18.152 - Introduction to PDEs, Fall 2004 Profs. Gigliola Staffilani and Andras Vasy

Partial solutions to problem set 8

Problems from Strauss, Walter A. *Partial Differential Equations: An Introduction.* New York, NY: Wiley, March 3, 1992. ISBN: 9780471548683.

Problem 87.3 $u_t = iu_{xx}$ w/ Dirichlet BC.
$u(x,t) = T(t)X(x)$: $T'(X) = iTX''$, so $\frac{T'}{iT} = \frac{X''}{X} = -\lambda$ (a constant, since the left-hand side depends on t and the right-hand side depends on x only).

The Dirichlet BC's give

$$X'' = -\lambda X, \qquad X(0) = X(l) = 0,$$

hence $\lambda = \lambda_n = \left(\frac{n\pi}{l}\right)^2$, where $n > 0$ integer, $X(x) = \sin\frac{n\pi x}{l}$.

Finally, $T' = -i\lambda T$, so $T(t) = Ae^{-i\lambda t}$.

$$u(x,t) = \sum A_n e^{-\frac{in^2\pi^2 t}{l^2}} \sin\frac{n\pi x}{l}.$$

Problem 87.4 $u_{tt} = c^2 u_{xx} - ru_t$. Dirichlet BC's:
Separate variables: $u(x,t) = X(x)T(t)$, so $T''X = c^2TX'' - rT'X$, so dividing through by c^2XT gives

$$\frac{T''}{c^2T} + \frac{rT'}{c^2T} = \frac{X''}{X} = -\lambda \qquad (c \text{ constant}),$$

since the left hand side depends on t only, and the right-hand side depends on x only.

Thus, together with Dirichlet BC's, this gives

$$X'' = -\lambda X, \quad X(0) = X(l) = 0,$$

so

$$X(x) = \sin\frac{n\pi x}{l}, \quad \lambda = \lambda_n = \left(\frac{n\pi}{l}\right)^2.$$

Also,

$$T'' + rT' = -\lambda c^2 T,$$

so

$$T'' + rT' + \lambda c^2 T = 0.$$

This is a constant coefficient ODE. The characteristic equation is $\mu^2 + r\mu + \lambda c^2 = 0$, and then the solutions are $Ae^{\mu_1 t} + Be^{\mu_2 t}$, where μ_1, μ_2 are the roots of the characteristic equation (assuming they are distinct.) Thus

$$\mu = \frac{-r \pm \sqrt{r^2 - r\lambda c^2}}{2} = -\frac{r}{2} \pm \sqrt{\frac{r^2}{4} - \lambda c^2}.$$

Taking $\lambda = \left(\frac{n\pi}{l}\right)^2$, we get

$$\mu = -\frac{r}{2} \pm \sqrt{\frac{r^2}{4} - \left(\frac{n\pi}{l}\right)^2 c^2}.$$

If $0 < r < \frac{2\pi c}{l}$, then for $n > 0$ integer (so $n \geq 1$), $\frac{r}{2} < \frac{n\pi c}{l}$, so $\frac{r^2}{4} < \frac{n^2\pi^2c^2}{l^2}$, so the square root gives an imaginary number.

We can either use the exponential solutions $e^{\mu_1 t}, e^{\mu_2 t}$, as above, or replace them by cos/sin:

$$T_n(t) = A_n e^{-\frac{rt}{2}} \cos\left(\sqrt{\left(\frac{n\pi c}{l}\right)^2 - \frac{r^2}{4}}t\right) + B_n e^{-\frac{rt}{2}} \sin\left(\sqrt{\left(\frac{n\pi c}{l}\right)^2 - \frac{r^2}{4}}t\right)$$

so the general solution is

$$\begin{aligned} u(x,t) &= \sum_{u=1}^{\infty} X_n(x)T_n(t) \\ &= \sum_{u=1}^{\infty} A_n e^{-\frac{rt}{2}} \cos\left(\sqrt{\left(\frac{n\pi c}{l}\right)^2 - \frac{r^2}{4}}t\right) \sin\left(\frac{n\pi x}{l}\right) \\ &\quad + \sum_{u=1}^{\infty} B_n e^{-\frac{rt}{2}} \sin\left(\sqrt{\left(\frac{n\pi c}{l}\right)^2 - \frac{r^2}{4}}t\right) \sin\left(\frac{n\pi x}{l}\right) \end{aligned}$$

Note that the argument of each sine/cosine is real. Since $r > 0$, $e^{-\frac{rt}{2}} \to 0$ as $t \to +\infty$; the damping will wipe out the oscillations $t \to +\infty$.

