12.086 / 12.586 Modeling Environmental Complexity
Fall 2008

Lecture notes for 12.086/12.586, Modeling Environmental Complexity

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*October 29, 2008*

# 1 Statistical topography

## 1.1 Self-affine surfaces

We now generalize our earlier discussion of random walks and consider *self-affine* functions $h(x)$ that satisfy

$$h(x) = b^{-\alpha} h(bx)$$

where $\alpha$ is called the *roughness* or *Hurst* exponent.

This relation is taken to be statistical rather than exact. Recall for, example, the case of the simple random walk with rms fluctuation $r = \langle h^2(x) \rangle^{1/2}$. Then

$$r(x) = b^{-1/2} r(bx) \qquad \Rightarrow \qquad \alpha = 1/2.$$

Note that anomalous diffusion corresponds to $\alpha \neq 1/2$.

In what follows we shall typically consider $h$ to represent the height of an interface or surface.

We first develop spectral representations of such surfaces.

### 1.1.1 Width and power spectra

Define

$$W(L) = \text{rms fluctuation of height over length } L. \tag{1}$$

The mean square width is then (assuming zero mean)

$$W^2(L) = \frac{1}{L} \int_0^L |h(x)|^2 \mathrm{d}x$$

For the simple random walk, we have

$$W^2(L) \propto L$$

For self-affine surfaces with roughness exponent $\alpha$,

$$W^2(L) \propto L^{2\alpha}.$$

The mean square width $W^2$ can also be obtained from the power spectrum. To show this, we use the Fourier transform relations

$$\hat{h}(k) = \frac{1}{L}\int_0^L h(x)e^{-ikx}\mathrm{d}x$$

and

$$h(x) = \sum_k \hat{h}(k)e^{ikx}$$

where

$$k = \frac{2\pi n}{L}, \qquad n = 0, \pm 1, \dots$$

Then

$$\begin{aligned} W^2(L) &= \frac{1}{L}\int_0^L \mathrm{d}x|h(x)|^2 \\ &= \frac{1}{L}\int_0^L \mathrm{d}x \sum_k \hat{h}(k)e^{ikx} \sum_{k'} \hat{h}^*(k')e^{-ik'x} \\ &= \frac{1}{L}\int_0^L \mathrm{d}x \sum_k \sum_{k'} \hat{h}(k)\hat{h}^*(k')e^{i(k-k')x} \\ &= \frac{1}{L}\int_0^L \mathrm{d}x \sum_k \hat{h}(k)\hat{h}^*(k) \\ &= \sum_k |\hat{h}(k)|^2 \end{aligned}$$

which is called *Parseval's relation.*

Now define the power spectral density

$$S(k) = \frac{|\hat{h}(k)|^2}{\Delta k} = \frac{L}{2\pi}|\hat{h}(k)|^2.$$

The scaling law $W(L) \propto L^{\alpha}$ suggests that $S(k)$ is also a power law. We write

$$S(k) \propto k^{-\beta} = \left(\frac{2\pi n}{L}\right)^{-\beta} \tag{2}$$

and estimate $\beta$ from

$$\begin{aligned} W^2(L) \propto L^{2\alpha} &\propto \frac{1}{L}\sum_k S(k) \\ &\propto \frac{1}{L}\sum_n \left(\frac{2\pi n}{L}\right)^{-\beta} \\ &\propto L^{\beta-1} \end{aligned}$$

Thus

$$\beta = 2\alpha + 1,$$

thereby relating the power-law decay of the power spectrum to the scaling of width with length.

#### 1.1.2 Wiener-Kintchine theorem and the autocorrelation function

Define the *autocorrelation function* (for real $h(x)$)

$$\Gamma(x) = \frac{1}{L}\int h(x')h(x'+x)\mathrm{d}x'$$

Substitute the Fourier transform for $h(x)$:

$$\begin{aligned} \Gamma(x) &= \frac{1}{L}\int \sum_{k,k'} \hat{h}(k)e^{ikx'}\,\hat{h}(k')e^{ik'(x'+x)}\,\mathrm{d}x' \\ &= \frac{1}{L}\int \sum_{k,k'} \hat{h}(k)\hat{h}(k')\,e^{i(k+k')x'}e^{ikx}\mathrm{d}x' \\ &= \frac{1}{L}\int \sum_{k} \hat{h}(k)\hat{h}(-k)e^{ikx}\mathrm{d}x' \\ &= \sum_k |\hat{h}(k)|^2 e^{ikx} \end{aligned}$$

This is the *Wiener-Kintchine theorem*: the autocorrelation is the Fourier transform of the power spectrum, and vice-versa.

### 1.1.3 Examples

**Random walk.** Consider random uncorrelated Gaussian jumps or steps $\eta(x)$ with zero mean, as one gets in a random walk. The autocorrelation of the jumps is

$$\begin{aligned}\Gamma(x) &= \left\langle \frac{1}{L}\int \eta(x')\eta(x'+x)\mathrm{d}x' \right\rangle \\ &= D\delta(x)\end{aligned}$$

where $D$ sets the scale of the noise $\eta$.

Using the Wiener-Kintchine theorem, the power spectrum of the jumps is

$$|\hat{\eta}(k)|^2 = \frac{1}{L}\int_{-L/2}^{L/2} D\delta(x)e^{-ikx}\mathrm{d}x = \frac{D}{L} = \text{const.},$$

which says that the power spectrum of white noise is flat.

We recognize these jumps as the derivative of the interface $h(x)$ traced out by a random walk. Since

$$\eta(x) = \frac{\mathrm{d}h}{\mathrm{d}x} = \sum_k [ik\hat{h}(k)]e^{ikx}$$

we see by the brackets that the Fourier transform of $\mathrm{d}h/\mathrm{d}x$ is $ik\hat{h}(k)$. Consequently

$$|\hat{\eta}(k)|^2 = k^2|\hat{h}(k)|^2 = \frac{D}{L}$$

and therefore

$$S(k) = \frac{L}{2\pi}|\hat{h}(k)|^2 = \frac{D}{2\pi k^2} \propto k^{-2}$$

for a simple random walk, corresponding to $\beta = 2$ and $\alpha = 1/2$.

Such an interface is sometimes called a *Brownian surface.*

**Fractional Brownian surfaces.** The case above can be generalized to $0 \leq \alpha \leq 1$. These are called *fractional Brownian surfaces.* (**figure**.)

The case $\alpha = 1/2$ is corresponds to pure diffusion.

The case $\alpha > 1/2$ is *persistent* or super-diffusive.

The case $\alpha < 1/2$ is *anti-persistent* or sub-diffusive.

The extreme case of $\alpha = 1$ is purely advective.

