18.152 - Introduction to PDEs, Fall 2004 Profs. Gigliola Staffilani and Andras Vasy

Partial solutions to problem set 5

Problems from Strauss, Walter A. *Partial Differential Equations: An Introduction.* New York, NY: Wiley, March 3, 1992. ISBN: 9780471548683.

Problem 54.2a) Let $\nu(x,t) = \int_{-\infty}^{\infty} H(s,t)u(x,s)ds$, where

$$H(s,t) = \frac{c}{\sqrt{4\pi kt}} e^{-\frac{s^2c^2}{4kt}} = \frac{1}{\sqrt{4\pi\kappa t}} e^{-\frac{s^2}{4\kappa t}}, \qquad \left(\kappa = \frac{k}{c^2}\right).$$

and u solves the wave equation on the whole line: $u_{tt} = c^2 u_{xx}$, i.e. $\partial_2^2 u = c^2 \partial_1^2 u$.

Notice that $H(s,t)$ is the Green's function for the heat equation with coefficient $\kappa = \frac{k}{c^2}$, so $\partial_t H = \kappa \partial_s^2 H$, i.e. $\partial_2 H = \kappa \partial_t^2 H$. Thus,

$$\begin{aligned}
k\partial_x^2 \nu &= \int_{-\infty}^{\infty} kH(s,t)(\partial_x^2 u)(x,s)ds \\
&= \int_{-\infty}^{\infty} \frac{k}{c^2} H(s,t)(\partial_s^2 u)(x,s)ds \\
&= \int_{-\infty}^{\infty} \frac{k}{c^2} (\partial_s H)(s,t)(\partial_s u)(x,s)ds \\
&= \int_{-\infty}^{\infty} \frac{k}{c^2} (\partial_s^2 H)(s,t)u(x,s)ds \\
&= \int_{-\infty}^{\infty} \frac{k}{c^2} (\partial_t H)(s,t)(\partial_s u)(x,s)ds \\
&= (\partial_t \nu)(x,t)
\end{aligned}$$

So ν indeed solves the heat equation.

Problem 54.2b) Let

$$w(x,t,r) = \int_{-\infty}^{\infty} H(r-s,t)u(x,s)ds = \int_{-\infty}^{\infty} H(s-r,t)u(x,s)ds.$$

(H is even in its first variable!) So $\nu(x,t) = w(x,t,0)$. But thinking of x as a parameter (i.e. fixing it), w is the solution of

$$\begin{cases} w_t = & \frac{k}{c^2} w_{rr} \\ w(x,0,r) = & u(x,r) \end{cases}$$

(H is the Green's function for this problem, r is the spatial variable!) Thus,

$$\lim_{t\to 0} w(x,t,r) = u(x,r).$$

Letting $r = 0$,

$$\lim_{t\to 0} \nu(x,t) = \lim_{t\to 0} w(x,t,0) = u(x,0)$$

indeed.

N.B. Part b) could be done directly, as in Section 2.4, but it is convenient to derive the result from the case of the already studied formula for the heat equation.

Problem 329.1

(6) $f(x) = H(a - |x|) = \begin{cases} 1 & \text{if } a - |x| > 0, \text{ i.e. } -a < x < a \\ 0 & \text{otherwise} \end{cases}$

$$\begin{aligned} \hat{f}(\xi) &= \int_{\mathbb{R}} e^{i\xi x} H(a - |x|)dx = \int_{-a}^{a} e^{-i\xi x}dx \\ &= -\frac{1}{i\xi}e^{-i\xi x}\Big|_{-a}^{a} = -\frac{1}{i\xi}\left(e^{-i\xi a} - e^{i\xi a}\right) \\ &= \frac{e^{i\xi a} - e^{-i\xi a}}{2i} \cdot \frac{2}{\xi} = \frac{2}{\xi}\sin \xi a. \end{aligned}$$

(7) $f(x) = e^{-a|x|}$, $a > 0$.

