12.086 / 12.586 Modeling Environmental Complexity
Fall 2008

Lecture notes for 12.086/12.586, Modeling Environmental Complexity

D. H. Rothman, MIT
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1 River networks

River networks are among the most beautiful of Nature's large-scale *scale-invariant* phenomena.

To see what we mean by scale-invariant, we briefly return to random walks.

1.1 Scale invariance of random walks

Define the rms excursion $r = \langle x^2(t) \rangle^{1/2}$. We have previously shown that

$$r \propto t^{1/2}.$$

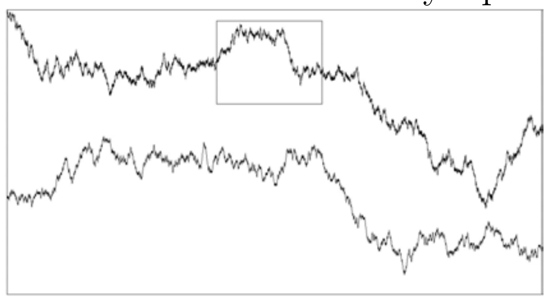
Now rescale time $t \rightarrow bt$ and note that

$$r(t) = b^{-1/2} r(bt).$$

This simple manipulation yields a remarkable observation: the statistics of the random walk are unchanged by the simultaneous rescaling

$$x \rightarrow b^{1/2} x, \qquad t \rightarrow bt.$$

This means that the random walk is statistically equivalent at all scales, e.g.



Here $b = 1/5$ and the inset is "blown up" by stretching the horizontal axis by a factor of 5 and the vertical axis by $\sqrt{5}$.

[Another example: go to any financial website that provides graphs of market fluctuations at a time scale of your choosing (days, weeks, months, years). Note that aside from long-term trends, all graphs look alike!]

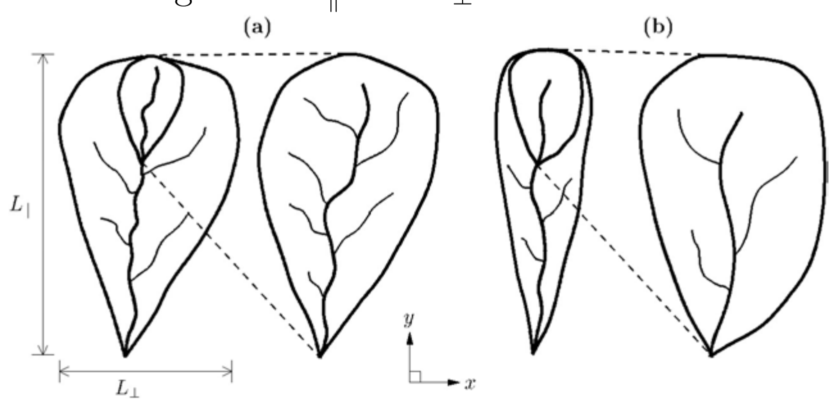
Although we present the random walk as an example of scale-invariance, note that space and time do not scale in the same way.

Later we will call such scaling *self-affine*, but for now it suffices to note the statistical equivalence at different scales.

1.2 Allometric scaling

River basins exhibit a similar phenomenon: they too have statistically similar shapes at all scales, but their dimensions scale differently.

We refer to the two lengths as $L_{\|}$ and $L_{\perp}$:



The area a is

$$a \propto L_{\parallel} L_{\perp}.$$

Measurements made from maps typically show

$$L_{\perp} \propto L_{\parallel}^{H}, \qquad 1/2 \lesssim H \lesssim 1$$

and therefore

$$a \propto L_{\parallel}^{1+H}$$

The case $H = 1$ in (a) yields geometric similarity or self-similarity: no matter what the size of a, all basins "look alike."

The case $H \neq 1$ in (b) is called *allometric scaling* (as opposed to "isometric"): dimensions grow at different rates.

Specifically,

$$\frac{L_{\perp}}{L_{\parallel}} \propto L_{\parallel}^{H-1} \propto \left(a^{\frac{1}{1+H}}\right)^{H-1} = a^{-\frac{1-H}{1+H}}.$$

Since observed values of H fall within $0 < H < 1$,

$$\frac{1-H}{1+H} > 0.$$

Therefore

- large river basins tend to be long and thin; and
- small river basins tend to be short and fat.

Upon appropriate rescaling, however, they all look the same.

1.3 Size distribution of river basins

Suppose we stand at a point x_0 chosen at random on a landscape. What is the size of the area a that drains into it?

We could estimate a by walking uphill from x_0, always following the steepest path. That brings us to a drainage divide, providing an estimate of $L_{\parallel}$.

