12.086 / 12.586 Modeling Environmental Complexity
Fall 2008

12.086/12.586 Problem Set 3
Percolation

Due: November 24

November 12, 2008

1. **Flood Plains** In central Africa there are large, flat plains which are home to elephants, baboons, wild dogs, and lions. In the dry season, the landscape has many small water holes. However, during the rainy season, the ponds flood and the landscape has a large interconnected water way (watch *Planet Earth* for footage of this). Thus, aquatic animals are isolated from each other during the dry season, but can mix in the wet season. A similar example has been found in the fossil record where a population of mollusks was seen to have periodic episodes of intense speciation when the climate was dry (leading to many isolated communities living in disconnected ponds) while being more homogeneous during wet climates. An interesting question is: how much rain does it take to transform a landscape of many small, disconnected ponds into one with at least one giant lake? We will answer this question using percolation theory. The initial ponds form in minima of the topography. In order for two ponds to be connected, water levels need to rise enough to swamp the land between them. If the water levels rise, a certain proportion p of the ridges will flood. In this case how large must p be for there is an infinitely large lake? This process is called *bond percolation*, rather than placing sites at random, we will place bonds between neighboring sites with probability p (Figure 1).
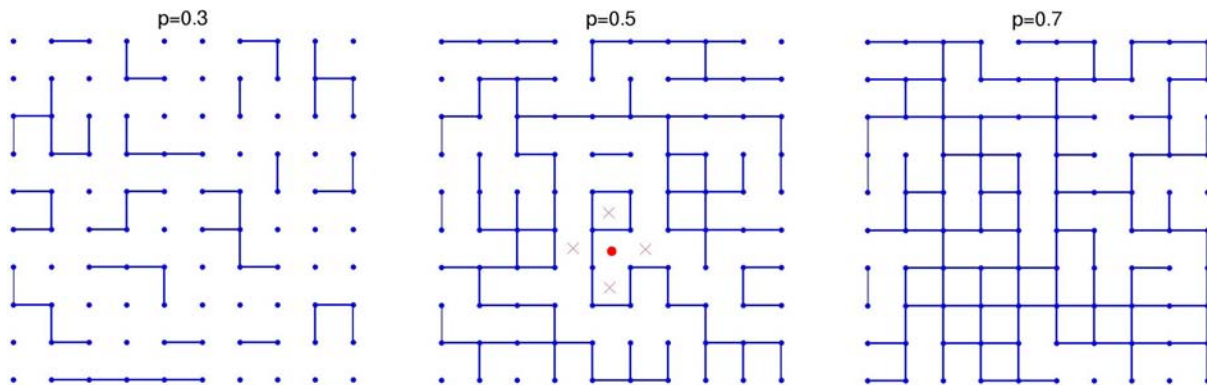

Figure 1: Bond Percolation for various values of p. The baboon is sitting at a site (red circle) between 4 ponds (blue circles). It could potentially move to any of the neighboring plots of dry ground (red 'X'), however the ridges that connect the red circle to the land to the North and West are flooded out (blue lines).

    (a) To gain some intuition for the problem begin by placing local minima of the topography (i.e. ponds) on a square lattice. Think about a baboon that is sitting on the ground

between 4 ponds. What is the probability $\hat{p}$ that the baboon can walk to the dry land to the East without getting wet? In the lingo, the baboon walks on the *dual* of the square lattice.

(b) Let $\hat{p}_c$ be the critical value of $\hat{p}$ such that there is an infinite tract of land where the baboons can walk without getting wet. What is $\hat{p}_c$ in terms of p_c? (Hint: Draw a picture)

(c) Given the last two results what is the value of p_c?