The opposite extreme, $\alpha = 0$, must be interpreted in context. If $\beta = 1$ (as predicted by the scaling relation), then

$$W^2(L) = \frac{1}{L}\sum_k S(k) \propto \frac{1}{L}\sum_n \left(\frac{2\pi n}{L}\right)^{-1} = \frac{1}{2\pi}\sum_n \frac{1}{n},$$

a harmonic series, which diverges.

To understand the divergence as a function of $L$, convert

$$\sum_n \to \int \mathrm{d}n \to \int \mathrm{d}k \frac{L}{2\pi}$$

Assume an upper wavenumber cutoff $k_{\max}$ and a lower wavenumber cutoff $2\pi/L$, corresponding to the system size. Substituting $n = kL/2\pi$,

$$W^2(L) \propto \int_{2\pi/L}^{k_{max}} k^{-1}\mathrm{d}k = \text{const} + \log L.$$

Thus $\alpha = 0$, $\beta = 1$ corresponds to a slow logarithmic divergence of the mean-square width.

Note, however, that any case in which $0 \leq \beta < 1$ gives an $L$-independent width (i.e., truly $\alpha = 0$), since

$$W^2(L) \propto \int_{2\pi/L}^{k_{\max}} k^{-\beta}\mathrm{d}k \sim \text{const} - k^{1-\beta}|_{2\pi/L} \to \text{const}.$$

as $L \to \infty$. The situation is analogous to our previous study of diverging means and variances of long-tailed distributions.

**Constructing self-affine surfaces.** Before discussing how such surfaces can arise in Nature, we present one way to *construct* a surface with a given $\alpha$ and $\beta$.

Our method is constructed in Fourier space:

- Choose the desired spectral decay so that

$$S(k) = |\hat{h}(k)|^2 \sim k^{-\beta}.$$

- Choose uncorrelated random phases

$$\phi(k) = \tan^{-1}\left(\frac{\mathrm{Im}\{\hat{h}(k)\}}{\mathrm{Re}\{\hat{h}(k)\}}\right)$$

- Compute the inverse Fourier transform.

For real $h(x)$ one has to honor the symmetries

$$\mathrm{Re}\{\hat{h}(k)\} = \mathrm{Re}\{\hat{h}(-k)\}, \qquad \mathrm{Im}\{\hat{h}(k)\} = -\mathrm{Im}\{\hat{h}(-k)\}$$

One way to generate these symmetries and the randomness of $\phi(k)$ automatically is to Fourier transform white noise and scale the amplitudes by $k^{-\beta}$ while retaining the original phase angles $\phi$.

## 1.2 Discrete models of growing self-affine surface

### 1.2.1 Random deposition

We first consider a growing interface that is *not* self-affine: *random deposition*.

For a one-dimensional substrate, the model is constructed as follows:

- Deposit particles at random horizontal locations $x_i$.
- Particles "land" on top of the most recently dropped particle.

There are no correlations, so for a lattice of length $L$, a unit increase in the height $h_i$ occurs, on average, every $L$ time steps.

The mean height therefore grows linearly with time, like

$$\bar{h}(t) \propto t$$

The individual heights have a Poisson distribution. Thus

$$\text{Var } h = \bar{h} \propto t$$

and consequently the width $W$ grows like

$$W(t) \propto t^{1/2}, \qquad \text{independent of } L.$$

### 1.2.2 Random deposition with surface diffusion

Now imagine that after each step of random deposition, particles diffuse around and stop at a site with a lower height.

The simplest such model proceeds as follows:

- Choose a random column $i$.
- Let the particle that falls stick to the top of column $i$, $i-1$, or $i+1$, whichever is smallest.

This diffusion process creates correlations and therefore a smoother surface.

The average height $h = \bar{h}$ again increases linearly with time.

However the width

$$W(L,h) = \left( \frac{1}{L} \sum_{i=1}^{L} (h_i - h)^2 \right)^{1/2}$$

depends on both $L$ and $h$.

We expect two scaling regimes:

- For $h \ll L$, the system sees an effective $L_{\text{eff}} < L$, and we expect the width of the interface to grow with $h$.

- For $h \gg L$, the roughness saturates to a constant and depends only on $L$.

This suggests the *finite-size* or *dynamical scaling* [1, 2] relation

$$W(L, h) \propto L^\alpha f(h/L^z)$$

The scaling function $f(x)$ is defined such that

$$f(x) \propto \begin{cases} x^\beta, & x \ll 1 \\ \text{const.}, & x \gg 1 \end{cases} .$$

Specifically,

$$W(L, h) \propto \begin{cases} h^\beta, & h \ll L^z \\ L^\alpha, & h \gg L^z \end{cases} .$$

We can relate $z$ to $\alpha$ and $\beta$ by noting that if

$$f(h/L^z) \propto h^\beta / L^{z\beta}, \qquad h \ll L^z$$

then, the $L$-dependence vanishes at early times if

$$\left(\frac{h}{L^z}\right)^\beta \propto \frac{h^\beta}{L^\alpha}$$

and therefore

$$z\beta = \alpha \quad \Rightarrow \quad \beta = \alpha/z.$$

Summary: at early times $t \ll h/L^z$,

$$W(h) \propto h^\beta \quad \text{or} \quad W(t) \propto t^\beta.$$

At late times $t \gg h/L^z$,

$$W(L) \propto L^\alpha, \quad \text{independent of } h \text{ or } t.$$

For random deposition with surface diffusion in one dimension, one finds empirically that [1, 2]

$$\alpha = 1/2, \qquad z = 2, \qquad \beta = 1/4.$$

Moreover, plots of

$$W(t)/L^{1/2} \quad \text{vs.} \quad h/L^2$$

fall on the same *scaling function* $f$ at all combinations of times $t \propto h$ and system sizes $L$.

### 1.2.3 Cluster aggregation

A popular model of cluster growth is the *Eden model*:

- Consider an aggregate of particles, on or off a lattice.
- Randomly choose an empty site next to the aggregate, i.e. a perimeter site. Add it to the cluster
- Repeat.

The result is qualitatively similar to, e.g., coffee drops and lichens **figures**.

To understand the growth process more clearly, consider a strip geometry of length $L$.

Populate the bottom row of a lattice with particles and add perimeter sites randomly.

*Result:* Simulations show that the growth again follows

$$W(L,t) \propto L^{\alpha} f(h/L^{z})$$

but now

$$\alpha = 1/2, \qquad z = 3/2, \qquad \beta = 1/3.$$

We see that the *static* or time-independent behavior is the same (i.e., the roughness exponent is again $\alpha = 1/2$) but the *dynamic* or time-dependent roughening is different.

## 1.3 Continuum models I: random deposition

Each of the models above can be described by a continuum model that yields quantitative predictions.