Obtaining the full area a, however, requires more work. A particularly labor intensive method would be to create a grid of, say, 1 m, and to place a person at each grid site above x_0. We then ask each person to follow his/her path of steepest *descent* from one grid site to the next. Then if N people eventually arrive at x_0, the drainage basin has size N m^2.

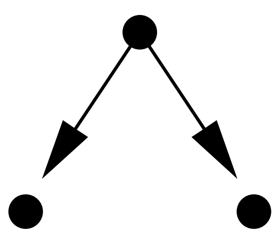
But what would a be a step to the right or left?

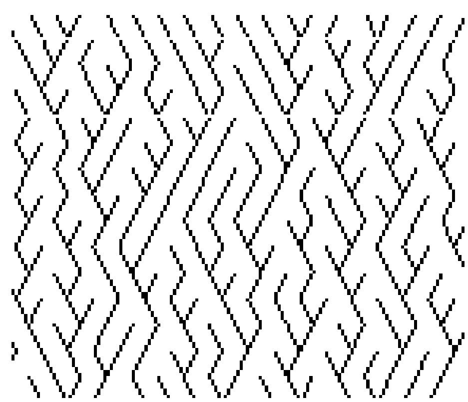
More generally, what is the probability distribution $P_a(a)$ of drainage areas?

To answer this question we create a model of a landscape made of random bumps. The bumps are smooth on a small scale but otherwise independently chosen from some well-behaved distribution like a Gaussian.

We then *tilt* the landscape so that all paths of steepest descent are always *directed* in the direction of the tilt.

We then populate the landscape as before, and follow each path of steepest descent. Supposing that the square grid is oriented 45° to the tilt direction, each step down will also be a step to the left or right, and the coalescence of the various paths will look like this:





If we then trace out any basin, we find that its left and right boundaries are particular realizations of a *random walk*.

This is Scheidegger's 1967 model of river networks [1].

Since for a random walk we have $\langle x^2 \rangle^{1/2} \propto t^{1/2}$, we substitute

$$t \to L_{\parallel}, \qquad x \to L_{\perp}$$

and conclude

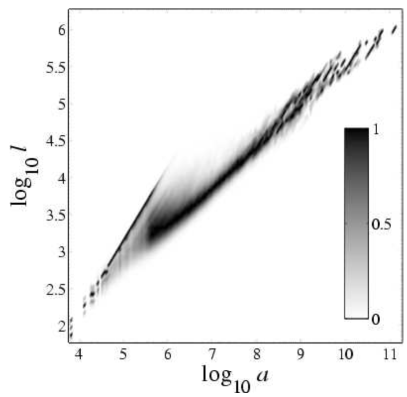
$$L_\perp \propto L_\parallel^{1/2} \qquad \Rightarrow \qquad a \propto L_\parallel^{3/2}.$$

Since our random walks are always *directed* toward the bottom boundary, the length l of the longest stream in each basin scales like $l \propto L_\parallel$, and therefore

$$l \propto a^{2/3}$$

Real rivers exhibit a similar scaling law, called *Hack's Law* [2]:

$$l \propto a^h, \qquad 0.57 \lesssim h \lesssim 0.60.$$



The correspondence between the two suggests that our model is reasonable.

To find $P_a(a)$, we write

$$\begin{aligned}\phi_l(y) &= \text{left basin boundary}\\ \phi_r(y) &= \text{right basin boundary}\end{aligned}$$

and assume that they both start at $y = 0$. Note that the difference

$$\phi(y) = \phi_l(y) - \phi_r(y)$$

is itself a random walk that not only begins at $y = 0$ but also ends at $y = 0$.

We ask a precise question: *What is the probability that $\phi(y)$ returns to its initial position for the first time after n steps*?

This is the classic *first-return* or *first-passage time* of a random walk. The answer, for large n, is [3, 4]

$$P(n) = \frac{1}{2\sqrt{\pi}} n^{-3/2}.$$

Since our random walks are directed, the main steam length $l \propto n$. Therefore

$$P_l(l) \propto l^{-3/2}.$$

To obtain the area distribution, we recall $l \propto a^{2/3}$ and write

$$\begin{aligned} P_a(a) &= P_l\big(l(a)\big)\frac{\mathrm{d}l}{\mathrm{d}a} \\ &\propto (a^{2/3})^{-3/2}a^{-1/3} \\ &= a^{-4/3}. \end{aligned}$$

Real rivers exhibit

$$P_a(a) \propto a^{-\tau}, \qquad \tau = 1.43 \pm 0.02,$$

suggesting once again that our model is reasonable.