(d) Return to the more realistic case. Assume now that topography is a gaussian surface (Figure 2) with mean elevation 0. If the water level rises to h, everything with elevation less than h will flood. How high must h be to ensure that there is an infinite lake? (Hint: A similar trick as before works since the topography is symmetric above and below the mean. How would the picture change if the topography were turned upside down and everything higher than h flooded?)
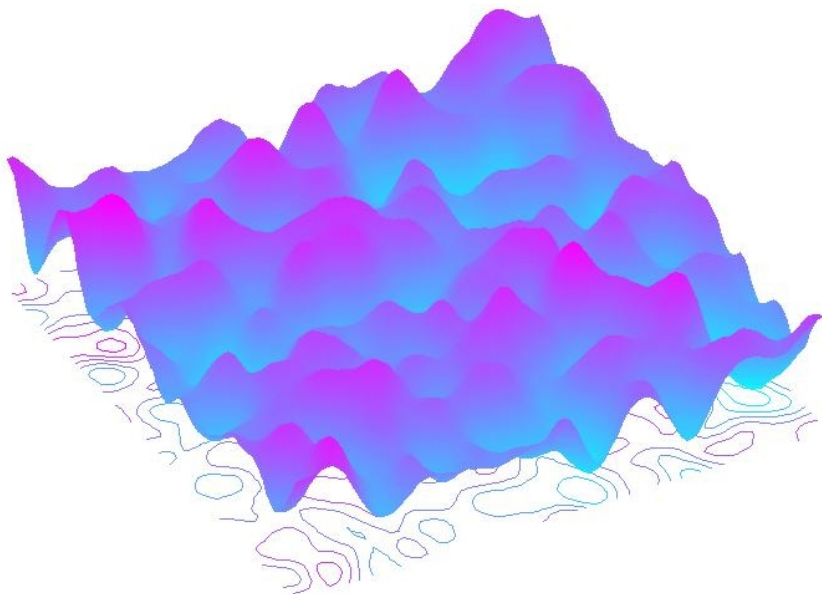

Figure 2: A Gaussian Surface

2. **Chemical Avalanches** It has been observed that when a liquid dissolves a solid at low temperature, the reaction does not happen at a constant rate. Rather, the dissolution front eats into the solid in fits and jumps called *chemical avalanches*. Here we will explore a model of dissolution based on two assumptions. First, any small volume of the material on the surface of the solid will react at a rate $R_i \propto e^{-E_i/k_B T}$. Here E_i is an activation energy and $k_B T$ is the thermal energy of the material. Second, the activation energy is a random variable $E_i = E_0 + p_i \Delta E$, where p_i is uniformly distributed between zero and one. For ease of notation, define Θ to be $k_B T/\Delta E$. Thus, $R_i = \tau_0^{-1} e^{-p_i/\Theta}$. Here τ_0 is a constant with units of time.

This phenomenon has been observed in the lab. For example, when $CuCl_2$ is dissolved in water and placed on a strip of magnesium, the magnesium replaces the copper, according to the reaction $Mg(s) + CuCl_2(aq) \rightarrow MgCl_2(aq) + Cu(s)$. Furthermore, during each reaction a small, magnetic pulse is produced. By measuring the magnitude of the magnetic field, it is possible to track how quickly the magnesium bar is dissolving. Interestingly, at low

