18.100B, Fall 2002, Homework 9

Due by Noon, Tuesday November 26
Rudin:

(1) Chapter 6, Problem 12

Proof. Suppose that $f \in \mathcal{R}(\alpha)$, let $C > 0$ be such that $|f(x)| \leq C$ for all $x \in [a, b]$. Given $\epsilon > 0$ there exists a partition P of $[a, b]$ such that

$$U(f, \alpha, P) - L(f, \alpha, P) = \sum_{i=1}^{n} (\alpha(x_i) - \alpha(x_{i-1}))(M_i - m_i) < \epsilon^2/2C \tag{1}$$

where M_i and m_i are the supremum and infimum of f over $[x_{i-1}, x_i]$. Consider the function given in the hint:

$$g(t) = \frac{x_i - t}{x_i - x_{i-1}} f(x_{i-1}) + \frac{t - x_{i-1}}{x_i - x_{i-1}} f(x_i), \ t \in [x_{i-1}, x_i]. \tag{2}$$

Note that the value at $t = x_i$ is independent of choice even if there are two intervals of which it is an end point. On $[x_{i-1}, x_i]$, g is continuous since it is linear there and it is continuous at each x_i, hence is continuous everywhere. On $[x_{i-1}, x_i]$, g takes values in $[m_i, M_i]$ since its maximum and minimum occur at the ends (it is linear) and these are values of f. Since f takes values in the same interval it follows that $f - g$ takes values in $[m_i - M_i, M_i - m_i]$. Thus

$$|f(x) - g(x)|^2 \leq |M_i - m_i|^2 \leq 2C(M_i - m_i) \text{ on } [x_{i-1}, x_i].$$

Estimating the integral on each segment of the partition we see that

$$\int |f(x) - g|^2 d\alpha \leq 2C \sum_{i \in I} (\alpha(x_i) - \alpha(x_{i-1})(M_i - m_i) < \epsilon^2$$

which implies that $\|f - g\|_2 < \epsilon$. □

(2) Chapter 6, Problem 15

Solution. By assumption f is real and continuously differentiable on $[a, b]$ hence so is $F(x) = xf^2(x)$. This has derivative $f^2(x) + 2xf(x)f'(x)$ so by the fundamental theorem of calculus

$$\int_a^b (f^2(x) + 2xf(x)f'(x))dx = F(b) - F(a) = 0$$

since $f(a) = f(b) = 0$. Thus

$$\int_a^b xf(x)f'(x)dx = -\frac{1}{2}\int_a^b f^2(x)dx = -\frac{1}{2}.$$

By Schwarz inequality

$$\frac{1}{4} = \left(\int_a^b xf(x)f'(x)dx\right)^2 \leq \int_a^b [f'(x)]^2 dx \cdot \int_a^b x^2 f^2(x)dx.$$

□

(3) Chapter 7, Problem 2

Proof. If f_n and g_n converge uniformly on a set E then they are uniformly Cauchy. Hence given $\epsilon > 0$ there exist N' and N'' such that

$$n, m > N' \Longrightarrow |f_n(x) - f_m(x)| < \epsilon/2, \; n, m > N'' \Longrightarrow |g_n(x) - g_m(x)| < \epsilon/2 \, \forall \, x \in E.$$

Taking $N = \max(N', N'')$ we see that

$$n, m > N \Longrightarrow |(f_n(x) + g_n(x)) - (f_m(x) + g_m(x))| < \epsilon \; \forall \; x \in E$$

so $f_n + g_n$ is uniformly Cauchy and hence uniformly convergent.

If both f_n and g_m are uniformly bounded, with $|f_n(x)|$, $|g_n(x)| \leq M$ for all $x \in E$ and all n then

$$|f_n(x)g_n(x) - f_m(x)g_m(x)| \leq \\ |f_n(x)g_n(x) - f_n(x)g_m(x)| + |f_n(x)g_m(x) - f_m(x)g_m(x)| \leq M\epsilon$$

if $n, m > N$ showing that $f_n g_n$ is uniformly Cauchy and hence uniformly convergent.

□

(4) Chapter 7, Problem 6

Proof. We may write the series as the sum of $\sum_n (-1)^n \frac{x^2}{n^2}$ and $\sum_n (-1) \frac{1}{n}$. The second series converges uniformly as a series of functions in x since it converges and does not depend on x. The first series converges uniformly on any bounded interval, using Theorem 7.10 and the convergence of $\sum_n \frac{1}{n^2}$. It follows that the sum of the series converges uniformly using the triangle inequality

$$|\sum_{n=p}^{m} (-1)^n \frac{x^2 + n}{n^2}| \leq |\sum_{n=p}^{m} (-1)^n \frac{x^2}{n^2}| + |\sum_{n=p}^{m} (-1)^n \frac{1}{n}|.$$

□

(5) Chapter 7, Problem 8

Proof. If $\sum_n |c_n|$ converges then for any $m \geq n$,

$$|\sum_{j=n}^{m} c_j I(x - x_j)| \leq \sum_{j=n}^{m} |c_j| \; \forall \; x \in [a, b]$$

By Theorem 7.10, it follows that the series converges uniformly on $[a, b]$. Given $\epsilon > 0$ there exists N such that

$$|\sum_{j \geq N} c_j I(y - x_j)| < \epsilon/3 \; \forall \; y \in [a, b].$$

If $x \neq x_n$ for any n then it follows that $\sum_{j<N} c_j I(y - x_j)$ is continuous at x, so there exists $\delta > 0$ such that

$$|x - y| < \delta \Longrightarrow |\sum_{j<N} c_j I(x - x_j) - \sum_{j<N} c_j I(y - x_j)| < \epsilon/3.$$

Then we see that, if $x - y| < \delta$,

$$|f(x) - f(y)| \leq |\sum_{j<N} c_j I(x - x_j) - \sum_{j<N} c_j I(y - x_j)| + |\sum_{j \geq N} c_j I(x - x_j)| + |\sum_{j \geq N} c_j I(y - x_j)| < \epsilon.$$

Thus, f is continuous at x. □