Consider first random deposition. Define uncorrelated Gaussian noise $\eta(x,t)$ with zero mean and correlations

$$\langle \eta(x,t)\eta(x',t')\rangle = D\,\delta(x-x')\delta(t-t')$$

Growth by random deposition with average velocity $v$ is then described by

$$\frac{\partial h}{\partial t} = v + \eta(x,t)$$

Averaging both sides, we obtain

$$\langle h \rangle = vt.$$

To determine the roughness, we transform $h$ to the comoving frame

$$h' = h - vt$$

and drop primes to obtain

$$\frac{\partial h}{\partial t} = \eta(x,t)$$

and note that $\langle h \rangle = 0$ at all times. Integrate to obtain

$$h(x,t) = \int \eta(x,t)\mathrm{d}t$$

The mean-square width evolves as

$$\begin{aligned} W^2(L,t) &= \frac{1}{L}\int_0^L h^2(x,t)\mathrm{d}x \\ &= \frac{1}{L}\int_0^L \mathrm{d}x \int \mathrm{d}t \int \mathrm{d}t'\, \eta(x,t)\eta(x,t') \\ &= \frac{1}{L}\int_0^L \mathrm{d}x \int \mathrm{d}t\, D \\ &= Dt \end{aligned}$$

where in the third relation we have loosely set $\delta(0) = 1$, yielding, as in Section 1.2.1, the result

$$W(L,t) \propto t^{1/2}, \quad \text{independent of } L.$$

## 1.4 Continuum models II: noisy diffusion

We next consider a continuum model of random deposition with surface diffusion.

We remain in the comoving frame as above but now add diffusion with diffusivity $\nu$:

$$\frac{\partial h}{\partial t} = \nu \nabla^2 h + \eta.$$

We call this the *noisy diffusion equation* [2]. It has been extensively studied by Edwards and Wilkinson [3] as a model of the surface formed by granular deposition.

#### 1.4.1 Fourier representation

We begin by Fourier transformation. We transform $h(x,t) \to \hat{h}(k,t)$ via

$$\hat{h}(k,t) = \frac{1}{L}\int_0^L h(x)e^{-ikx}\mathrm{d}x$$

and similarly transform $\eta(x,t) \to \hat{\eta}(k,t)$. Noting that

$$\widehat{\frac{\partial^2 h(x,t)}{\partial x^2}} = -k^2\hat{h}(k,t)$$

we obtain

$$\frac{\partial}{\partial t}\hat{h}(k,t) = -\nu k^2 \hat{h}(k,t) + \hat{\eta}(k,t).$$

Multiply both sides by $e^{\nu k^2 t}$:

$$e^{\nu k^2 t}\frac{\partial}{\partial t}\hat{h}(k,t) = \big(-\nu k^2 \hat{h}(k,t) + \hat{\eta}(k,t)\big)e^{\nu k^2 t}$$

and bring the first term on the RHS over to the LHS to obtain

$$\frac{\partial}{\partial t}\left(\hat{h}(k,t)e^{\nu k^2 t}\right) = \hat{\eta}(k,t)e^{\nu k^2 t}.$$

Integrate both sides and divide by $e^{\nu k^2 t}$:

$$\hat{h}(k,t) = e^{-\nu k^2 t}\int_0^t \mathrm{d}t' \hat{\eta}(k,t')e^{\nu k^2 t'}.$$

### 1.4.2 The width $W(L,t)$

We seek the width $W(L,t)$ via Parseval's relation

$$W^2(L,t) = \sum_k |\hat{h}(k,t)|^2,$$

which yields

$$W^2(L,t) = \sum_k \int_0^t \mathrm{d}t' \int_0^{t'} \mathrm{d}t''\, \hat{\eta}(k,t')\hat{\eta}^*(k,t'')e^{-\nu k^2(2t-t'-t'')}.$$

Note that

$$\langle \hat{\eta}(k,t')\hat{\eta}^*(k,t'')\rangle = |\hat{\eta}(k)|^2\delta(t'-t'') = \frac{D}{L}\delta(t'-t'')$$

where the second relation was found in Section 1.1.3. Inserting above, we obtain

$$\begin{aligned} W^2(L,t) &= \frac{D}{L}\sum_k \int_0^t \mathrm{d}t' e^{-2\nu k^2(t-t')} \\ &= \frac{D}{L}\sum_k \frac{1}{2\nu k^2}\left(e^{2\nu k^2 t}-1\right)e^{-2\nu k^2 t} \\ &= \frac{D}{L}\sum_k \frac{1}{2\nu k^2}\left(1-e^{-2\nu k^2 t}\right) \end{aligned}$$

Now substitute $k = 2\pi n/L$ and perform the sum over $n$:

$$W^2(L,t) = \frac{D}{L}\sum_n \frac{1-e^{-2\nu(2\pi/L)^2 t}}{2\nu(2\pi/L)^2 n^2}$$

Rearrange:

$$W^2(L,t) = L\left(\frac{D}{8\nu\pi^2}\right)\sum_n \frac{1}{n^2}\left(1-e^{-8\nu\pi^2 n^2(t/L^2)}\right)$$

Note that the sum converges (since $\sum_n n^{-2} = \pi^2/6$). Simplify by writing

$$W(L,t) = L^\alpha f(t/L^z),$$

where $\alpha = 1/2$, $z = 2$, and

$$f(x) = \left[ \frac{D}{8\pi^2 \nu} \sum_n \frac{1}{n^2} \left( 1 - e^{-8\nu\pi^2 n^2 x} \right) \right]^{1/2}.$$

By our previous reasoning, the $L$-dependence must vanish at short times so that

$$W(L,t) \propto t^{\alpha/z} = t^{\beta}, \qquad \beta = 1/4.$$

At long times, we have the explicit result

$$\lim_{t\to\infty} W(L,t) = \left( \frac{DL}{48\nu} \right)^{1/2}$$

### 1.4.3 Evolution of the power spectrum

It is also of interest to investigate the time-evolution of the power spectrum

$$S(k,t) = L|\hat{h}(k,t)|^2 = \frac{D}{2\nu k^2} \left( 1 - e^{-2\nu k^2 t} \right)$$

which we have obtained here by removing the $k$-summation from $W^2(L,t)$.

At long times

$$S(k,t) \sim 1/k^2, \qquad t \to \infty$$

At short times $t$ such that $\nu k^2 t$ is small,

$$S(k,t) \sim \frac{D}{2\nu k^2} \left( 2\nu k^2 t \right) = Dt, \qquad \nu k^2 t \ll 1$$

independent of the wavenumber $k$.

Thus for wavenumbers $k \ll (\nu t)^{-1/2}$, the spectrum of height fluctuations appears to rise uniformly and linearly with time.

On the other hand, $S(k,t)$ saturates to

$$S(k,t) \sim \frac{D}{2\nu k^2} \quad \text{for} \quad k \gg (\nu t)^{-1/2}.$$

Define $L_\parallel$ to be the horizontal scale corresponding to the crossover wavenumber:

$$L_\parallel \sim \sqrt{\nu t}$$

On the other hand, the vertical scale $L_\perp$ of roughening corresponding to $L_\parallel$ is

$$L_\perp \sim W(L_\parallel) \sim L_\parallel^\alpha = L_\parallel^{1/2} \sim (\nu t)^{1/4},$$

consistent with our previous result $\beta = 1/4$.

