\mu(\Delta) = \langle E(\Delta)e, e\rangle = \|E(\Delta)e\|^2, \tag{C.5}$$

for every Borel subset Δ of $\mathrm{sp}(a)$. Note that if $E(\Delta) = 0$, then $\mu(\Delta) = 0$ and that, conversely, if $\mu(\Delta) = 0$, it must follow that $E(\Delta) = 0$ since e is a separating vector for $vN(a)$. In other words, μ and E are mutually absolutely continuous.

Definition 42. *A* scalar-valued spectral measure *for a normal element a with spectral measure E is a positive Borel measure μ defined on sp(a) with the property that μ and E are mutually absolutely continuous.*

The measure given in (C.5) does more than guarantee the existence of a scalar-valued spectral measure; indeed, one can show that every scalar-valued spectral measure for a is of this form.

Working with the scalar-valued spectral measures, one can derive specialized versions of the functional calculus and the Spectral Theorem in the setting of von Neumann algebras. The essential difference is that instead of considering the *continuous* functions defined on the spectrum of a normal operator a, one may now pass to *essentially bounded* functions on $\mathrm{sp}(a)$. In particular, for μ a scalar-valued spectral measure of a, to each $\phi \in L^\infty(\mu)$ one may associate an element $\phi(a) \in vN(a)$ and compute its spectrum as follows.

Theorem 29 (Functional calculus in von Neumann algebras). *If $\mathscr{H}$ is a separable Hilbert space, a_N a normal operator on $\mathscr{H}$, and μ a scalar-valued spectral measure for a, then there exists a map $\rho : \mathscr{L}^\infty(\mu) \to vN(a)$ given by $\rho(\phi) = \phi(a)$ that is both an isometric $*$-isomorphism and a homeomorphism, with the weak* topology on $\mathscr{L}^\infty(\mu)$ and the weak operator topology on $W^*(a)$.*

Theorem 30 (Spectral Mapping Theorem in von Neumann algebras). *If a is a normal operator on a separable Hilbert space, μ a scalar-valued spectral measure for a, and $\phi \in \mathscr{L}^\infty(\mu)$, then the spectrum of $\phi(a)$ equals the μ-essential range of ϕ.*

C.3.2 Factors

This section considers the notion of an "irreducible" von Neumann algebra, referred to as *a factor*. For an algebra $\mathcal{A}$, the *center* of $\mathcal{A}$ is the subset of the commutant of $\mathcal{A}$ that is contained in $\mathcal{A}$, i.e. the center of $\mathcal{A}$ is given as $\{a \in \mathcal{A} \mid ab = ba, \forall\, b \in \mathcal{A}\}$. Then, a factor is a von Neumann algebra with a trivial center:

Definition 43. *A von Neumann algebra whose center is $\mathbb{C}1$ is called a factor.*

Clearly, $\mathscr{B}(\mathscr{H})$ is a factor. Similarly, if $\mathscr{H} = \mathscr{H}_1 \otimes \mathscr{H}_2$ and if $\mathcal{A} = \mathscr{B}(\mathscr{H}_1) \otimes 1$, then $\mathcal{A}' = 1 \otimes \mathscr{B}(\mathscr{H}_2)$ and $\mathcal{A}$ and $\mathcal{A}'$ are both factors.

Another important example of a factor is a *free group factor*. Let G be a discrete group and, revisiting Example 33, let $C^*(G)$ be the C^* algebra of G. Recall the Hilbert-space representation of $C^*(G)$ given by left-multiplication by G on $\ell^2(G)$, with the inner product given by $\langle a_1, a_2 \rangle \mapsto \tau(a_1 a_2^*)$. Then, one may also consider the *group von Neumann algebra* of G, denoted $vN(G)$ and given by the WOT closure of the subalgebra of $\mathscr{B}(\ell^2(G))$ corresponding to $C^*(G)$.

If F_n is the free group with n generators, then $\mathbb{C}F_n$ is the complex *free group algebra* on n generators and $vN(F_n)$ is the *free group factor* of order n. While it is clear that for $n \neq m$, $\mathbb{C}F_n$ and $\mathbb{C}F_m$ are two very different objects, it is far from obvious what happens at the level of the corresponding von Neumann algebras. In fact, the following is one of the major outstanding conjectures in the theory of operator algebras:

$$\text{For } n \neq m \ (n, m \geq 2)\text{, is it true that } vN(F_n) \simeq vN(F_m)?$$

The above problem is referred to as the *free group factors isomorphism problem.* It is in search of its resolution, and specifically in search of invariants for distinguishing classes of von Neumann algebras related to free group factors, that the theory of free probability (discussed in Chapter 3) was born.

C.3.3 Projections

Unlike C^* algebras, which need not even contain a single non-trivial projection, von Neumann algebras abound in projections. The latter can be used to derive some distinguishing features of von Neumann algebras.

