18.100B (ANALYSIS I)
PRIMER: SETS AND BASIC LOGIC

This primer collects some mathematical nomenclature we shall use throughout the course.

Sets. For our purposes, it is sufficient to exclusively deal with *"naive set theory"* whose terminology is as follows.[1]

Sets are collections of elements (or members). If A is any set, we mean by $x \in A$ that x is an element of A; if x is not an element of A, we write $x \notin A$.

The set which contains no element is called the *empty set* and denoted by $\emptyset$. Any set containing at least one element is referred to as *non-empty*.

If A and B are sets such that every element of A is an element of B, we say that A is a *subset* of B, which we denote by $A \subseteq B$; or occasionally by $B \supseteq A$. (Alternative notations for $\subseteq$ and $\supseteq$ are $\subset$ and $\supset$, respectively.) Note that the empty set is always a subset, i. e., we have that $\emptyset \subseteq A$ holds for every A.

If both $A \subseteq B$ and $B \subseteq A$ hold, we write $A = B$; and $A \neq B$ otherwise. We call A a *proper subset* of B if $A \subseteq B$ and $A \neq B$, and we denote this by writing $A \subsetneq B$.

In many instances, sets are either given by explicitly listing their elements, e. g., $A = \{1, 2, 3\}$, or by stating properties that all elements have in common, e. g., $A = \{x : x \text{ is an even integer}\}$.

Prominent examples of sets are: the natural numbers $\mathbb{N} = \{0, 1, 2, ...\}$; the integers $\mathbb{Z} = \{..., -2, -1, 0, 1, 2, ...\}$; the rational numbers $\mathbb{Q}$; the real numbers $\mathbb{R}$; and the complex numbers $\mathbb{C}$. Note that $\mathbb{N} \subsetneq \mathbb{Z} \subsetneq \mathbb{Q} \subsetneq \mathbb{R} \subsetneq \mathbb{C}$.

Given two sets A and B, we define their *union*

$$A \cup B = \{x : x \in A \text{ or } x \in B\},$$

and their *intersection*

$$A \cap B = \{x : x \in A \text{ and } x \in B\}.$$

One can prove the following rules:

$$A \cap B = B \cap A, \quad A \cup B = B \cup A; \tag{1}$$

$$A \cap (B \cap C) = (A \cap B) \cap C, \quad A \cup (B \cup C) = (A \cup B) \cup C; \tag{2}$$

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C), \quad A \cup (B \cap C) = (A \cup B) \cap (A \cup C); \tag{3}$$

$$A \cap \emptyset = \emptyset, \quad A \cup \emptyset = A. \tag{4}$$

Furthermore, the set

$$A \setminus B = \{x : x \in A \text{ and } x \notin B\}$$

is called the *difference* of A with respect to B. Note that $A \setminus A = \emptyset$, as well as $A \setminus B = A$ whenever $A \cap B = \emptyset$.

[1] For further reading on axiomatic set theory, see e. g. `http://en.wikipedia.org/wiki/Axiomatic_set_theory`

Be aware of the fact that sets can contain sets. The *power set*

$$\mathcal{P}(A) = \text{set of all subsets of } A$$

furnishes an illuminating example for this, e. g.,

$$\mathcal{P}(\{1,2\}) = \{\emptyset, \{1\}, \{2\}, \{1,2\}\}.$$

Later in the course we will identify the power set of the natural numbers, $\mathcal{P}(\mathbb{N})$, with the set of real numbers $\mathbb{R}$.

Finally, we turn to the *cartesian product* defined by

$$A \times B = \{(x,y) : x \in A \text{ and } y \in B\},$$

with A and B being two sets. Here (x,y) denotes the ordered pair, where we say that $(x_1, y_1) = (x_2, y_2)$ holds if and only if $x_1 = x_2$ and $y_1 = y_2$. Also, we remark that $A \times B = \emptyset$ holds if and only if $A = \emptyset$ or $B = \emptyset$.

Furthermore, we define the *n-fold cartesian product* of the sets $A_1, \ldots, A_n$ by

$$A_1 \times \cdots \times A_n = \{(x_1, \ldots, x_n) : x_1 \in A_1 \text{ and } \cdots \text{ and } x_n \in A_n\ \}.$$

A useful example of this construction is given by $\mathbb{R}^n$ which is the n-fold cartesian product of $\mathbb{R}$ with itself.

Basic Logic. Supplementing Rudin's textbook, we briefly recall some basic notations related to mathematical logic.[2]

Symbol	**Meaning**	**Example**
$\exists$	there exists	$\exists q \in \mathbb{Q}$ such that $q^2 = 4$
$\nexists$	there does not exist	$\nexists q \in \mathbb{Q}$ such that $q^2 = 2$
$\exists!$	there exists only one	$\exists! n \in \mathbb{N}$ such that $12.1 < n < 13.1$
$\forall$	for all	$\forall x \in \mathbb{R}$ we have that $x^2 \geq 0$
$\Rightarrow$	implies that	$x > 0 \quad \Rightarrow \quad x^2 > 0$
$\Leftrightarrow$	is equivalent to	$x \neq 0 \quad \Leftrightarrow \quad 1/x \neq 0$
$\sim$, !, $\neg$	negation	X is true $\quad \Leftrightarrow \quad \neg X$ is false

We remark that the implication $X \Rightarrow Y$ can be rephrased by saying that "if X, then Y" or "when X is true, Y is true" or "Y is true only if X is true". In this case, we say that X is *sufficient* for Y, and that Y is *necessary* for X. It is a beginner's common mistake to confuse sufficient and necessary.

A ubiquitous word in the mathematical literature is *"iff"* which serves as an abbreviation for *"if and only if"*. For instance, the equivalence $X \Leftrightarrow Y$ means that "X is true iff Y is true".

[2]An easy-to-read and informal introduction to mathematical logic can be found Terence Tao's new textbook "Analysis, Volume I", Chapter 12, pp. 118; see `http://www.math.ucla.edu/~tao/preprints/books.html` where you can download sample chapters (including Chapter 12).