#### 1.4.4 Two dimensions

The above results correspond only to growth on a one-dimensional substrate.

In two dimensions, the power spectrum

$$S(k,t) \rightarrow S(\vec{k},t), \quad \vec{k} = [k_1, k_2].$$

The isotropy of the growth process requires radial symmetry such that

$$S(|\vec{k}|,t) = S(\vec{k},t)$$

The form of the power spectrum remains the same:

$$S(|\vec{k}|) \propto \frac{D}{2\nu|\vec{k}|^2}\left(1 - e^{-2\nu|\vec{k}|^2 t}\right)$$

Since

$$|\vec{k}|^2 = \frac{4\pi^2(n_1^2 + n_2^2)}{L^2}$$

time is again scaled like $1/L^2$ and we retain the same dynamic exponent $z = 2$.

However the roughness exponent changes. To see how, we estimate the width as we did previously as in Section 1.1.3:

$$W^2(L) \propto \int_{1/L}^{k_{\max}} \mathrm{d}\vec{k}\, S(\vec{k}) \propto \int_{1/L}^{k_{\max}} \mathrm{d}k_1 \int_{1/L}^{k_{\max}} \mathrm{d}k_2 \frac{1}{k_1^2 + k_2^2}$$

Converting to radial wavenumber $k_r = |\vec{k}|$, we have

$$W^2(L) \propto \int_{1/L}^{k_{\max}} \frac{1}{k_r^2}\, k_r\, \mathrm{d}k_r = \text{const.} + \log L.$$

Thus the mean square width grows only logarithmically with $L$, and the static roughness exponent $\alpha = 0$.

We conclude, therefore, that noisy diffusion *cannot* explain two-dimensional self-affine surfaces.

## 1.5 Continuum models III: the KPZ equation

We now consider interfaces that exhibit *normal growth*, i.e., they grow everywhere in a direction perpendicular to their local tangent, i.e., *outward*.

Among our three discrete models, normal growth corresponds to cluster aggregation—growth everywhere along a perimeter with equal probability.

We also include, as above, diffusive smoothing and noise.

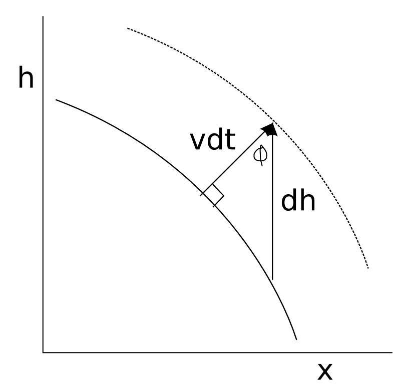
The resulting continuum model is

$$\frac{\partial h(x,t)}{\partial t} = \nu \nabla^2 h(x,t) + \frac{\lambda}{2} |\nabla h(x,t)|^2 + \eta(x,t)$$

### 1.5.1 Origin of the nonlinearity

Let

- $v\delta t$ = displacement in growth direction after short time $\delta t$.
- $\delta h$ = displacement in $h$-direction.




Consider $\delta h$ to be the hypotenuse of a right triangle. Then

- One side is $v\delta t$.
- The other side is $v\delta t \tan\phi = v\delta t|\nabla h|$

The identity $\tan\phi = \nabla h$ can be verified by splitting the present right triangle into two component right triangles and identifying $\phi$ within the lower triangle.

From Pythagoras,

$$(\delta h)^2 = (v\delta t)^2 + (v\delta t|\nabla h|)^2$$

and therefore

$$\begin{aligned} \delta h &= v\delta t \left(1 + |\nabla h|^2\right)^{1/2} \\ &\simeq v\delta t \left(1 + \frac{1}{2}|\nabla h|^2 + \ldots\right) \end{aligned}$$

In the limit $\delta t \to 0$,

$$\frac{\partial h}{\partial t} = v + \frac{v}{2}|\nabla h|^2 + \mathcal{O}\left(|\nabla h|^4\right)$$

Transform to the comoving frame $h' = h - vt$, drop primes, and keep only the lowest-order nonlinear term. Then

$$\frac{\partial h}{\partial t} = \frac{\lambda}{2}|\nabla h|^2$$

where $\lambda = v =$ average normal velocity.

Assume also that bumps tend to diffuse with diffusivity $\nu$ and that there is "noise" equivalent to random deposition. Then

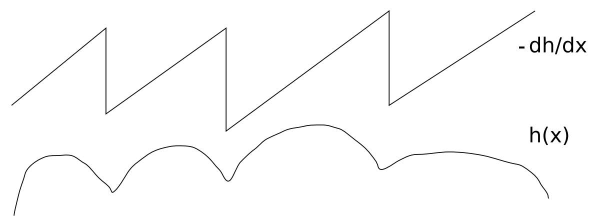
$$\frac{\partial h}{\partial t} = \nu\nabla^2 h + \frac{\lambda}{2}|\nabla h|^2 + \eta,$$

known as the KPZ equation [4].

#### 1.5.2 Qualitative behavior

The nonlinear term approximates growth in which each bump expands spherically outward, similar to the "Huygens secondary source" of geometrical optics.

The growth of these bumps—and in particular the way the way in which the growing surface becomes progressively smoother—is analogous to shapes taken by a rough surface covered by snow.




Note that for true normal growth the curve of slopes would not be piecewise linear. However the parabolic approximation is particularly interesting.

To see this, take the gradient of both sides of the noiseless growth equation:

$$\frac{\partial}{\partial t}\nabla h = \nu\nabla^2\nabla h + \frac{\lambda}{2}\nabla|\nabla h|^2$$

Substitute $u = -\nabla h$ to obtain

$$\frac{\partial u}{\partial t} = \nu\nabla^2 u - \frac{\lambda}{2}\nabla u^2,$$

which is revealingly rewritten as

$$\frac{\partial u}{\partial t} + \lambda u \cdot \nabla u = \nu\nabla^2 u.$$

showing that slopes $-u$ advect at velocity $\lambda u$.

The case $\lambda = 1$ is *Burgers' equation*, a pressure-free Navier-Stokes equation, used to model gas dynamics and traffic.

Note that, left undisturbed by noise, one parabola or shock eventually overtakes all others and the surface becomes smooth and flat, just as with a heavy snowfall.

Thus the role of noise, as in the noisy diffusion problem, is to provide a constant source of "roughness."

### 1.5.3 Roughness exponent $\alpha$ via the Fokker-Planck equation [2, 5]

The nonlinearity of the KPZ equation makes calculation of the roughness exponent $\alpha$ considerably more difficult.