Problem 87.5: Same as in 87.4, except now for $n = 1$, the roots

$$\mu = -\frac{r}{2} \pm \sqrt{\frac{r^2}{4} - \frac{\pi^2 c^2}{e^2}}$$

are real, though the square root is still imaginary for $n \geq 2$. Thus, the solution is

$$\begin{aligned} u(x,t) &= A_1 e^{-\frac{rt}{2} + \sqrt{\frac{r^2}{4} - \frac{\pi^2c^2}{e^2}}t} \sin\frac{\pi x}{l} + B_1 e^{-\frac{rt}{2} - \sqrt{\frac{r^2}{4} - \frac{\pi^2c^2}{e^2}}t} \sin\frac{\pi x}{l} \\ &\quad + \sum_{n=2}^{\infty} A_n e^{-\frac{rt}{2}} \cos\left(\sqrt{\left(\frac{n\pi c}{l}\right)^2 - \frac{r^2}{4}}t\right) \sin\left(\frac{n\pi x}{l}\right) \\ &\quad + \sum_{n=2}^{\infty} B_n e^{-\frac{rt}{2}} \sin\left(\sqrt{\left(\frac{n\pi c}{l}\right)^2 - \frac{r^2}{4}}t\right) \sin\left(\frac{n\pi x}{l}\right) \end{aligned}$$

Problem 90.1 $u_t = ku_{xx}, u(0,t) = u_x(1,t) = 0$.
Separation of variables gives

$$T' = -k\lambda T, \quad X'' = -\lambda X, \quad X(0) = 0, \quad X'(l) = 0.$$

Thus,

$$X(x) = C\cos\sqrt{\lambda}x + D\sin\sqrt{\lambda}x.$$

$X(0) = 0 \Rightarrow C = 0$.
$X'(l) = 0 \Rightarrow \cos\sqrt{\lambda}l = 0$ ($D = 0$ would be trivial). $\Rightarrow \sqrt{\lambda}l = \frac{\pi}{2} + n\pi, n \geq 0$ integer.

$$\lambda = \lambda_n = \left(\frac{(n+\frac{1}{2})\pi}{l}\right)^2 \quad n \geq 0 \text{ integer.}$$

$$X_n(x) = \sin \frac{(n+\frac{1}{2})\pi x}{l},$$

so

$$u(x,t) = \sum_{n=0}^{\infty} A_n e^{-k(n+\frac{1}{2})^2\pi^2 t/l^2} \sin \frac{(n+\frac{1}{2})\pi x}{l}.$$

NB: It is easy to see that $\lambda = 0, \lambda < 0$, or $\lambda \notin \mathbb{R}$ doesn't give new solution.

Problem 90.3: Separation of variables gives

$$X'' = -\lambda X, \quad X(-l) = X(l), \quad X'(-l) = X'(l).$$

Thus, $X = C\cos\sqrt{\lambda}x + D\sin\sqrt{\lambda}x$, so $X(-l) = X(l)$ gives

$$C\cos -\sqrt{\lambda}l + D\sin -\sqrt{\lambda}l = C\cos\sqrt{\lambda}l + D\sin\sqrt{\lambda}l.$$

But cosine is even, sine is odd, so this gives

$$-D\sin\sqrt{\lambda}l = D\sin\sqrt{\lambda}l.$$

Similarly, $X'(-l) = X'(l)$ gives

$$C\sqrt{\lambda}\sin(-\sqrt{\lambda}l) = C\sqrt{\lambda}\sin\sqrt{\lambda}l,$$

i.e.

$$-C\sin\sqrt{\lambda}l = C\sin\sqrt{\lambda}l.$$

Thus $\sqrt{\lambda}l = n\pi$, i.e. $\sqrt{\lambda}_n = \left(\frac{n\pi}{l}\right)^2$,

$$X_n(x) = C_n\cos\frac{n\pi x}{l} + D_n\sin\frac{n\pi x}{l}.$$

$\lambda = 0$ gives $X_0(x) = C_0$. Since $T' = -\lambda kT$, we get

$$u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{l} + b_n\sin\frac{n\pi x}{l}\right)e^{-\left(\frac{n\pi}{l}\right)^2 kt}.$$

N.B.: Again, $\lambda < 0$ or $\lambda \notin \mathbb{R}$ doesn't give new solutions.