$$\begin{aligned} \hat{f}(\xi) &= \int_{\mathbb{R}} e^{i\xi x}e^{-a|x|}dx = \int_{-\infty}^{0} e^{-i\xi x}e^{ax}dx + \int_{0}^{\infty} e^{-i\xi x}e^{-ax}dx \\ &= \frac{1}{a - i\xi}e^{(a-i\xi)x}\Big|_{-\infty}^{0} + \frac{-1}{a + i\xi}e^{-(a+i\xi)x}\Big|_{0}^{\infty} \\ &= \frac{1}{a - i\xi} + \frac{1}{a + i\xi} = \frac{a + i\xi + a - i\xi}{(a - i\xi)(a + i\xi)} = \frac{2a}{a^2 + \xi^2}. \end{aligned}$$

Problem 329.2

iii) $g(x) = f(x - a)$. Let $y = x - a$.

$$\hat{g}(\xi) = \int_{\mathbb{R}} e^{-i\xi x}f(x - a)dx = \int_{\mathbb{R}} e^{-i\xi(y+a)}f(y)dy = e^{-i\xi a}\int_{\mathbb{R}} e^{-i\xi y}f(y)dy = e^{-i\xi a}\hat{f}(\xi).$$

More generally, this is valid for functions on $\mathbb{R}^n, a \in \mathbb{R}^n$.

iv) $g(x) = e^{iax}f(x)$.

$$\hat{g}(\xi) = \int_{\mathbb{R}} e^{-i\xi x}e^{iax}f(x)dx = \int_{\mathbb{R}} e^{-i(\xi-a)x}f(x)dx = \hat{f}(\xi - a).$$

Again, this is valid for functions on $\mathbb{R}^n$.

vi) $g(x) = f(ax), a > 0$.

$$\hat{g}(\xi) = \int_{\mathbb{R}} e^{-i\xi x}f(ax)dx = \int_{\mathbb{R}} e^{-i\frac{\xi}{a}y}f(y)\frac{dy}{a} = \frac{1}{a}\hat{f}\left(\frac{\xi}{a}\right).$$

For $\mathbb{R}^n$, with $x = \frac{y}{a}$, $dx = \frac{dy}{a^n}$, so $\hat{g}(\xi) = \frac{1}{a^n}\hat{f}(\frac{\xi}{a}$.

Problem 329.8 Let $\chi_n = \begin{cases} \frac{1}{2a} & \text{if } -a < x < a \\ 0 & \text{if } |x| > a \end{cases} = \frac{1}{2a}H(a - |x|)$.

From 329.1 (6),

$$\hat{\chi}_a = \frac{1}{2a} \cdot \frac{2}{\xi} = \frac{\sin a\xi}{a\xi}.$$

Thus, $\hat{\chi}_a(\xi) = f(a\xi)$ where $f(\eta) = \frac{\sin\eta}{\eta}$. Thus, $f \in C^\infty(\mathbb{R})$; it is C^∞ at $\eta = 0$ since sine vanishes there. Therefore $\hat{\chi}_a$ converges to $f(0) = 1$ uniformly on compact sets as $a \to 0$; in particular, it converges weakly to the function 1 as $a \to 0$.

Problem 329.9 $-u_{xx} + a^2 u = \partial, a > 0$. Take the F.T.; use $\hat{\delta} = 1 : \xi^2 \hat{u} + a^2 \hat{u} = 1$, so $\hat{u} = \frac{1}{\xi^2 + a^2}$.

But from 329.1 (7), the inverse F.T. of $\frac{1}{\xi^2+a^2}$ is $\frac{1}{2a} e^{-a|x|}$, so $u = \frac{1}{2a} e^{-a|x|}$.

Problem 333.1 $u_t = k u_{xx} + \mu u_x$, $(x, t) \in \mathbb{R} \times (0, \infty)$. $u(x, 0) = \varphi(x)$
Take the F.T. in x. Then $\hat{u}_t = -k|\xi|^2 \hat{u} + i\xi\mu\hat{u}$. (We don't need $|\xi|^2$; we can write ξ^2 here.)
So $\hat{u}(\xi, 0) = \hat{\varphi}(\xi)$.