The simplest approach utilizes standard results in stochastic dynamics [6] that associate the evolution of a dynamical variable—say, $h$—to the evolution of its pdf $P(h,t)$.

First, consider the general case of a *Langevin equation* for the single variable $h = h(t)$:

$$\frac{\partial h}{\partial t} = G(h) + \eta(t), \qquad \langle \eta(t)\eta(t')\rangle = D\delta(t-t')$$

Integration of the Langevin dynamics provides the future value of $h$ from its present value, via the fluctuating "force" $\eta$. By integrating over all possible paths taken by a particular $h$-trajectory, the Langevin dynamics may be associated with the *Fokker-Planck equation* [6, 7]

$$\frac{\partial P}{\partial t} = -\frac{\partial}{\partial h}[G(h)P] + \frac{D}{2}\frac{\partial^2 P}{\partial h^2},$$

where $P = P(h,t)$ is the probability of obtaining $h = h(t)$.

**Relation of Fokker-Planck equation to Langevin equation** To better understand the Fokker-Planck equation, write the continuity equation for the probability density $P$,

$$\frac{\partial P}{\partial t} = -\frac{\partial J(h,t)}{\partial h}$$

where $J(h,t)$ is the probability flux

$$J(h,t) = G(h)P - \frac{D}{2}\frac{\partial P}{\partial h}$$

We interpret the first term as the advection or drift of probability, and the second as diffusion.

The drift arises from the deterministic part of the Langevin equation. For a small time $\Delta t$, $h$ changes by

$$\Delta h = G(h)\Delta t + \int_0^{\Delta t} \eta(t')\mathrm{d}t'$$

yielding the drift

$$\frac{\langle \Delta h\rangle}{\Delta t} = G(h).$$

The diffusive term comes from the noise. To see this, expand $(\Delta h)^2$ to first order in $\Delta t$,

$$(\Delta h)^2 \simeq 2G(h)\Delta t \int_0^{\Delta t} \eta(t')\mathrm{d}t' + \int_0^{\Delta t} \mathrm{d}t' \int_0^{\Delta t'} \mathrm{d}t''\eta(t')\eta(t'').$$

Then

$$\frac{\langle(\Delta h)^2\rangle}{\Delta t} = D$$

Thus the first and second terms in the flux $J(h,t)$ of the probability $P(h,t)$ derive from the first and second moments, respectively, of $\Delta h$ as computed from the Langevin equation.

The utility of the Fokker-Planck equation is that it allows computation of the long-term behavior of $P$ (e.g., the steady state) from the short-term Markovian dynamics.

**Continuous form** For interfaces $h(x,t)$, we have the additional complication that $h$ depends on $x$ (as does $\eta$). Conceptually, we can discretize space in units of $l$ so that

$$\xi_i = l^d h(x_i).$$

Then the Fokker-Planck equation describes the evolution of $P(\xi)$, $\xi = \{\xi_i\}$:

$$\frac{\partial P}{\partial t} = -\sum_i \frac{\partial}{\partial \xi_i}[G_i(\xi)P] + \frac{D}{2}\sum_i \frac{\partial^2 P}{\partial \xi_i^2}.$$

To convert sums to integrals, we must define functional derivatives

$$\frac{\delta F(h)}{\delta h(x_i)} = \lim_{l\to 0}\frac{1}{l^d}\frac{\partial F(\xi)}{\partial \xi_i}$$

Writing the Fokker-Planck equation in the continuum limit $l \to 0$ then yields, for $P(h)$, $h = h(x)$,

$$\frac{\partial P}{\partial t} = -\int \mathrm{d}x \frac{\delta}{\delta h}[G(h)P] + \frac{D}{2}\int \mathrm{d}x \frac{\delta^2 P}{\delta h^2}.$$

We now identify

$$G(h) = \nu\nabla^2 h + \frac{\lambda}{2}(\nabla h)^2.$$

Inserting above, we obtain

$$\frac{\partial P}{\partial t} = -\int \mathrm{d}x \left[ \left( \nu \nabla^2 h + \frac{\lambda}{2} (\nabla h)^2 \right) \frac{\delta P}{\delta h} \right] + \frac{D}{2} \int \mathrm{d}x \, \frac{\delta^2 P}{\delta h^2},$$

where we have used expressions like

$$\frac{\delta}{\delta h} \nabla^2 h = \nabla^2 \frac{\delta h}{\delta h} = 0.$$

**Steady solution** In one dimension, the steady solution, for which $\partial P/\partial t = 0$, is

$$P(h) = \exp\left( -\frac{\nu}{D} \int \mathrm{d}x \left( \frac{\mathrm{d}h}{\mathrm{d}x} \right)^2 \right).$$

This can be verified by writing $P = P(x, h, h')$ and recalling the variational derivative

$$\frac{\delta P}{\delta h} \equiv \frac{\partial P}{\partial h} - \frac{\mathrm{d}}{\mathrm{d}x} \frac{\partial P}{\partial h'}$$

The first term $(\partial P/\partial h)$ vanishes leaving

$$\frac{\delta P}{\delta h} = \frac{2\nu}{D} \frac{\mathrm{d}^2 h}{\mathrm{d}x^2} P$$

Inserting into the second term on the RHS of the Fokker-Planck equation, we obtain

$$\frac{\partial P}{\partial t} = -\int \mathrm{d}x \left[ \left( \nu \frac{\mathrm{d}^2 h}{\mathrm{d}x^2} + \frac{\lambda}{2} \left( \frac{\mathrm{d}h}{\mathrm{d}x} \right)^2 \right) \frac{\delta P}{\delta h} \right] + \int \mathrm{d}x \, \nu \, \frac{\mathrm{d}^2 h}{\mathrm{d}x^2} \frac{\delta P}{\delta h}$$

The two terms in $h''$ exactly cancel, leaving only

$$\begin{aligned} \frac{\partial P}{\partial t} &= P \int \mathrm{d}x \, \frac{\nu\lambda}{2D} \left( \frac{\mathrm{d}h}{\mathrm{d}x} \right)^2 \frac{\mathrm{d}^2 h}{\mathrm{d}x^2} \\ &= P \int \frac{\nu\lambda}{2D} \, \mathrm{d}x \, \frac{\mathrm{d}}{\mathrm{d}x} \left( \frac{\mathrm{d}h}{\mathrm{d}x} \right)^3 \\ &= 0, \end{aligned}$$

where in the last relation we have taken the system size to be large and set the surface integral (i.e., an integral of a divergence) to zero.

A remarkable consequence of this calculation that the nonlinear ($\lambda$) term does not contribute to the steady distribution $P(h)$.

Thus in one dimension, the stationary states of the noisy diffusion and KPZ equations are identical, yielding the roughness exponent

$$\alpha = 1/2 \qquad (d = 1),$$

which, of course, is consistent with the Gaussian solution $P(h)$.

