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12.086 / 12.586 Modeling Environmental Complexity
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12.086/12.586 Problem Set 1
Random Walks and First Passage Times
Due: October 1

In this first problem set, the idea is to introduce a couple of neat ideas about random walks that might come up in models (e.g. your final projects). It is possible to do everything here with paper, pencil, and a couple cups of coffee. It is also possible to do them with a computer. A couple of these only require you to give a reasonable, intuitive argument. Use any method you like. Additionally, you can email th TA at petroffa@mit.edu for hints or any questions. The TA also have office hours on Tuesdays 1-3 is 54-621. I am usually around (and always happy to talk about cool models) so if these give you grief just email me and we can meet.
-Alex

1. **Return Probability**

   (a) Loris are small primates which live in South East Asia, Sri Lanka, and India (Figure 1). One particular group of them feed on nectar from certain flowers that grow in the canopy. The problem is that these flowers are distributed far apart and once a loris feeds on a flower it takes a while for the flower to produce more nectar. Thus an individual often wastes a lot of energy to visit an empty flower. The question is: how much stress does an individual Loris put on the total number of flowers? One could conceive of similar problems facing organisms that live in a number of different dimensions (e.g. a nematode grazing on a flat surface), so generalize the problem to d many dimensions. Make the simplifying assumption grazers scour the area randomly (unbiased random walk). How does the volume V of likely positions visited by the grazer scale in time? How does the density ρ of grazed sites scale in time? Discuss the result in d =1,2, and 3 in the limit that time goes to infinity. (Hint: How does the mean-square distance vary as a function of time?)

Figure 1: A loris

Courtesy of PDClipart.org.

   (b) Many random walks in nature are biased. One of the famous examples is a bacterium moving up a chemical gradient. A bacterium measures the local concentration of the chemical, takes a step in a random direction, and measures the change in the chemical concentration. Depending on the change in concentration, the bacterium adjusts the step size. Thus, the bacterium moves up (or down if it is toxic) the gradient with probability $q = \frac{1}{2} + \epsilon$. Since this is a random process there is some probability that the bacterium will move up the gradient, and then back down to where it started. If we trap a bacterium in a gradient, what is the probability that it will never return to where it started? Assume that the gradient can be modeled as a one-dimensional lattice (Hint: Let p be the probability that the bacterium does return. Suppose that the bacterium takes a step up the gradient to 1. It will then either return to 0 or go a random walk which might eventually take it back to 1. The probability that it returns to 1 is the same as returning to 0. Use this to write p in terms of p and ϵ.)

(c) Optional: Can you think of an example like the first which takes place on a fractal? How could the Loris interact as to avoid visiting empty flowers? (Hint: in real life they do)

2. **First Passage in One Dimension** In the last problem we focused on "'will a random walk ever hit the same place twice?"' This time, the question is "'how long until a random walker returns to a point?"' In the lectures we saw that this question is related to the distribution of river basin sizes; however the problem has many other applications. In one-dimension, the size of a population can be thought to take a random walk (do births outnumber deaths at a given point). The first passage to zero corresponds to the time when the population goes extinct. Brain cells behave similarly. A neuron is constantly being bombarded with chemicals that make it either easier or harder for it to carry a charge. When the charge reaches a certain threshold, the neuron fires. This problem has been considered by Gerstein and Mandelbrot (*Random walk models for the spike activity of a single neuron* Biophysical Society 1964) among others. There are also analogues in higher dimensions (e.g. when will a Loris return to a flower). To make things mathematically simpler we will only consider one-dimensional systems on a lattice. We will find the probability a random walker will return after a time T.

   (a) Will constraining ourselves to one-dimension influence the final result?

   (b) Will constraining ourselves to a lattice influence the final result?

   (c) There is code on stellar that simulates a random walk and finds the time of first return. Use this code (or write your own or do the analysis) to generate an *ensemble* of first return times (about 10000 test cases works well). Use this data to estimate the probability of a random walk having a particular return time $F(T)$. In general, $F \sim T^{-\nu/2}$. Estimate ν for the one dimensional case. Where do you suppose the 2 in the exponent come from? (If you are using matlab, once you have an ensemble of times and you want a histogram, use [freq,x]=hist(X), and get ν by fitting $\log(freq) \propto -\nu/2\ \log(T)$). (Hint: bin the first return time logarithmically!)

   (d) What is the expected time $\langle T \rangle$ until the walker returns to zero? What does this mean?