18.100B, Fall 2002, Homework 5

Due by Noon, Tuesday October 8. Rudin:

(1) Chapter 3, Problem 1

Solution: The sequence is supposed to be in $\mathbb{R}^n$. We use the triangle inequality in the form $|b| = |b - a + a| \leq |a - b| + |a|$ which implies that $|b| - |a| \leq |a - b|$. Reversing the roles of a and b we also see that $|a| - |b| \leq |a - b|$ and so $||a| - |b|| \leq |a - b|$.

If $\{s_n\}$ converges to s then given $\epsilon > 0$ there exists N such that $n > N$ implies $|s_n - s| < \epsilon$. By the triangle inequality $||s_n| - |s|| \leq |s_n - s|$ so $\{|s_n|\}$ converges to $|s|$.

(2) Chapter 3, Problem 20

Solution: Let $\{p_n\}$ be a Cauchy sequence in a metric space X. By assumption, some subsequence $\{p_{n(k)}\}$ converges to $p \in X$. Thus, given $\epsilon > 0$ there exits K such that $k > K$ implies that $d(p, p_{n(k)} < \epsilon/2$ for all $k > K$. By the Cauchy condition, given $\epsilon > 0$ there exists M such that $n, m > M$ implies $d(x_n, x_m) < \epsilon/2$. Now, consider $N = n(l)$ for some $l \geq K$ such that $n(l) > M$, which exists since $n(k) \to \infty$ with k. For this choice,

$$n > N \Longrightarrow d(p_n, p) \leq d(p_n, p_{n(l)}) + d(p_{n(l)}, p) < \epsilon$$

shows that $\{p_n\}$ converges to p.

(3) Chapter 2, Problem 21

Note that the problem should say that $\{E_n\}$ is a sequence of closed, bounded and *non-empty* sets in a complete metric space with $E_n \supset E_{n+1}$ and if $\lim_{n\to\infty} \operatorname{diam}(E_n) = 0$, where $\operatorname{diam}(E) = \left(\sup_{p,q\in E} d(p,q)\right)$, then $\bigcap_{n=1}^{\infty} E_n$ consists of exactly one point.

Solution: If $p, q \in \bigcap_n E_n$ then $p, q \in E_n$ for all n, so $d(p,q) \leq \operatorname{diam}(E_n) \to 0$ with n, so $d(p,q) = 0$ and there can be at most one point in the intersection. So, suppose $\{p_n\}$ is any sequence with $p_n \in E_n$. By the convergence of $\operatorname{diam}(E_n)$ to 0, given $\epsilon > 0$ theren exists N such that $n > N$ implies $\operatorname{diam}(E_n) < \epsilon$ for all $n > N$. Since $E_m \subset E_N$ if $m \geq N$ it follows that $d(p_n, p_m) \leq \operatorname{diam}(E_N) < \epsilon$ if $n, m > N$ and hence the sequence is Cauchy. The assumption that X is complete implies that this sequence converges to a limit p. Since $p_n \in E_N$ of $n > N$ and each E_N is closed, $p \in E_N$ for all N and hence $p \in \bigcap_n E_n$ which therefore consists of exactly one point.

(4) Chapter 2, Problem 22.

Solution: Let $\{G_n\}$ be a sequence of dense open subsets of a complete metric space X. We can assume that $X \neq \emptyset$ otherwise the question is trivial. We construct a sequence of open balls $E_k = B(p_k, \epsilon_k) \subset G_k$, $\epsilon_k > 0$ with $B(p_k, 2\epsilon_k) \subset G_k \cap E_{k-1}$ for all $k > 1$. Choose $\epsilon_1 > 0$ and a point $p_1 \in G_1$ such that $E_1 = B(p_1, 2\epsilon_1) \subset G_1$; this is possible since $E_1 \neq \emptyset$ is open. From the density of G_2 in X, p_1 is a limit point of G_2, so there exists $p_2 \in E_1 \cap G_2$ and hence $\epsilon_2 > 0$ such that $B(p_2, \epsilon_2) \subset E_1 \cap G_2$. Now, proceed in this way, supposing we have chosen p_l and $\epsilon_l > 0$ for $l = 1, \dots, k-1$ such that with $E_l = B(p_l, \epsilon_l)$ we have $B(p_l, 2\epsilon_l) \subset E_{l-1} \cap G_l$ for each $l = 2, \dots, k-1$. Then, from the density of G_k in X we can choose $p_k \in E_{k-1} \cap G_k$ such that $B(p_k, 2\epsilon_k) \subset E_{k-1} \cap G_k$. The closed set $\{p; d(p, p_l\} \leq \epsilon_l\}$ satisfies the conditions of Problem 21; they are non-empty, and decreasing, in fact $B(p_k, 2\epsilon_k) \subset B(p_{k-1}, \epsilon_{k-1})$ implies $2\epsilon_k \leq \epsilon_{k-1}$ so $\operatorname{diam}(E_k) \to 0$ as $k \to \infty$.

Thus there is a point in $\bigcap_k E_k$, and hence in $\bigcap_k G_k$. In fact we could do this with the center of the first ball arbitrarily close to a given point $p \in X$, and with $\epsilon_1 > 0$ arbitrarily small, so it follows that $\bigcap_k G_k$ is dense (of course it need not be open).

This is Baire's theorem, the intersection of a countable set of open dense subsets of a complete metric space is dense.