In higher dimensions, however, the $\lambda$-term does not vanish. (Integration by parts reveals contributions that are *not* surface integrals.)

Indeed, no solution is known in higher dimensions. Numerical simulations suggest, however, that $\alpha \simeq 0.4$ in $d = 2$.

#### 1.5.4 Scaling argument for time dependent roughening ($\beta$ and $z$) [8]

Recall the dynamical scaling relation

$$W(L,t) \propto L^\alpha f(t/L^z)$$

where

$$W(L,t) \propto \begin{cases} t^\beta, & t \ll L^z \\ L^\alpha, & t \gg L^z \end{cases}, \qquad \beta = \alpha/z.$$

Consider a typical bump or bulge on the surface, with lateral correlation length $L_\parallel$ and vertical correlation length $L_\perp$, equal to the width $W$. Then

$$L_\perp \propto L_\parallel^\alpha \quad \text{for} \quad t \gg L_\parallel^z$$

The typical width $L_\parallel$ grows with time in such a way that $L_\parallel^z \sim t$ or

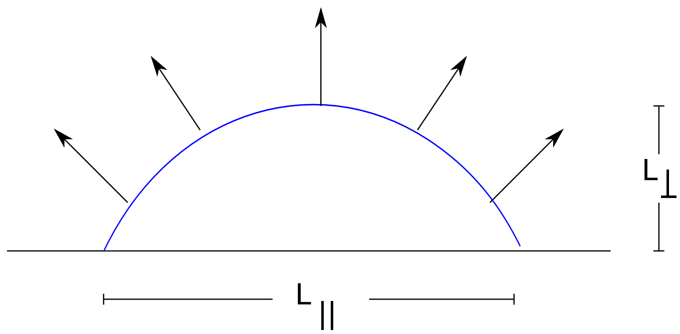
$$L_\parallel \sim t^{1/z}$$

Note that $z = 2$, as found for the noisy diffusion equation, corresponds to diffusive growth where

$$L_\parallel \sim t^{1/2}.$$

Thus $z \ll 2$ corresponds to a kind of “superdiffusive” spreading. We expect this here because bumps always grow in the normal direction.

Consider such a bump and its growth:




The characteristic slope is

$$|\nabla h| \sim L_\perp / L_\parallel.$$

As noted in Section 1.5.2, slopes advect horizontally at a velocity proportional to slope. Therefore the characteristic bump of size $L_\parallel$ widens like

$$\frac{\mathrm{d}L_\parallel}{\mathrm{d}t} \propto |\nabla h| \sim \frac{L_\perp(L_\parallel)}{L_\parallel} \propto L_\parallel^{\alpha-1}$$

Separating variables, we have

$$L_\parallel^{1-\alpha}\mathrm{d}L_\parallel \sim \mathrm{d}t \quad \Rightarrow \quad L_\parallel^{2-\alpha} \sim t$$

and therefore

$$L_\parallel \sim t^{\frac{1}{2-\alpha}}.$$

But above we also found $L_\parallel \sim t^{1/z}$. Therefore $z = 2 - \alpha$ or

$$\alpha + z = 2.$$

This result appears general: it depends only on normal growth, but not dimensionality. In $d = 1$ we therefore have

$$\alpha = 1/2, \quad z = 3/2 \quad (d = 1)$$

Thus at early times,

$$W(t) \propto t^\beta = t^{1/3}, \qquad t \ll L^z = L^{3/2}$$

since $\beta = \alpha/z$. We also have, using our earlier result, that

$$L_\parallel \sim t^{1/z} = t^{2/3},$$

which is faster than $t^{1/2}$ and therefore *superdiffusive*.

### 1.5.5 Summary and applications

**Summary** The continuum growth equation

$$\frac{\partial h}{\partial t} = \nu \nabla^2 h + \frac{\lambda}{2}|\nabla h|^2 + \eta(x,t)$$

contains within it three classes of surface growth:

1. $\nu = \lambda = 0$: random deposition, wherein

$$W(L,t) \sim t^{1/2}, \quad \text{independendent of } L \text{ and } d.$$

2. $\nu > 0$, $\lambda = 0$: noisy diffusion. The width $W(L,t) = L^\alpha f(t/L^z)$, where

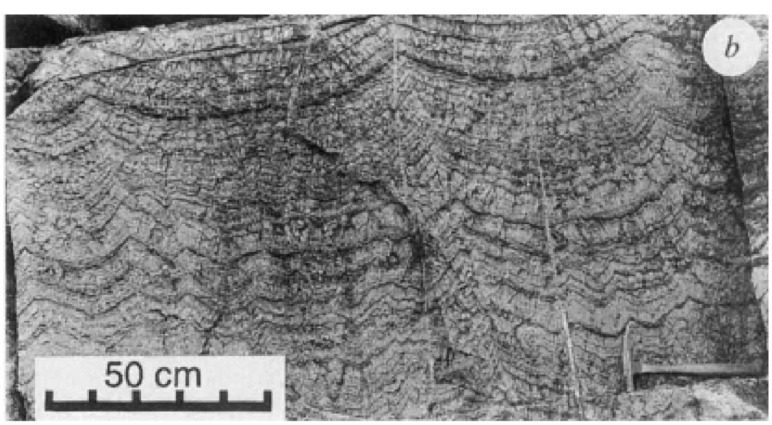
$$\begin{aligned} \alpha &= 1/2, \quad z = 2 \quad (d = 1) \\ \alpha &= 0, \quad\;\; z = 2 \quad (d = 2) \end{aligned}$$

3. $\nu > 0$, $\lambda \neq 0$: normal growth, yielding

$$\begin{aligned} \alpha &= 1/2, \quad z = 3/2 \quad (d = 1) \\ \alpha &\simeq 0.4, \quad z \simeq 1.6 \quad (d = 2) \end{aligned}$$

**Applications** Of the plentitude, we cite two of earth-science interest:

- *Earth's topography* [9,10]. Widespread observations of power-law scaling of width with length. Suggests applicability of KPZ model and variants to "normal erosion."
- *Stromatolites and the early history of life* [11].



> The power spectra of the layer "heights" scales like $S(k) \sim k^{-2}$ over nearly three orders of magnitude, implying $\alpha = 1/2$. Suggests that these "trace" fossils of early microbial activity could be of purely physical origin.

Note that nearly all natural surfaces occur in $d = 2$. Thus the lack of power-law scaling in noisy diffusion suggests a dominant role for normal growth.

## 1.6 Gaussian surfaces [12, 13]

Faced with a particular measure of surfaces (e.g., a correlation between slope and drainage area), how can we know if it is a consequence of the dynamics of surface growth or the mere construction of a "typical" surface?

To answer such a question, we consider a class of surfaces that includes self-affine surfaces but is more general.