First note that the projctions in $\mathscr{B}(\mathscr{H})$ form an ortholattice. Indeed, given two projections p and q, let

$$p \leq q \iff p\mathscr{H} \subseteq q\mathscr{H}$$

$$p \wedge q = \text{ orthogonal projection onto } p\mathscr{H} \cap q\mathscr{H}$$

$$p^{\perp} = 1 - p$$

$$p \vee q = (p^{\perp} \wedge q^{\perp})^{\perp}.$$

If $\mathcal{A}$ is a von Neumann algebra, a non-zero projection $p \in \mathcal{A}$ is said to be *minimal* if there is no lesser non-trivial projection in $\mathcal{A}$, i.e. if $q \leq p$ implies that either $q = 0$ or $q = p$. Pursuing the point of view of a non-commutative measure theory, a minimal projection is analogous to an atom of a measure.

The presence of a minimal projection leads to the first distinguishing feature that a factor may have. In particular:

Definition 44. *A factor containing a minimal projection is said to be as a* type I *factor.*

Given any Hilbert space $\mathscr{H}$, any one-dimensional projection is clearly minimal, and $\mathscr{B}(\mathscr{H})$ is therefore a type I factor. It is also instructive to consider the following finite-dimensional example, which extends naturally to the setting where the Hilbert spaces are separable.

Example 35 Let $\mathscr{H}_1, \mathscr{H}_2$ be two finite-dimensional Hilbert spaces and consider the subalgebra $\mathscr{B}(\mathscr{H}_1) \otimes 1$. An element of the algebra can be written as $A \otimes I$, where I is the identity matrix on $\mathscr{H}_2$ and A a matrix on $\mathscr{H}_1$. Then, the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & 0 \end{pmatrix}$$

is a minimal projection. As we have previously seen, $\mathscr{B}(\mathscr{H}_1) \otimes 1$ is a factor, so it is therefore a type I factor. •

In the spirit of the above example, there is an easy classification for all type I factors, obtained as follows.

Theorem 31. *If $\mathcal{A}$ is a type I factor on a separable Hilbert space $\mathscr{H}$, then there are* *separable Hilbert spaces* $\mathscr{H}_1$ *and* $\mathscr{H}_2$ *and a unitary* $U : \mathscr{H} \to \mathscr{H}_1 \otimes \mathscr{H}_2$ *such that* $U\mathcal{A}U^* = \mathscr{B}(\mathscr{H}) \otimes 1$.

The characterization of factors based on the properties of the projections they contain can be pursued further. In particular, a factor is said to be of *type II* if it has no minimal projection, but contains a non-zero *finite* projection. However, rather than discussing finite projections and type II factors at large, we will instead focus on the type II_1 factors, which can be described in terms of *traces*.

C.3.4 Traces

This section deals with the generalization of the notion of the matrix trace to the general setting of bounded linear operators on a Hilbert space. In this general setting, a *trace* need not exist and otherwise need not be unique, but it is precisely the trace-related properties that will present another key distinguishing feature of factors.

Definition 45. *A linear functional* $tr : \mathcal{A} \to \mathbb{C}$ *with the property that for all* $a, b \in \mathcal{A}$, $tr(ab) = tr(ba)$, *is said to be a* trace. *A trace* tr *on* $\mathcal{A}$ *is said to be* positive *if* $tr(a^*a) \geq 0$ *for all* $a \in \mathcal{A}$, faithful *if* $tr(a^*a) = 0 \implies a = 0$, *and* normalized *if* $tr(1) = 1$.

In the finite-dimensional setting, the usual matrix trace is indeed a trace. In contrast, one can show that there exists no trace on $\mathscr{B}(\mathscr{H})$ when $\mathscr{H}$ is infinite-dimensional. As the reader may also recall, for any free group F_n on n generators, there is a canonical trace on $vN(F_n)$ – it is the functional τ_{F_n} described in Example 33.

Beyond the existence of the trace, one may also wish to consider its continuity. For the continuity of the trace to be a useful distinguishing feature, the strong topology is generally too fine, the the weak topology ends up being too coarse, and it is instead the *ultraweak operator topology* that is just right.[2] In particular, the ultraweak operator

[2]Note, that the ultraweak operator topology is finer than the weak operator topology, but is generally not comparable to the strong operator topology.

topology is generated by the sets

$$\left\{A \in \mathscr{B}(\mathscr{H}); \quad \left|\sum_{i=1}^{\infty}\langle (A - A_0)x_i, y_i\rangle\right| < \epsilon\right\}$$

taken over all $A_0 \in \mathscr{B}(\mathscr{H})$, $\epsilon > 0$, and all sequences $(x_i)_{i\in\mathbf{N}}$ and $(y_i)_{i\in\mathbf{N}}$ in $\ell^2(\mathscr{H})$.

Definition 46. *A type II_1 factor is an infinite-dimensional factor $\mathcal{A}$ on $\mathscr{H}$ admitting a non-zero positive trace that is ultraweakly continuous.*

One may verify that τ_{F_n} is ultraweakly continuous and $vN(F_n)$ is therefore a type II_1 factor. Note that the uniqueness of the trace is equivalent to factoriality in the following sense:

Theorem 32. *If $\mathcal{A}$ is a von Neumann algebra with a positive ultraweakly continuous faithful normalized trace tr, then $\mathcal{A}$ is a type II_1 factor if and only if tr is the unique ultraweakly continuous normalized trace on $\mathcal{A}$.*

All factors that are neither type I nor type II are referred to as the *type III factors*. Note that the only trace on type III factors takes value ∞ on non-zero positive elements.

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