12.086 / 12.586 Modeling Environmental Complexity
Fall 2008

12.086/12.586 Problem Set 2
Scale-free phenomena and Surface Growth

Due: October 20

October 2, 2008

1. **White Collar Crime** One fantastically complex system is the accounting records of a large company. In addition to the multitude of complex mechanisms that influence accounts, people also lie. Is there any way one can identify "normal" accounts just by looking at them? One tool that is used to find evidence of "certain irregularities in accounting practices" is the probability distribution for the first digit of randomly chosen number. One may guess that the first digit of a randomly chosen stock price would be as likely to be 3 as to be 9; however empirically 1 occures about 30% of the time while 9 occures only about 5% of the time. There is an empirical p.d.f that describes this. To find cooked books, auditors compare the observed frequency of first digits from accounting records to the empirical p.d.f.. In fact, the first digit of numbers that describe many natural systems (e.g. river drainage areas, city populations, addresses, physical constants) all have what appears to be the same probability distribution. If one posits the existence of some universal probability distribution for the first digits of naturally occurring numbers, it is possible to calculate the observed probability distribution. The essential observation is that if there is a universal probability distribution function $P(x)$ for the first digit, then the shape of the function cannot change by rescaling the data (i.e. assume the probability that Microsoft's stock is selling for something that starts with a one does not depend on if we are trading in dollars or yen). Thus, $P(\lambda x) = f(\lambda)P(x)$.

   (a) Use this constraint along with the requirement that P always be normalized to show that $f = \lambda^{-1}$ and $P \propto x^{-1}$.

   (b) What is the probability that the first digit is a d? (Hint: find the probability that a number picked between 1 and 10 will be between d and $d + 1$.

   (c) Benford (1938) collected data from a variety of different collections of numbers (e.g. addresses, physical constants, death rates). Here is the observed pdf for the first digit of data points from 741 various measures of the cost of things. $P(d = 1) = 32.4$, $P(d = 2) = 18.8$, $P(d = 3) = 10.1$, $P(d = 4) = 10.1$, $P(d = 5) = 9.8$, $P(d = 6) = 5.5$, $P(d = 7) = 4.7$, $P(d = 8) = 5.5$, $P(d = 9) = 3.1$. Compare this result to theory.

   (d) (Optional) Generalize this result to first digits in base b.

2. **Diffusion on an ice cube tray** In class we considered the case of diffusion on a comb. We found that an unbiased random walker could get stuck as is wandered back and forth on a tooth of the comb. We again consider a random walker that gets stuck in a trap. The idea is that we place a marble into an ice cube tray and shake it randomly. If we shake it gently the marble spends a very long each hole in the tray. The harder we shake the tray, the more frequent the marble gets kicked into a new hole. If we shake the tray very hard the marble jumps on every shake. In this final case, the marble does not even "see" the holes and simply jumps randomly across the tray; resulting in an unbiased random walk.

   (a) The example of a marble in an infinite ice cube tray is clearly contrived. Give an example of a real system you suspect might be similar. (Example: Thermal agitation effectively shakes a system. The free-energy of a system can often be visualized with many local minima, in which a system can get stuck.)

   (b) Exponential distributions are common in nature (e.g. the time between the decay of radioactive atoms). They also describe the probability that particle taken from a system will have some specific energy. For example, the average nitrogen molecule on Earth is moving at about 500 m $\sec^{-1}$, however there are some moving at 50000 m $\sec^{-1}$. The probability that we can find such a molecule is proportional to e^{-E/k_BT}, where E is the kinetic energy of an N_2 molecule moving at 50000 m $\sec^{-1}$ and T is temperature. This observation gives rise Arrenius' Law which predicts the time τ a particle spends in a well of depth V is proportional to e^{V/k_BT}. How does the diffusion constant depend on the temperature of the particle (i.e. how hard we are shaking the tray)? How does the mean-square distance of the marble scale with time? Give a plot of the scaling exponent as a function of the temperature of the particle.

   (c) To make the system sub-diffusive we need to add randomness to topography as well. Suppose that the probability of landing in a well of depth V is e^{-V/V_0}, where V_0 is the average depth of a well. (If you like, you can interpret V_0 as the thermal energy of the topography just like k_BT is the thermal energy of the particle). Given that τ is a function of V, use this probability distribution to show that the probability the marble is stuck in a well for a time τ is proportional to $\tau^{-(1+\mu)}$. What is μ?

   (d) How does the mean-square distance of the marble scale with time? Give a plot of the scaling exponent as a function of the temperature of the particle (Hint: For what values of μ is $\langle\tau\rangle$ finite?).

   (e) (Optional) Can you think of a physical system in which the step sizes have a power law distribution as would be the case for a Lévy walk?

   (f) Recall that $\langle X^2\rangle = \langle \ell^2\rangle \langle\tau\rangle$. Combine the prediction of $\langle \ell^2\rangle$ from Lévy walks with this result to find the scaling of a $\langle X^2\rangle$ in time for a continious time random walk with power law step sizes.

