18.100B, Fall 2002, Homework 8 solutions

Was due by Noon, Tuesday November 19 Rudin:

(1) Chapter 6, Problem 5

Solution. 1. No, it is not true that a bounded function, f on $[a, b]$ with $f^2 \in \mathcal{R}(\alpha)$ is necessarily in $\mathcal{R}(\alpha)$ itself. We need a counterexample to see this. Take the function $f = 1$ at rational points and $f = -1$ at irrational points. This is not integrable by the preceeding question (the difference between upper and lower sums is always $2(b - a)$). On the other hand $f^2 = 1$,

2. If f is real-valued and bounded and $f^3 \in \mathcal{R}(\alpha)$ then $f \in \mathcal{R}(\alpha)$ as follows from Theorem 6.11 with $\phi(t) = t^{1/3}$ the unique real cube root. □

(2) Chapter 6, Problem 7

Solution. (a) If $f \in \mathcal{R}$ on $[0, 1]$ then

$$\int_c^1 f(x)dx = \int_0^1 f(x)dx - \int_0^c f(x)dx$$

and if $|f| \le M$ then $|\int_0^c f(x)dx| \le 2Mc$ so $\int_c^1 f(x)dx \longrightarrow \int_0^1 f(x)dx$ as $c \downarrow 0$. [It is enough to say that $\int_c^1 f(x)dx$ depends continuously on c by Theorem 6.20.

(b) Consider $g(x) = x^{-3/2}$, $x > 0$ and $g(0) = 0$. This is definitely not integrable since it is not bounded. Moverover the integral over $[c, 1]$ does not converge since

$$\int_c^1 x^{-3/2}dx = -2(1 - c^{-\frac{1}{2}}) \longrightarrow \infty$$

as $c \downarrow 0$. Now consider the function

$$f(x) = \begin{cases} x^{-3/2} & \frac{1}{2k} \le x < \frac{1}{2k-1} \\ -x^{-3/2} & \frac{1}{2k+1} \le x < \frac{1}{2k} \end{cases}, \ 1 \le k.$$

For any $c > 0$ this function is integrable on $[c, 1]$ since it is bounded and has only a finite number of points of discontinuity. The integral over any of the intervals $[\frac{1}{2k}, \frac{1}{2k-1}]$ is $-2((2k - 1)^{\frac{1}{2}} - (2k)^{\frac{1}{2}}$ and over $[\frac{2k+1}{,} \frac{1}{2k}]$ is $2((2k)^{\frac{1}{2}} - (2k+1)^{\frac{1}{2}})$. Both of these are bounded in absolute value by $Ck^{-\frac{1}{2}}$. Combining the two integrals shows that the integral over $[\frac{2k+1}{,} \frac{1}{2k-1}]$ is $2(2(2k)^{\frac{1}{2}} - (2k + 1)^{\frac{1}{2}} - (2k - 1)^{\frac{1}{2}}) \le Ck^{-3/2}$ (by Taylor's theorem applied to $x = 0$ for $2 - (1+x)^{\frac{1}{2}} - (1-x)^{\frac{1}{2}}$ with $x = 1/2k$. Thus if N is the largest integer such that $2N \le c$ then

$$|\int_c^1 fdx - \int_{\frac{1}{2N+1}}^1 fdx| \le CN^{-\frac{1}{2}} \to 0 \text{ as } N \to \infty$$

and

$$\int_{\frac{1}{2N+1}}^1 fdx = \sum_{k=1}^{N} \int_{\frac{1}{2N+1}}^{\frac{1}{2k-1}} fdx$$

converges by comparison to $\sum\limits_{k=1}^{\infty} k^{-3/2} < \infty$. This shows that $\int_c^1 f dx$ converges as $c \to 0$.

Note that if f is bounded and integrable on $[c, 1]$ for every $c > 0$ then it is integrable on $[0, 1]$, so you cannot do this with a bounded function. □

(3) Chapter 6, Problem 10, (a),(b) and (c).

Proof. (a) If $u = 0$ or $v = 0$ this is obvious so we can assume that both are positive. Since p and q are both positive and $p = \frac{q}{q-1}$ both of them must lie in the interval $1 < p < \infty$. Now divide through the inequality we want by v^q and set $a = u^p/v^q$. It follows that $uv^{1-q} = a^{1/p}$ since $q/p = q - 1$. Thus we only need to show that

$$a^{1/p} \leq \frac{1}{p}a + \frac{1}{q},\ 0 < a < \infty. \tag{1}$$

The continuous function $\frac{1}{p}a + \frac{1}{q} - a^{1/p}$ is positive at 0 and tends to ∞ as $a \to \infty$. Thus if it has an interior minimum in $(0, \infty)$ it will have to be at a point where the derivative vanishes, namely $\frac{1}{p} = \frac{1}{p}a^{1/p-1}$ which is to say $a = 1$. Since it takes the value 0 there it is in fact non-negative, meaning (1) holds. This proves the inequality

$$uv \leq \frac{u^p}{p} + \frac{v^q}{q}$$

with equality only where $a = 1$, which is $u^p = v^q$ (including the case where both are zero).

(b) If $0 \leq f \in \mathcal{R}(\alpha)$ and $0 \leq g \in \mathcal{R}(\alpha)$ then f^p and $g^q \in \mathcal{R}(\alpha)$ by Theorem 6.11. It also follows that $fg \in \mathcal{R}(\alpha)$ and, using (†)

$$\int_a^b fg d\alpha \leq \frac{1}{p}\int_a^b f^p d\alpha + \frac{1}{q}\int_a^b g^q d\alpha = 1.$$

(c) If f and g are complex-valued in $\mathcal{R}(\alpha)$ then $|f|$ and $|g|$ are non-negative elements of $\mathcal{R}(\alpha)$ and $fg \in \mathcal{R}(\alpha)$. Moverover

$$|\int_a^b fg d\alpha| \leq \int_a^b |f||g| d\alpha.$$

If $I = \int_a^b |f|^p \neq 0$ and $J = \int_a^b |g|^q \neq 0$ then apply the conclusion of the previous part to $|f|/c$ and $|g|/d$ where $c^p = I$ and $d^q = J$. This gives the desired result

$$|\int_a^b fg d\alpha| \leq cd = \left(|\int_a^b |f|^p d\alpha|\right)^{1/p} \left(|\int_a^b |g|^q d\alpha|\right)^{1/q}.$$

On the other hand if one of these intgrals vanishes, say the first since we can always reverse the roles of p and q, then

$$\int_a^b |f|(c|g|) d\alpha \leq c^q \frac{1}{q}\int_a^b |g|^q d\alpha$$

for any $c > 0$ and sending $c \to 0$ shows that $\int_a^b |f||g| d\alpha = 0$ so the inequality still holds. □

(4) Chapter 6, Problem 11

Setting $p = q = 2$ in the previous problem we see that

$$(\int_a^b |uv| d\alpha)^2 \leq \int_a^b |u|^2 d\alpha \int_a^b |v|^2 d\alpha.$$

Now multiply out

$$\int_a^b |u+v|^2 d\alpha = \int_a^b |u|^2 d\alpha + \int_a^b (\bar{u}v + u\bar{v}) d\alpha$$
$$+ \int_a^b |v|^2 d\alpha \leq (\left(\int_a^b |u|^2 d\alpha\right)^{\frac{1}{2}} + \left(\int_a^b |v|^2 d\alpha\right)^{\frac{1}{2}})^2.$$

This means $\|u+v\|_2 \leq \|u\|_2 + \|v\|_2$. Now setting $u = f - g$ and $v = h - g$ gives the general case.