The ODE is $\hat{u}_t = (-k|\xi|^2 + i\xi\mu)\hat{u}$, hence

$$\hat{u}(\xi, t) = f(\xi) e^{(-k|\xi|^2 + i\xi\mu)t}.$$

Setting $t = 0$ and writing $\hat{u}(\xi, 0) = \varphi(\xi)$ gives

$$\hat{u}(\xi, t) = \varphi(\xi) e^{-k\xi^2 + i\mu\xi)t}.$$

Let $S_\mu(x, t)$ be the inverse Fourier transform of $g(\xi, t) = e^{-k\xi^2 + i\mu\xi)t}$; then

$$u(x, t) = \int_{\mathbb{R}} S_\mu(x - y, t)\varphi(y) dy.$$

But $g(\xi, t) = e^{i\mu t\xi} e^{-kts^2}$, so

$$F^{-1} g(x, t) = (F^{-1} e^{-kt\xi^2}(x + \mu t)$$

(this is 329.2 (iii) with $a = -\mu t$), so

$$u(x, t) = \frac{1}{\sqrt{4\pi kt}} \int_{\mathbb{R}} e^{-\frac{(x+\mu t-y)^2}{4kt}} \varphi(y) dy,$$

where we used that the inverse F.T. of $e^{kt|\xi|^2}$ is $S(x, t) = \frac{1}{\sqrt{4\pi kt}} e^{-\frac{x^2}{4kt}}$.

Problem 333.2 We wish to solve $u_{xx} + u_{yy} = 0$ in $\mathbb{R}_x \times (0, \infty)_y$.

$$\frac{\partial}{\partial y}(x, 0) = h(x).$$

Take F.T. $i - x$, we get

$$= \xi^2 \hat{u}_{yy} = 0, \qquad \hat{u}_y(\xi, 0) = \hat{h}(\xi).$$

The general solution of the ODE is

$$\hat{u}(\xi, y) = f(\xi) e^{|\xi| y} + g(\xi) e^{-|\xi| y}$$

(the absolute values are added to make the argument below easier; one could have written a linear combination of $e^{\xi y}$ & $e^{-\xi y}$ as well, and consider cases separately). For the I.F.T. to make sense, $\hat{u}$ needs to be tempered in ξ, so we need $f(\xi) = 0$, so

$$\hat{u}(\xi, y) = g(\xi) e^{-|\xi| y}.$$

Then $\hat{u}(\xi,0)=\hat{h}(\xi)$ gives $-|\xi|g(\xi)=\hat{h}(\xi)$, so

$$\hat{u}(\xi,y)=-\frac{\hat{h}(\xi)}{|\xi|}e^{-|\xi|y}.$$

Thus,

$$\hat{u}(\xi,y)=\hat{h}(\xi)e^{-|\xi|y},$$

and this can be inverse-Fourier transformed easily. Namely, by a calculation analogous to 329.1 (7), the IFT of $e^{-|\xi|y}$ is $\frac{1}{2\pi}\frac{2y}{y^2+x^2}$, so

$$\begin{aligned}
u_y(x,y) &= \int_{\mathbb{R}} \frac{1}{\pi}\frac{y}{y^2+(x-z)^2}h(z)dz \\
&= \frac{1}{2\pi}\int_{\mathbb{R}} \frac{\partial}{\partial y}\ln(y^2+(x-z)^2)h(z)dz \\
&= \frac{\partial}{\partial y}\left(\frac{1}{2\pi}\int_{\mathbb{R}} \ln(y^2+(x-z)^2)h(z)dz\right)
\end{aligned}$$

This implies that

$$u(x,y)=\frac{1}{2\pi}\int_{\mathbb{R}} \ln(y^2+(x-z)^2)h(z)dz+\nu(x).$$

But $u_{xx}+u_{yy}=0$, and the same holds for the first term on the right-hand side, so $\nu_{xx}+\nu_{yy}=0$ as well, so $\nu''=0$, hence $\nu(x)=Ax+B$.

Notice that such a function ν is indeed harmonic, and it satisfies the homogenous boundary condition, and so the solution of the problem is not unique. We thus get

$$u(x,y)=\frac{1}{2\pi}\int_{\mathbb{R}} \ln(y^2+(x-z)^2)h(z)dz+Ax+B,$$

A,B arbitrary.