Partial solutions to problem set 9

Problems from Strauss, Walter A. *Partial Differential Equations: An Introduction.* New York, NY: Wiley, March 3, 1992. ISBN: 9780471548683.

Problem 107.2 (and Problem 107.5 at the same time!) Let $\varphi(x) = x^2$, $0 \leq x \leq 1 = l$.
a) Fourier sine series:

$$\varphi(x) = \sum_{n=1}^{\infty} A_n \sin n\pi x.$$

$$\begin{aligned}
A_n &= 2\int_0^1 \varphi(x) \sin n\pi x dx = 2\int_0^1 x^2 \sin n\pi x dx \\
&= -\frac{2}{n\pi} x^2 \cos n\pi x \Big|_0^1 - \left(-\frac{2}{n\pi}\right) \int_0^1 2x \cos n\pi x dx \\
&= -\frac{2}{n\pi} \cos n\pi + \frac{4}{n\pi}\left(\frac{1}{n\pi}\right) x \sin n\pi x \Big|_0^1 - \frac{4}{n\pi}\left(\frac{1}{n\pi}\right) \int_0^1 \sin n\pi x dx \\
&= -\frac{2}{n\pi}((-1)^n) + \frac{4}{n^2\pi^2}(0-0) - \frac{4}{n^2\pi^2}\left(-\frac{1}{n\pi}\right) \cos n\pi x \\
&= \frac{2}{n\pi}((-1)^{n+1}) + \frac{4}{n^3\pi^3}((-1)^n - 1) \\
&= \begin{cases} -\frac{2}{n\pi}, & \text{if } n \text{ is even} \\ \frac{2}{n\pi} - \frac{8}{n^3\pi^3}, & \text{if } n \text{ is odd.} \end{cases}
\end{aligned}$$

b) Fourier cosine series:

$$\varphi(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos n\pi x.$$

If $n \neq 0$:

$$\begin{aligned}
A_n &= 2\int_0^1 \varphi(x) \cos n\pi x dx = 2\int_0^1 x^2 \cos n\pi x dx \\
&= \frac{2}{n\pi} x^2 \sin n\pi x \Big|_0^1 - \left(\frac{2}{n\pi}\right) \int_0^1 2x \sin n\pi x dx \\
&= -\frac{4}{n\pi}\left(-\frac{1}{n\pi}\right) x \cos n\pi x \Big|_0^1 - \left(-\frac{4}{n\pi}\right)\left(-\frac{1}{n\pi}\right) \int_0^1 \cos n\pi x dx \\
&= \frac{4}{n^2\pi^2} \cos n\pi - \frac{4}{n^2\pi^2}\left(\frac{1}{n\pi}\right) \sin n\pi x \Big|_0^1 \\
&= \frac{4}{n^2\pi^2}(-1)^n \\
&= \begin{cases} \frac{4}{n^2\pi^2}, & \text{if } n \text{ is even}, n \neq 0 \\ -\frac{4}{n^2\pi^2}, & \text{if } n \text{ is odd.} \end{cases}
\end{aligned}$$

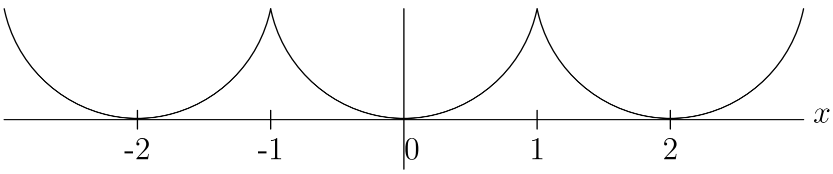
$$A_0 = 2\int_0^1 x^2 dx = 2\frac{x^3}{3}\Big|_0^1 = \frac{2}{3}.$$

Thus, the two Fourier series for x^2, $x \in [0,1)$ are

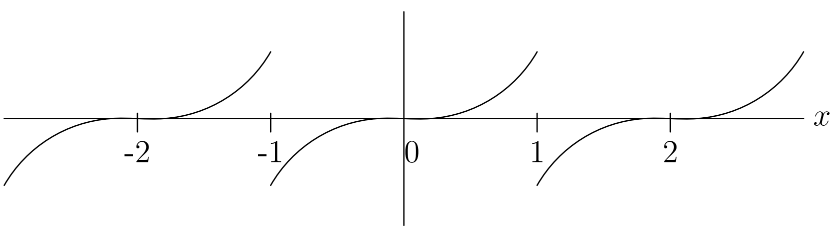
$$x^2 = \sum_{n\geq 1,\ \text{odd}} \left(\frac{2}{n\pi} - \frac{8}{n^3\pi^3}\right) \sin n\pi x - \sum_{n\geq 2,\ \text{even}} \frac{2}{n\pi} \sin n\pi x.$$

$$x^2 = \frac{1}{3} + \sum_{n=1}^{\infty} (-1)^n \frac{4}{n^2\pi^2} \cos n\pi x.$$

Notice that the coefficients of the Fourier cosine series decay faster as $n \to \infty$ since they represent a continuous function on the whole real line:



while the Fourier sine series represents a discontinuous one:
The coefficients can also be obtained by direct integration (cf. 107.5).



