

From Rudin, Chapter 1.

Exercise 1 If  $s$  and  $r \neq 0$  are rational then so are  $s + r$ ,  $-r$ ,  $1/r$  and  $sr$  (since the rationals form a field). So if  $r$  is rational and  $x$  is real, then  $x + r$  rational implies  $(x + r) - r = x$  is rational. An irrational number is just a non-rational real number, so conversely if  $x$  is irrational then  $x + t$  must be irrational. Similarly if  $rx$  is rational then so is  $(rx)/r = x$ ; thus if  $x$  is irrational then so is  $rx$ .

Exercise 3 [(a)] If  $x \neq 0$  then  $x^{-1}$  exists and if  $xy = xz$  then

$$y = (x^{-1}x)y = x^{-1}(xy) = x^{-1}(xz) = (x^{-1}x)z = z$$

using first (M5) then (M2), (M3), the given condition, (M3) and (M5).

[(b)] Is (a) with  $z = 1$ .

[(c)] Multiply by  $x^{-1}$  so  $x^{-1} = x^{-1}(xy) = (x^{-1}x)y = 1y = y$  using associativity and definition of inverse.

[(d)] The identity for  $x^{-1} = 1/x$ ,  $x \cdot x^{-1}$  gives by commutativity  $x^{-1} \cdot x = 1$  which means  $1/(1/x) = x$  by the uniqueness of inverses.

Exercise 5 If  $A$  is a set of real numbers which is bounded below then  $\inf A$  is by definition a lower bound, i.e.  $\inf A \leq a$  for all  $a \in A$  and if  $\inf A \geq b$  for any other lower bound  $b$ . We already know that if it exists it is unique. Now if  $A$  is bounded below then

$$(1) \quad -A = \{-x; x \in A\}$$

is bounded above. Indeed if  $b \leq x$  for all  $x \in A$  then  $-b \geq -x$  for all  $x \in A$  which means  $-b \geq y$  for all  $y \in -A$ . Now, if  $\sup(-A)$  is the least upper bound of  $-A$  it follows that  $-\sup(-A)$  is a lower bound for  $A$  since

$$x \in A \implies -x \in -A \implies \sup(-A) \geq -x \implies -\sup(-A) \leq x.$$

As noted above, if  $b$  is any lower bound for  $A$  then  $-b$  is an upper bound for  $-A$  so  $-b \geq \sup(-A)$  and  $b \leq -\sup(-A)$ . This is the definition of  $\inf A$  so

$$\inf A = -\sup(-A).$$