1.4 Scaling relation

Gathering our results, we have

$$\begin{aligned} l &\propto a^h \\ P_a(a) &\propto a^{-\tau} \\ P_l(l) &\propto l^{-\gamma} \end{aligned}$$

where, comparing Scheidegger's random-walk model to real observations, we have

Exponent	Scheidegger	Real world
h	2/3	0.58 ± 0.02
τ	4/3	1.43 ± 0.02
γ	3/2	1.8 ± 0.1

Since we are looking at areas and lengths, we expect that, whatever their real values, the exponents h, τ, and γ should be related to each other. We find this relation by writing

$$\begin{aligned} P_a(a) &= P_l\big(l(a)\big)\frac{\mathrm{d}l}{\mathrm{d}a} \\ &\propto (a^h)^{-\gamma}a^{h-1} \\ &= a^{-[1-h(1-\gamma)]}, \end{aligned}$$

which yields

$$\tau = 1 - h(1 - \gamma).$$

This is an example of a *scaling relation*: an algebraic relationship that gives the explicit dependence between exponents. In this case, it says that we need only know two exponents to obtain the third. (Further work [5] shows that there is only one independent exponent.)

Our scaling relation is readily verified by substituting the value for Scheidegger's model:

$$\frac{4}{3} = 1 - \frac{2}{3}\left(1 - \frac{3}{2}\right)$$

As for the real world, we have

$$1.43 \simeq 1 - 0.58(1 - 1.8) = 1.46.$$

Overall we see that the statistical geometry of river networks—the power-law scaling, and the relations between the exponents—is well described by Scheidegger's random walks.

Note that we have said nothing about how real rivers form.

Instead we note that their tendency to aggregate (i.e., connect) as they flow downhill is the main ingredient necessary to account for their principal geometric features *after* they form.

We suspect that this reflects the universal properties of random walks: i.e., that the mean square fluctuation in one dimension scales roughly like the mean fluctuation in the other.

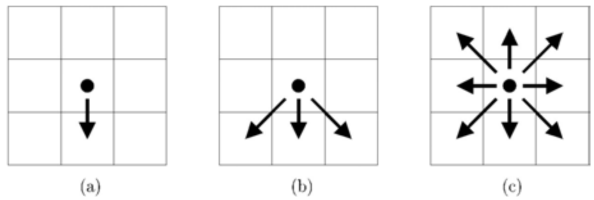
If this is true, it means that natural landscapes contain no particularly special features that suggest that the typical paths of steepest descent are *unlike* random walks.

That the Scheidegger model does not capture the correct values of the exponents τ, h, and γ suggests nevertheless that something important is missing.

We proceed to consider what that might be.

1.5 Universality classes

Let us consider our random-walk model as a member of a wider class of models based on the kind of path taken by flowing water:



- (a) Non-convergent.
- (b) Directed random walks (Scheidegger).
- (c) Undirected random walks.

Now consider Hack's law $l \propto a^h$.

The trivial non-convergent case (a) yields $h = 1$.

Directed random walks (b) yield $h = 2/3$.

Undirected random walks (c) yield $h = 5/8$ [6].

Remaining exponents are obtained from our previous scaling relation, augmented by the additional relation $\tau = 2 - h$ [5].

Each of these cases may be considered *universality classes*. These classes are delineated by qualitative conditions: here, the way in which flowing water can aggregate. These qualitative conditions then lead to specific quantitative predictions of exponents.

Which universality class is "correct"? One could argue that undirected random walks are the most realistic class, and that it is no surprise that their Hack exponent, $h = 5/8$, is the closest to the observed $h \simeq 0.58$.

Although such reasoning may possibly be valid, it misses the main lesson: *Aggregating random walks capture the main features of river network geome-*

try.

All the rest comes down to getting the exponents "right." But because it is not precisely clear what it would mean to be correct, it is both more interesting and safer to simply learn our lesson and ask if it has any applicability elsewhere.

1.6 Sandpiles and self-organized criticality

Scheidegger's model of rivers has been rediscovered in many different contexts.

A particularly interesting example is the directed sandpile model.

But we first discuss the "classical" undirected sandpile model of Bak, Tang, and Wiesenfeld [7].

On a grid of size $L \times L$, assign a number

$$Z(x, y) = \text{number of sand grains at location } x, y$$

At each time step, choose a site randomly and add one grain of sand:

$$Z(x, y) \rightarrow Z(x, y) + 1.$$

Continue to repeat these random additions until a site is "unstable" and topples.

The toppling rule: when $Z(x, y) \geq Z_c$,

$$Z(x, y) \rightarrow Z(x, y) - 4$$

and a grain of sand is added to each of the nearest neighboring sites:

$$\begin{aligned} Z(x \pm 1, y) &\rightarrow Z(x \pm 1, y) + 1 \\ Z(x, y \pm 1) &\rightarrow Z(x, y \pm 1) + 1. \end{aligned}$$

When sand is transferred to neighboring sites, the neighboring sites themselves can also topple. As can their neighbors, etc. The toppling continues until no more sites are unstable. If a boundary of the system is reached, particles fall off the edge.