Thus, the Fourier sine series for $\varphi(x) = x$ on $(0,1)$ is from the book, Ex. 3,

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{2}{n\pi} \sin n\pi x.$$

Integrating from 0 to x gives

$$\sum_{n=1}^{\infty} (-1)^n \frac{2}{n^2\pi^2} (\cos n\pi x - 1),$$

where we used the fact that $\int_0^x \sin n\pi x dx = -\frac{1}{n\pi} \cos n\pi x \Big|_0^x = \frac{1}{n\pi} - \frac{1}{n\pi} \cos n\pi x$.

On the other hand, $\int_0^x x' dx' = \frac{x^2}{2}$, so

$$\sum_{n=1}^{\infty} (-1)^n \frac{2}{n^2\pi^2} (\cos n\pi x - 1) = \frac{x^2}{2},$$

i.e.

$$x^2 = \sum_{n=1}^{\infty} (-1)^n \frac{4}{n^2\pi^2} \cos n\pi x - \sum_{n=1}^{\infty} \frac{4}{n^2\pi^2} (-1)^n.$$

Rather than finding $\sum_{n=1}^{\infty} \frac{4}{n^2\pi^2} (-1)^n$, we find the A_0 coefficient of the Fourier cosine series as before, so $A_0 = \frac{1}{3}$, so

$$x^2 = \frac{1}{3} + \sum_{n=1}^{\infty} (-1)^n \frac{4}{n^2\pi^2} \cos n\pi x,$$

as before. (Note that this gives $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2\pi^2} = \frac{1}{12}$, as in 5b!) Similarly, integrating the Fourier cosine series for $\varphi(x) = x$ on (0,1), i.e.

$$x = \frac{1}{2} - \frac{4}{\pi^2} \sum_{n\geq 1,\text{ odd}} \frac{1}{n^2} \cos\frac{n\pi x}{1}$$

gives

$$x^2 = x - \frac{8}{\pi^2} \sum_{n\geq 1,\text{ odd}} \frac{1}{n^3\pi} \sin n\pi x,$$

where we used the fact that

$$\int_0^x \cos\frac{n\pi x'}{1} dx' = \frac{1}{n\pi} \sin n\pi x.$$

The second part agrees with the part of the Fourier sine series for x^2 we had before that involves $\frac{1}{n^3}$ terms. The $\frac{1}{n}$ part, on the other hand, is just the Fourier sine series for $\varphi(x) = x$, in perfect agreement with our latest result!

Problem 107.8: Solve

$$(1)\begin{cases} \nu_t &= k\nu_{xx} \quad 0 < x < 1 \\ \nu(0,t) &= 0 \\ \nu(1,t) &= 1 \\ \sigma(x,0) &= \varphi(x) = \begin{cases} \frac{5x}{2} & \text{for } 0 < x < \frac{2}{3} \\ 3 - 2x & \text{for } \frac{2}{3} < x < 1. \end{cases} \end{cases}$$

To get homogenous boundary conditions we find $U = U(x)$ such that

$$(2)\begin{cases} U'' &= 0 \\ U(0) &= 0 \\ U(1) &= 1 \end{cases}$$

So U is simply a steady-state solution of (1) (without the initial conditions). Then let

$$u(x,t) = \nu(x,t) - U(x),$$

so

$$(3)\begin{cases} u_t &= ku_{xx} \quad \text{since } U'' = 0 \\ u(1,t) &= \nu(1,t) - U(1) = 0 \\ u(0,t) &= \nu(0,t) - U(0) = 0 \\ u(x,0) &= \varphi(x) - U(x) = \begin{cases} \frac{5x}{2} - U(x) & \text{if } 0 < x < \frac{2}{3} \\ 3 - 2x - U(x) & \text{if } \frac{2}{3} < x < 1. \end{cases} \end{cases}$$

First we solve (2):

$$\begin{aligned} U'' = 0 &\Rightarrow U = Ax + B, \\ U(0) = 0 &\Rightarrow B = 0, \\ U(1) = 1 &\Rightarrow A = 1, \\ \text{so} &\quad U(x) = x \end{aligned}$$