### 1.6.1 Preliminaries

We consider surfaces in $d = 2$, with the Fourier transform pair

$$\hat{h}(\vec{k}) = \frac{1}{L^2}\int_0^L\int_0^L h(\vec{r})e^{-i\vec{k}\cdot\vec{r}}\mathrm{d}\vec{r}, \qquad \vec{k} = \frac{2\pi}{L}(n_1, n_2), \quad n_1, n_2 = 0, \pm 1, \ldots$$

and

$$h(\vec{r}) = \sum_{\vec{k}} \hat{h}(\vec{k})e^{i\vec{k}\cdot\vec{r}}.$$

Now write

$$\hat{h}(\vec{k}) = A(\vec{k})e^{i\phi(\vec{k})},$$

where

$$A(\vec{k}) = |\hat{h}(\vec{k})| \quad \text{and} \quad \phi(\vec{k}) = \tan^{-1}\left(\frac{\mathrm{Im}\{\hat{h}(\vec{k})\}}{\mathrm{Re}\{\hat{h}(\vec{k})\}}\right)$$

Recall that real $h(\vec{r})$ requires

$$\phi(-\vec{k}) = -\phi(\vec{k})$$

Consequently the inverse Fourier transform simplifies to a sum of cosines:

$$\begin{aligned} h(\vec{r}) &= \sum_{\vec{k}} A(\vec{k})\, e^{i\phi(\vec{k})} e^{i\vec{k}\cdot\vec{r}} \\ &= \sum_{\vec{k}} A(\vec{k}) \cos(\vec{k}\cdot\vec{r} - \phi(\vec{k})) \end{aligned}$$

### 1.6.2 Random phases imply Gaussian heights

*Assumption.* The $\phi(k)$ are uniformly distributed over the range $-\pi$ to $\pi$ and statistically independent (except for the symmetry relation above).

To see a consequence of this assumption, consider the (arbitrary) point $\vec{r} = 0$. Substitute into the inverse Fourier transform above to obtain

$$h(0) = \sum_{\vec{k}} A(\vec{k}) \cos\phi(\vec{k})$$

But this is just a sum of uncorrelated random numbers. Invoking the central limit theorem, $P[h(0)]$ must be Gaussian.

Now simplify notation so that

$$A_n = A(\vec{k}_n), \qquad \theta_n = \vec{k}_n \cdot \vec{r} - \phi(\vec{k}_n).$$

Then any height $h$ is given by

$$h = \sum_n A_n \cos\theta_n$$

But $\theta_n$ is also an uncorrelated random phase because the $\phi_n$'s are uncorrelated and distributed over the entire interval $[-\pi, \pi]$.

Thus we again have a sum of independent random numbers, so that the pdf of any height $h$ is also Gaussian.

We can obtain the mean square height $\langle h^2\rangle$ from

$$\langle h^2\rangle = \left\langle \left( \sum_n A_n \cos\theta_n \right)^2 \right\rangle$$

which we can rewrite as

$$\langle h^2 \rangle = \left\langle \sum_{n,n'} A_n A_{n'} \cos\theta_n \cos\theta_{n'} \right\rangle .$$

The average is performed over the (uniform) distribution of the $\theta_n$'s. Since the $\theta_n$'s are uncorrelated, there are contributions to the sums only when $n = n'$ or (because cosines are even) $n = -n'$. Therefore

$$\langle h^2 \rangle = 2 \sum_n A_n^2 \langle \cos^2\theta_n \rangle = \sum A_n^2$$

We recognize this as Parseval's theorem, but note that it has been derived here merely from the assumption of random phases.

Consequently we see that random phases not only predict a Gaussian distribution of heights but also the mean-square width $W^2 = \langle h^2 \rangle$. Therefore all heights are distributed according to the Gaussian pdf

$$P(h) \propto e^{-h^2/2W^2} .$$

We see, therefore, that the assumption of random phases is equivalent to assuming that each height $h(\vec{r})$ results from the sum of a large number of independent contributions.

Note that this requires no assumptions concerning the power spectrum $|\hat{h}(k)|^2$ nor the (associated) autocorrelation function.

That is, heights $h(\vec{r})$ can be correlated even though phases $\phi(k)$ are not.

Consequently random phases lead to a *Gaussian surface* that is completely defined by the power spectrum or the autocorrelation function, which set the variance and covariance of the heights.

#### 1.6.3 Distribution of gradients

Taking the gradient of our previous expression for $h$ and setting $\vec{k}_n = (p_n, q_n)$, we have

$$\begin{aligned} h_x = \frac{\partial h}{\partial x} &= -\sum_n A_n p_n \sin\theta_k \\ h_y = \frac{\partial h}{\partial y} &= -\sum_n A_n q_n \sin\theta_k, \end{aligned}$$

both obviously distributed as Gaussians (being sums of independent random numbers), with mean-square fluctuations

$$\langle h_x^2 \rangle = \left\langle \left( -\sum_n A_n p_n \sin\theta_n \right)^2 \right\rangle = \sum_n A_n^2 p_n^2$$

and (similarly)

$$\langle h_y^2 \rangle = \sum_n A_n^2 q_n^2$$

We also have

$$\langle h_x h_y \rangle = \left\langle \left( \sum_n A_n p_n \sin\theta_n \right) \left( \sum_{n'} A_{n'} q_{n'} \sin\theta_{n'} \right) \right\rangle = \sum_n A_n^2 p_n q_n$$

Thus each component of slope is Gaussian, but the components of slope are correlated. (As they must be, to satisfy $\nabla \times \nabla h = 0$.)

#### 1.6.4 Slopes are uncorrelated to heights

We now seek the cross-correlation

$$\langle h h_x \rangle = \left\langle \left( \sum_n A_n \cos\theta_n \right) \left( -\sum_{n'} A_{n'} p_{n'} \sin\theta_{n'} \right) \right\rangle$$

Keeping only terms for which $n = n'$,

$$\langle h h_x \rangle = -\sum_n A_n^2 p_n \langle \cos\theta_n \sin\theta_n \rangle = 0,$$

where the sum vanishes because of the orthogonality of sines and cosines averaged over the uniform distribution of $\theta_n$.

A similar calculation results in

$$\langle hh_y \rangle = 0.$$

We thus obtain the remarkable result that heights and slopes are uncorrelated.

These results are easily generalized to obtain

$$\left\langle \frac{\partial^{n+m} h}{\partial x^n \partial y^m} \frac{\partial^{n'+m'} h}{\partial x^{n'} \partial y^{m'}} \right\rangle = 0 \quad \text{when} \quad (n+m) - (n'+m') \text{ is odd.}$$

### 1.6.5 Slope-area relations [14, 15]

The utility of Gaussian surfaces derives from their role as a "null model" for which specific predictions can be made.

As an example, we consider the oft-cited geomorphological correlation between slope and drainage area.

A good "proxy" for drainage area is the extent to which elevation contours curve: high curvature should correlate positively with large drainage area.