(An alternative formulation, due to Grassberger [7], considers a room full of bureaucrats. When their paper work piles up beyond a critical level, they pass off one unit of work to each of their neighboring bureaucrats...)

1.6.1 Avalanches

The interest in such a model is in the *avalanches.* Define the avalanche size

$$s = \text{number of sites that topple in a single event.}$$

Simulations show a power-law distribution of s:

$$P(s) \sim s^{-\tau}, \qquad \tau \simeq 1.1$$

The importance of such a relation is that there is no characteristic avalanche size (i.e., the distribution's mean and variance exist only if one assumes a finite system size L).

One can find data for real avalanches, or perform experiments with laboratory sandpiles, that, at least in some cases, confirm this picture.

However its importance derives not from its detailed correspondence to real sandpiles but instead the apparent ubiquity of such power-law phenomena in natural systems.

1.6.2 Earthquakes

Consider, for example, an *earthquake fault.* Model it on a grid, so that $Z(x, y)$ represents stress supported by a local *asperity* (i.e., friction due to surface roughness).

Regional stress can be modeled by successively adding one unit of stress and random sites. When stress exceeds a critical threshold, the asperity breaks such that 4 units of stress are removed and one unit is distributed to each of the nearest neighbors.

This model is of course merely a reinterpretation of the sandpile model. But the distribution of avalanches, reinterpreted as earthquakes, has a remarkable correspondence to the *Gutenberg-Richter scaling law*: the number N of earthquakes of magnitude m (amount of energy released) scales like

$$N(m) \sim m^{-b}, \qquad b \simeq 1$$

1.6.3 Self-organized criticality

When the temperature of a liquid is lowered to it freezing point, density fluctuations occur at all scales. This *scale-free* behavior is characteristic of all *equilibrium critical phenomena*, and is independent of the particular type of phase transition.

The lack of a characteristic fluctuation size is a general property of *equilibrium critical phenomena*, wherein long-range correlations exist at the critical temperature T_c.

Bak's sandpile model is interesting because it exhibits such critical behavior without any "tuning" of the temperature. The system instead "self-organizes" to the critical state. Consequently the phenomenon is known as *self-organized criticality.*

1.6.4 Directed sandpiles

We proceed to show how Scheidegger's rivers relate to Bak's avalanches.

We make one key change to Bak's sandpile: we assume that the "tilt" of the pile requires that sand flow only downhill [8, 9].

We tilt a square lattice diagonally, and add particles at random sites on the top row. We take $Z_c = 2$, so that when $Z(x, y) \geq Z_c$,

$$Z(x, y) \to Z(x, y) - 2$$

and we add "sand" to the neighboring sites in the nearest lower row:

$$Z(x \pm 1, y - 1) \to Z(x \pm 1, y - 1) + 1$$

If those sites are now unstable, the sand is transferred to the nearest neighbors below once again, and so on.

The avalanching continues until all sites are once again stable such that $Z \leq Z_c = 2$ everywhere.

The correspondence with the Scheidegger model follows from identifying the paths taken by unstable particles of sand with those of Scheidegger's raindrops or random walkers.

Specifically, we once again see that the avalanche "basin" is once again bounded by directed random walks, and we inherit not only the same first-passage problem of Scheidegger but also the same exponents.

In particular, we have the avalanche size distribution

$$P(s) \sim s^{-\tau}, \qquad \tau = 4/3.$$

Finally, we note that Takayasu's model of random aggregation [10] is also equivalent to Scheidegger's model. In this case particles move about randomly on a 1D lattice, with constant reinjection of mass. In steady state, the distribution of mass is the same as the basin distribution or the avalanche distribution.

1.7 The lesson learned

The main lesson here is that the allometry imposed by random walks can account for a wide variety of apparently unrelated phenomena in which size distributions decay as power laws, suggesting that there is no characteristic size or fluctuation.

But this does not in itself mean that rivers, sandpiles, etc. are truly the result of some mechanism that generates random walks. And at the same time it also leads one to question whether the notion of "self-organized criticality" is necessary to understand each of these problems.

What is clear, however, is that a useful first step in analyzing such problems is to create minimal models as we have here, to analyze their predictions, and then to ask whether both the assumptions of the model and its predictions correspond to reality.

The qualitative correspondence of the predicted power laws with observations is a powerful indication that something about these models is correct.

Yet important elements may also be missing:

- the way in which the real problem generates the random-walk geometry; and
- the possibility that the deviation of observed exponents from predicted exponents reflects some fundamental flaw.

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