3. **A Snow Drift** A snow flake falls from the cloud and gently drifts down to the ground. When it hits the ground it sticks to where it lands. As a result the snow drift grows

in the normal direction. What sorts of scaling relations do we expect for the height h the snow drift depending on the time t and the scale L on which we observe the drift. The same system can be used to model a game of Tetris played by an inexperienced person.

(a) Justify the growth equation $\partial_t h = \lambda\sqrt{1+(\nabla h)^2} + \eta$, where η is uncorrelated gaussian noise. Since snow drifts tend to be reasonably flat, expand about $(\nabla h) \ll 1$.

(b) Use the technique of dynamic rescaling to find α, β, and z.

(c) There is code on the web site that simulates this process. Take a look.

4. **Solid-on-Solid** We are going to develop a few techniques to measure α, β, and z from a real data set. To do so, it is useful to use a model that is simple enough to analyze exactly. The essential aspects of noisy surface growth we have considered so far are diffusion and surface normal growth. The important idea about diffusion is that sharp parts disappear quickly. To force diffusion into our toy model we will consider a noisy system in which local extrema are eliminated but flat surfaces are not changed. At each time step, an exremum is picked at random. If it is a local maximum, its height is decreased by two units with probability p_- ; thus, flipped maxima become local minima. Similarly, local minima are raised two units with probability p_+ to become local maxima. Figure 1 shows this process.

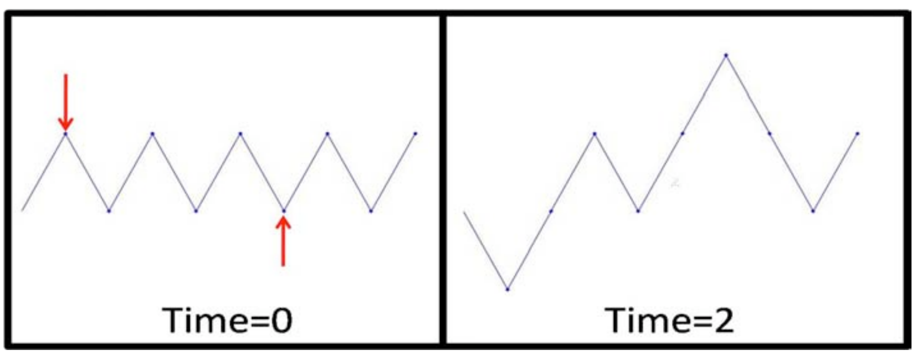

Figure 1: the difference between two time steps of the solid-on-solid model is shown. Red arrows point to the maximum and minimum that will flip in the next two time steps.

(a) If the height of the i^{th} node is h_i, what is the largest possible magnitude of $s_i = h_i - h_{i-1}$?

(b) Show that when $p_+ = p_-$, h_i can be expressed as a random walk of slopes s_i. Given this observation, what is the roughness exponent?

(c) Check this result numerically. Write your own code to simulate this process (or use the code posted on the web site). Simulate an ensemble of surfaces ($p_+ = p_-$). By looking at the power spectrum, calculate the roughness exponent.

(d) By plotting the initial rise of the root-mean-square variation of the surface about its mean, find β.

(e) Given α and β, what is z?

(f) Just by looking at a growing surface in a simulation or in the lab, it is seldom obvious if the growth is linear or non-linear. There is a trick to find the importance of surface normal growth $\lambda(\nabla h)^2$ in the growth law. The idea is to tilt the surface to create an average surface-normal direction. Thus, tilting the surface will increase the average velocity v if surface normal growth is important. Assume that if $\partial_t h = \nu\nabla^2 h + \frac{\lambda}{2}(\nabla h)^2 + \eta$, where η is noise. Impose a mean slope $\langle\nabla h\rangle = m$ by tilting the system to a slope of m. Show that $v = \langle\partial_t h\rangle = L^{-1}\int_0^L \partial_t h\ dx$ is $v(0) + \frac{\lambda}{2}m^2$, where $v(0)$ is a constant. Assume periodic boundary conditions. (Hint: one way of doing this is to use Gauss' Law to get rid of the diffusion term and assume that $\nabla h = m + \phi$, where ϕ represents the fluctuations about the mean value.)

(g) Write the average velocity of the Solid-on-Solid model in terms of the probability of picking a maximum Π_- times the expected change in height for a maximum, plus a similar term for picking a minimum.

(h) Now give $\Pi_\pm$ in terms of the number N_+ points on the lattice where $s_i > 0$ and the number N_- points on the lattice where $s_i < 0$.(Hint: Ignore the few points for which $s_i = 0$. What is $N_+ + N_-$? Given that there is a total slope of m, what is $N_+ - N_-$?)

(i) Combine the last two results to find v in terms of $p_\pm$ and m.

(j) Given the two forms of v, show that $\lambda = p_- - p_+$ Do you expect the same roughness exponent for $p_+ = p_-$ as for $p_+ \neq p_-$? Do you expect the same dynamic exponent z? Redo parts (c)-(e) for $(p_+ \neq p_-)$.