18.100B, Fall 2002, Homework 4, Solutions

Was due by Noon, Tuesday October 1. Rudin:

(1) Chapter 2, Problem 22

Let $\mathbb{Q}^k \subset \mathbb{R}^k$ be the subset of points with rational coefficients. This is countable, as the Cartesian product of a finite number of countable sets. Suppose that $x = (x_1, \dots, x_k) \in \mathbb{R}^k$. By the density of the rationals in the real numbers, given $\epsilon > 0$ there exists $y_i \in \mathbb{Q}^k$ such that $|x_i - y_i| < \epsilon/k$, $i = 1, \dots, k$. Thus if $y = (y_1, y_2, \dots, y_k)$ then

$$|x - y| \le \sqrt{k} \max_{i=1}^k |x_i - y_i| < \epsilon$$

shows the density of $\mathbb{Q}^k$ in $\mathbb{R}^k$. Thus $\mathbb{R}^k$ is separable.

(2) Chapter 2, Problem 23

Given a separable metric space X, let $Y \subset X$ be a countable dense subset. The product $A = Y \times \mathbb{Q}$ is countable. Let $\{U_a\}$, $a \in A$, be the collection of open balls with center from Y and rational radius. If $V \subset X$ is open then for each point $p \in V$ there exists $r > 0$ such that $B(p, r) \subset V$. By the density of Q in X there exists $y \in Q$ such that $p \in B(y, r/2)$. Moreover there exists $q \in \mathbb{Q}$ with $r/2 < q < r$. Then $x \in B(y, q)$. Thus each point of V is in an element of one of the U_a's which is contained in V, so

$$V = \bigcup_{U_a \subset V} U_a.$$

It follows that the $\{U_a\}_{a \in A}$ form a base of X (actually now more usually called an *open basis*).

(3) Chapter 2, Problem 24

By assumption X is a metric space in which every infinite set has a limit point.

For each positive integer n choose points $x_1(n)$, $x_2(n)$, ... successively with the property that $d(x_j(n), x_k(n)) \ge 1/n$ for $k < j$. After a finite number of steps no futher choice is possible. Indeed, if there were an infinite set of points E satisfying $d(x, x') \ge 1/n$ for all $x \ne x'$ in E then E could have no limit point – since a limit point $q \in X$ would have to satify $d(q, p_i) < 1/2n$ for an infinite number of (different) $p_i \in E$ and this would imply that $d(p_1, p_2) \le d(p_1, q) + d(q, p_2) < 1/n$ which is a contradiction. Let $Y \subset X$ be the countable subset, as a countable union of finite sets, consisting of all the $x_j(n)$, for all n. Then Y is dense in X. To see this, given $p \in X$ and $\epsilon > 0$ choose $n > 1/\epsilon$. If $p = x_j(n)$ for some j then it is in Y. If not then for some j, $d(p, x_j(n)) < 1/n$, otherwise it would be possible to choose another $x_j(n)$ contradicting the fact that we have chosen as many as possible. Then $d(p, q) < \epsilon$ for some $q \in Y$ which is therefore dense and X is therefore separable.

(4) Chapter 2, Problem 26.

By assumption, X is a metric space in which every infinite subset has a limit point. By the problems above it is separable, and hence has a countable open basis, $\{U_i\}$. Let $\{V_a\}_{a \in A}$ be an arbitrary open cover of X. Each V_a is a union of U_j's by the definition of an open basis. For each j such that U_j is in one of these unions, choose a V_{a_j} which contains it. Then for every $b \in A$, V_b must be contained in a union of the U_{a_j}'s, hence in the union of the V_{a_j}'s which therefore form a countable subcover of the original open cover V_a. Consider the successive open sets

$$\bigcup_{i=1}^{N} V_{a_i}.$$

If one of these contains X then we have found a finite subcover of the V_a's. So, suppose to the contrary that

$$F_N = X \setminus \bigcup_{i=1}^{N} V_{a_i} \neq \emptyset \ \forall \ N.$$

The F_N's are decreasing as N increases. Let $E \subset X$ be a set which contains one point from each F_N. It must be an infinite set, since otherwise some fixed point would be in F_N for arbitrary large, hence all, N but

$$\bigcap_{N \in \mathbb{N}} F_N = \emptyset \tag{1}$$

since together all the V_{a_i} do cover X. By the assumed property of X, E must have a limit point p. For each N, all but finitely many points of E lie in F_N, so p must be a limit point of F_N for all N, but each F_N is closed so this would mean $p \in F_N$ for all N, contradicting (1).