Thus, (3) becomes

$$(3)\begin{cases} u_t &= ku_{xx} \quad \text{since } U''=0 \\ u(0,t) &= u(1,t)=0 \\ u(x,0) &= \hat{\varphi}(x) = \begin{cases} \frac{3x}{2} & \text{for } 0<x<\frac{2}{3} \\ 3-3x-U(x) & \text{for } \frac{2}{3}<x<1. \end{cases} \end{cases}$$

So now the situation is simple: we find the Fourier sine series for $\hat{\varphi}$ (Dirichlet BC's).

$$\hat{\varphi}(x) = \sum_{n=1}^{\infty} A_n \sin n\pi x, \text{ and then}$$

$$u(x,t) = \sum_{n=1}^{\infty} A_n (\sin n\pi x) e^{-(n\pi)^2 kt},$$

$$\nu(x,t) = u(x,t) + U(x) = u(x,t) + x.$$

So

$$\begin{aligned} A_n &= 2\int_0^1 \hat{\varphi}(x)\sin n\pi x dx \\ &= 2\int_0^{\frac{2}{3}} \frac{3x}{2}\sin n\pi x dx + 2\int_{\frac{2}{3}} 1(3-3x)\sin n\pi x dx \\ &= 3x\left(-\frac{1}{n\pi}\right)\cos n\pi x\Big|_0^{\frac{2}{3}} - 3\left(-\frac{1}{n\pi}\right)\int_0^{\frac{2}{3}}\cos n\pi x dx \\ &\quad + 2(3-3x)\left(-\frac{1}{n\pi}\right)\cos n\pi x\Big|_{\frac{2}{3}}^1 - 2\int_{\frac{2}{3}}^1\left(-\frac{1}{n\pi}\right)(-3)\cos n\pi x dx \\ &= -\frac{2}{n\pi}\cos\frac{2n\pi}{3} + \frac{3}{n\pi}\left(\frac{1}{n\pi}\right)\sin n\pi x\Big|_0^{\frac{2}{3}} + \frac{2}{n\pi}\cos\frac{2n\pi}{3} - \frac{6}{n\pi}\left(\frac{1}{n\pi}\right)\sin n\pi x\Big|_{\frac{2}{3}}^1 \\ &= \frac{3}{n^2\pi^2}\sin\frac{2n\pi}{3} + \frac{6}{n^2\pi^2}\sin\frac{2n\pi}{3} \\ &= \frac{9}{n^2\pi^2}\sin\frac{2n\pi}{3}, \text{ and 4, 5 are as above.} \end{aligned}$$

Problem 113.7: $\varphi = \varphi(x)$ is a function on $(-l,l)$. Let $x' = \frac{\pi}{l}x$, and let $\hat{\varphi}(x') = \varphi(x)$, i.e. $\hat{\varphi}(x') = \varphi(\frac{l}{\pi}x')$.

(Note that as x' moves in $(-\pi,\pi)$, x moves in $(-l,l)$, so $\hat{\varphi}$ is indeed defined in $(-\pi,\pi)$.) The Fourier series for $\hat{\varphi}$ is

$$\begin{aligned} \hat{\varphi}(x') &= \frac{A_0}{2} + \sum_{n=1}^{\infty}(A_n\cos nx' + B_n\sin nx') \\ A_n &= \frac{1}{\pi}\int_{-\pi}^{\pi}\hat{\varphi}(x')\cos nx' dx' \\ B_n &= \frac{1}{\pi}\int_{-\pi}^{\pi}\hat{\varphi}(x')\sin nx' dx'. \end{aligned}$$

Thus $dx' = \frac{\pi}{l} dx$, so we get $(\hat{\varphi}(x') = \varphi(x)')$.

$$\begin{aligned} A_n &= \frac{1}{l} \int_{-\pi}^{\pi} \varphi(x) \cos \frac{n\pi x}{l} dx \\ B_n &= \frac{1}{l} \int_{-\pi}^{\pi} \varphi(x) \sin \frac{n\pi x}{l} dx. \end{aligned}$$

By changing the variables in the integral, and then

$$\varphi(x) = \hat{\varphi}(x') = \hat{\varphi}\left(\frac{\pi x}{l}\right) = \frac{A_0}{2} + \sum_{n=1}^{\infty} \left(A_n \cos \frac{n\pi x}{l} + B_n \sin \frac{n\pi x}{l} \right)$$

just as expected!

Problem 113.1: $\varphi(x) = |\sin x|$ on $(-\pi, \pi)$ is even, since

$$\varphi(-x) = |\sin(-x)| = |-\sin x| = |\sin x| = \varphi(x).$$

So all coefficients of sines in the Fourier series must be 0; φ is given by its Fourier cosine series.