temperatures the bar does not dissolve at a constant rate. Figure 3 shows the magnetic field as a function of time and a histogram of the size of magnetic pulses (these figures are from Claycomb *et al. Phys. Rev. Lett.* **87** (2001) 178303). We observe that there is a power law distribution for the probability of that an event of a given size occurs (i.e. the probability that N many reactions occur together scales like $N^{-\mu}$). These avalanches are phenomenologically similar to the avalanches we studied in Per Bak's sand pile. In fact, Claycomb *et al.* develop a sand pile model to describe this experiment.
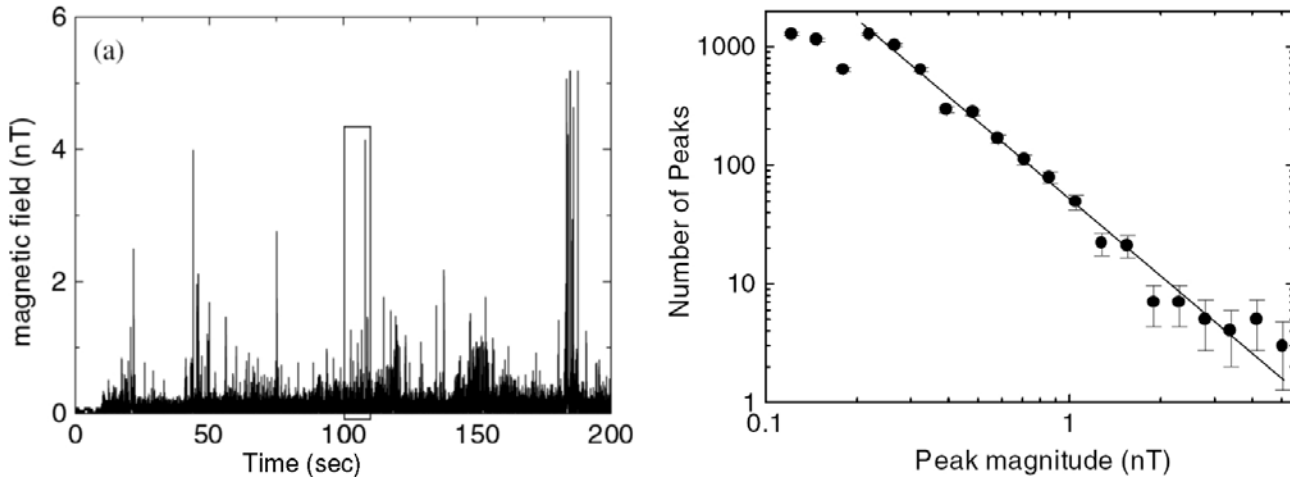

Figure 3: Left: Strength of the magnetic field (proportional to the number of reactions) as a function of time. Right: Histogram of the size of peak values.

(a) First find the probability that a particular site on the surface will be the next to dissolve. The probability that any particular site will dissolve in the next short amount of time Δt is $R_i \Delta t$. How long should Δt be to predict that one site somewhere on the surface dissolves in this period of time? Given that the probability P_i of a particular surface site dissolving is proportional to it reaction rate, what must the proportionality constant be?

(b) Given the result from the last part, we can simulate the reaction. At each time step we first look at the reaction rates on the surface to find how long we need to wait before a reaction will occur. We then select which site will dissolve using the probability distribution. We remove the selected site and write down that an amount of time Δt elapsed during that time step. The simulation finishes when half of the availible points have been removed. Write code that will simulate this process in two dimensions (or use the code on the website). Run the simulation at high temperature and low temperature. Describe qualitatively the difference between the two.

(c) For a given temperature Θ, what proportion π of sites will take longer than τ to dissolve? (Hint: what is the probability that a randomly selected point will?). To gain some intuition for the system, we can paint all of the points that decay faster than τ red (hot) and those that decay more slowly blue (cool). Blue points are placed with probability π and red points are placed with probability $1 - \pi$. There is code online to let you see the hot and cold points for a given choice of τ and Θ. Take a look.

(d) For a given Θ and τ we can understand the dynamics of the front in terms of two phenomena *walls* and *holes*. A wall is a connected set of points that decays slowly and spans the system. A hole is a connected set of points that decay quickly. What happens

3

when the front encounters a wall? What is the largest value of τ for which there still exists a wall somewhere in the system?

(e) We may suspect that the reaction is always limited by that amount of time it takes to "break on through to the other side" of the slowest reacting wall. If this is the case how does the total time it takes to run the reaction depend on temperature? Confirm this suspicion numerically.

(f) What happens when a front encounters a hole? Describe the relation of holes to avalanches. How do the existence of large avalanches effect the claim that there is some characteristic time for the reaction to finish. Does the small possibility of large avalanches effect the average time it takes for the reaction to run or our confidence that a specific experiment will take that long to finish (i.e. the variance in the run time)?

(g) The code online gives as an output the p.d.f. for the size of avalanches as a function of the number of elements in it. What shape is the distribution for large Θ (e.g. $\Theta = 1$)? What shape is the distribution for small Θ (try $\Theta = 500^{-1}$). Plot the mean and variance of this distribution as a function of temperature to show that there is a critical temperature below which avalanches destroy any confidence in our guess for the mean run time.