First, recall the formula for the curvature of a one-dimensional function $y(x)$:

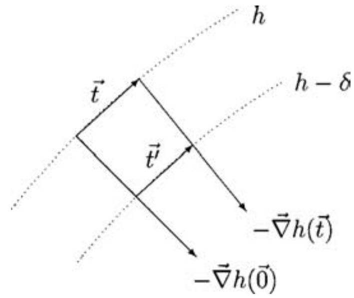
$$\kappa = \frac{y''}{[1 + (y')^2]^{3/2}}$$

The curvature $\kappa$ of a contour line has a somewhat similar form:

$$\kappa = \frac{h_y^2 h_{xx} - 2h_x h_y h_{xy} + h_x^2 h_{yy}}{\left(h_x^2 + h_y^2\right)^{3/2}}.$$

Its relation to first and second height derivatives derives from its relation to the topographic *convergence* $p$, the relative contraction of a contour segment per unit height:

$$p = \frac{|\vec{t}| - |\vec{t'}|}{|\vec{t}|\delta}$$




Here $\delta$ makes $p$ independent of the contour spacing.

The convergence is related to the curvature by a factor of slope:

$$p = \frac{\kappa}{|\nabla h|}$$

Thus $\kappa$ describes how contours curve, and the slope $|\nabla h|$ determines the available downhill-path-length per unit height.

Now rewrite the curvature as

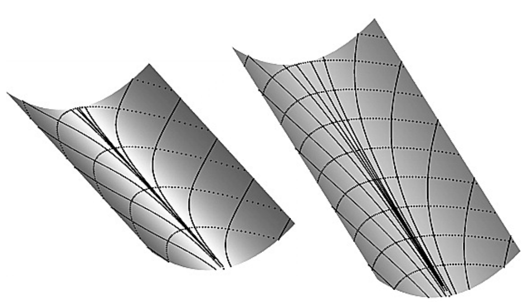
$$\kappa = \frac{(\nabla h)^t \begin{pmatrix} h_{yy} & -h_{xy} \\ h_{xy} & h_{xx} \end{pmatrix} \nabla h}{|\nabla h|^3}$$

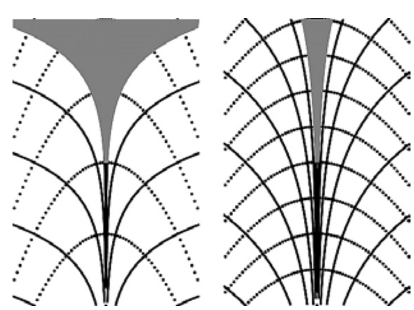
Since first derivatives are independent of second derivatives, the second derivatives are essentially a prefactor, yielding

$$\kappa^2 \propto \frac{1}{|\nabla h|^2} \quad \Rightarrow \quad |\kappa| \propto \frac{1}{|\nabla h|}.$$

We thus have an explicit relationship between curvature and slope.

Qualitatively this is easy to understand: consider a tilted U-shaped valley so that water flows (ultimately) down the valley's center.





Curvature, convergence, and aggregation vary with the tilt:

- *No tilt.* Elevation contours are merely parallel to the valley floor.
- *Infinitesimal tilt.* Elevation contours have infinite curvature at their apex, thus convergence and areal aggregation are also maximal.
- *Steep (nearly vertical) tilt.* Elevation contours are nearly perpendicular to the valley floor; curvature, convergence, and areal aggregation are minimal.

The utility of the Gaussian hypothesis is that it quantifies this qualitative observation.

In the specific case of the slope-area relation, we identify curvature or convergence with area and realize immediately that there *must* be an inverse correlation between slope and drainage area.

#### 1.6.6 Quantitative null hypothesis

In empirical studies in which mechanistic relations are unknown or poorly understood, it is common to seek relationships by searching for correlations.

Typically one tabulates, say, two quantities $X$ and $Y$, and asks whether the fluctuations $\tilde{X}$ and $\tilde{Y}$ from their means are correlated, i.e., if $\langle \tilde{X}\tilde{Y} \rangle \neq 0$.

But this first requires defining what is meant by "correlated." A quantifiable definition of correlated is the following:

> $X$ is correlated to $Y$ if we *measure* $|\langle \tilde{X}\tilde{Y} \rangle| > q$, where $|\langle \tilde{X}\tilde{Y} \rangle| > q$ would occur only $P\%$ of the time if $X$ and $Y$ are *un*correlated random processes.

The notion of "uncorrelated random processes" amounts to a *null hypothesis* which one seeks to falsify by finding a small value of $P$.

The assumption of random phases also amounts to kind of null hypothesis: it is what one might expect in the absence of any particular surface growth mechanism.

Consequently we can identify "interesting" aspects of real surfaces by quantifying characteristics that would be *unexpected* under the Gaussian assumption. Such characteristics would presumably be associated with deterministic (i.e., physical) mechanisms rather than mere statistics.

*Example:* Is a particular combination $h = h^*$, $h_x = h_x^*$, and $h_y = h_y^*$ "unusual"?

To answer this question, note that our results

$$\langle h h_x \rangle = \langle h h_y \rangle = 0 \quad \text{and} \quad \langle h_x h_y \rangle \neq 0$$

yield the pdf

$$p(h, h_x, h_y) = p(h) p(h_x, h_y)$$

where

$$p(h) = \frac{1}{(2\pi W)^{1/2}} e^{-h^2/2W^2}$$

and

$$p(h_x, h_y) \propto \frac{1}{2\pi \sigma_{h_x} \sigma_{h_y} (1-\rho^2)^{1/2}} \exp\left[ -\frac{1}{2(1-\rho^2)} \left( \frac{h_x^2}{\sigma_{h_x}^2} + \frac{h_y^2}{\sigma_{h_y}^2} - \frac{2\rho h_x h_y}{\sigma_{h_x} \sigma_{h_y}} \right) \right],$$

and $\rho$ is the correlation coefficient

$$\rho = \frac{\text{Cov}(h_x, h_y)}{\sigma_{h_x} \sigma_{h_y}}.$$

Then we say that $h = h^*$, $h_x = h_x^*$, and $h_y = h_y^*$ is unusual if

$$\int_{h^*}^{\infty} \mathrm{d}h \int_{h_x^*}^{\infty} \mathrm{d}h_x \int_{h_y^*}^{\infty} \mathrm{d}h_y \, p(h, h_x, h_y) \, < \, P,$$

where $P \ll 1$ is some suitably small number.

Such procedure may be useful in analyzing ecological problems, such as the spatial distribution of the abundance of phytoplankton at the sea surface.

A pitfall, however, is that even on Gaussian surfaces, a fraction $P$ of sites will be deemed unusual.

Thus such a technique is best employed as a way of testing whether a particular *set* of observed features of a landscape is inconsistent with the simple Gaussian assumption—i.e., much more prevalent than one would expect from a random Gaussian fluctuation.

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