Partial solutions to problem set 10

Problems from Strauss, Walter A. *Partial Differential Equations: An Introduction.* New York, NY: Wiley, March 3, 1992. ISBN: 9780471548683.

Problem 118.2: First, notice that all polynomials with only even degree terms are orthogonal to all polynomials with only odd degree terms on [-1,1]: indeed: they are even respectively odd functions, and f even, g odd $\Rightarrow \int_{-1}^{1} f(x)g(x)dx = 0$ since fg is odd.

In particular, x is orthogonal to 1. Next, x^2 is orthogonal to x since x is odd, x^2 even), so we only need to worry about the constants.

So we take (by Gram-Schmidt),

$$Q_2(x) = x^2 - \frac{(x^2, 1)}{(1, 1)} 1$$

in place of x (in general, we would take $x^2 - \frac{(x^2,x)}{(x,x)} x - \frac{(x^2,1)}{(1,1)} 1$,, but $(x^2, x) = 0$ as we just observed.) Now,

$$(x^2, 1) = \int_{-1}^{1} x^2 dx = \frac{x^3}{3}\Big|_{-1}^{1} = \frac{2}{3};$$

$$(1, 1) = \int_{-1}^{1} 1 dx = 2,$$

so $Q_2(x) = x^2 - \frac{1}{3}$. (Any multiple of Q_2 also works, e.g. $3x^2 - 1$.)

Next to find a cubic polynomial orthogonal to $Q_0(x) = 1, Q_1(x) = x, Q_2(x) = x^2 - \frac{1}{3}$, note that orthogonality to Q_0 and Q_2 is automatic if we have only odd degree terms, so applying the Gram-Schmidt process to x^3 gives

$$Q_3(x) = x^3 - \frac{(x^3, x)}{(x, x)} x.$$

Now

$$(x, x) = \int_{-1}^{1} x^2 dx = \frac{2}{3}, \quad (x^3, x) = \int_{-1}^{1} x^4 dx = \frac{2}{5},$$

so $Q_3(x) = x^3 - \frac{3}{5}x$ (again, e.g. $5x^3 - 3x$ would also work).