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Optimal Parametric Auctions
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Abstract

We study the problem of an auctioneer who wants to maximize her profits. In our model, there are n buyers with private valuations drawn from independent distributions F_1, \dots, F_n . When these distributions are known to the seller, Myerson's optimal auction [21] is a well known mechanism that maximizes revenue. However, in many cases it is too strong to assume that the seller knows these distributions.

We propose an alternative model where the seller only knows the mean μ_i and variance σ_i^2 of each distribution F_i . We call mechanisms that only use this information *parametric auctions*. We construct such auctions for all single-dimensional downward closed environments. For a very large class of distributions, including (but not limited to) distributions with a monotone hazard rate, our auctions achieve a constant fraction of the revenue of Myerson's auction.

When the seller has absolutely no knowledge about the distributions, it is well known that no auction can achieve a constant fraction of the optimal revenue when the players are not identically distributed. Our parametric model gives the seller a small amount of extra information, allowing her to construct auctions for which (1) she does not know the full distribution of valuations, (2) no two bidders need to be drawn from identical distributions and (3) the revenue obtained is a constant fraction of the revenue in Myerson's optimal auction.

For digital goods environments we present a different parametric auction that not only gives a better approximation to the optimal auction, but that is also optimal in a new sense, which we call *maximin optimality*. Informally, an auction is maximin optimal if it maximizes revenue in the worst case over an adversary's choice of the distribution. We show that our digital parametric is maximin optimal among the class of posted price mechanisms.

1 Introduction

We study the problem of selling a good in an auction, when the seller has limited information about what buyers are willing to pay. Specifically, we consider the problem of selling a good to n buyers, where each buyer is interested in purchasing one unit of the good. Each player i has a value v_i for the good, drawn from a distribution F_i . Starting with the work of Myerson [21], one traditionally assumes that the seller knows these distributions F_1, \dots, F_n . This assumption allows the seller to design an auction that maximizes her revenue.

Parametric Auctions. The assumption that the seller has full knowledge of the distributions F_1, \dots, F_n is very strong. We use a strictly weaker assumption, namely

the seller only knows the mean μ_i and standard deviation σ_i of each F_i .

We call auctions where the seller only needs to know these parameters of the distribution *parametric auctions*. We construct such an auction for a very general setting: single-dimensional downward closed environments. We show that our auction obtains a significant fraction of the revenue when (1) the distributions F_1, \dots, F_n

have a monotone hazard rate, (2) we are in a matroid environment and the distributions are regular and, (3) we are in a digital goods environment and the distribution is *arbitrary*.

Our Benchmark. Letting \mathbf{F} be the distribution from which valuations are drawn, our benchmark is the revenue obtained by the optimal auction on \mathbf{F} . We denote this benchmark by $Rev(OPT, \mathbf{F})$. More generally, for any auction A and any distribution \mathbf{F} we denote by $Rev(A, \mathbf{F})$ the expected revenue that A obtains when valuations are drawn from \mathbf{F} , and by $Rev_i(A, \mathbf{F})$ the expected revenue that auction A obtains from player i . Thus, $Rev(A, \mathbf{F}) = \sum_{i=1}^n Rev_i(A, \mathbf{F})$. For every auction A that we construct, we will seek to give a lower bound on the *competitive ratio*

$$\frac{Rev(A, \mathbf{F})}{Rev(OPT, \mathbf{F})}.$$

The idea of evaluating auctions by analyzing their competitive ratio was introduced by [17], and has been used widely in the literature. An incomplete list of applications includes prior-free auctions [15, 16], prior-independent auctions [11], simple¹ auctions [19], auctions with correlated bidders (where the optimal auction is not known) [25, 12, 22] and auctions where bidders have multi-dimensional types [6].

1.1 Performance relative to the Optimal Auction

Our Results for Digital Goods with Monotone Hazard Rate Distributions. We construct a parametric auction \mathcal{A} for the digital goods setting. When each distribution F_i has a monotone hazard rate, we can show that

$$\frac{Rev(\mathcal{A}, \mathbf{F})}{Rev(OPT, \mathbf{F})} \geq \frac{1}{e} > 36\%.$$

We remark that our results hold no matter how many players there are, and what asymmetries may exist among them. (Notice that the best prior-free auctions can only obtain a constant fraction of Myerson's revenue when players are identically and independently distributed.)

Our Results for Digital Goods with Arbitrary Distributions. Our auction \mathcal{A} guarantees good revenue only when each distribution F_i has a monotone hazard rate. For arbitrary independent distributions, we construct a different digital auction \mathbb{A} . The competitive ratio of \mathbb{A} will be constant in the number of players and will not depend on the asymmetries of the players. However, it will be a function of the ratios $\frac{\mu_i}{\sigma_i}$. This is intuitive because if σ_i is very large compared with μ_i , then the seller has very little information about F_i , and one cannot expect her to collect significant revenue from player i .

More concretely, we can give the following bound on the expected revenue that our auction \mathbb{A} obtains from each player i

$$\frac{Rev_i(\mathbb{A}, \mathbf{F})}{Rev_i(OPT, \mathbf{F})} \geq \rho\left(\frac{\mu_i}{\sigma_i}\right),$$

where $\rho(\cdot)$ is a function of $\frac{\mu_i}{\sigma_i}$ that we define in section 4. We remark that

- $\rho\left(\frac{\mu_i}{\sigma_i}\right)$ is always positive.
- When each F_i has a monotone increasing hazard rate, $\rho\left(\frac{\mu_i}{\sigma_i}\right) > 10.5\%$.
- More generally, for a very wide class of distributions \mathcal{F}_c , the function $\rho\left(\frac{\mu_i}{\sigma_i}\right)$ is bounded below by a constant. Let c be a constant and define $\mathcal{F}_c = \{F : \mathbb{E}[F] = \mu, Var(F) = \sigma^2, \frac{\mu}{\sigma} > c > 0\}$. We call distributions in this class *c-informative*. When F_i is *c-informative*, we have $\rho\left(\frac{\mu_i}{\sigma_i}\right) > \rho(c) > 0$. We remark that the class \mathcal{F}_c is very general. Any distribution F over a bounded domain $[0, B]$ with mean $\mu > cB$ is in \mathcal{F}_c . Furthermore, any distribution with a monotone hazard rate belongs to the class \mathcal{F}_1 .

¹The authors study, among others, Vickrey auctions with reserve prices, which are much simpler than the Myerson optimal auction when distributions are not identical.

This last observation implies that for all F_1, \dots, F_n belonging to the class \mathcal{F}_c , our auction \mathbb{A} has a competitive ratio

$$\frac{Rev(\mathbb{A}, \mathbf{F})}{Rev(OPT, \mathbf{F})} > \rho(c).$$

Our results for Downward Closed Environments with Monotone Hazard Rate Distributions.

We construct a parametric auction \mathcal{B} for downward closed single-dimensional settings. When each distribution F_i has a monotone hazard rate, our auction \mathcal{B} obtains a competitive ratio

$$\frac{Rev(\mathcal{B}, \mathbf{F})}{Rev(OPT, \mathbf{F})} \geq 1.23\%.$$

Our results for Matroid Settings with Regular Distributions. More generally, when the distributions F_1, \dots, F_n are regular², belong to the class \mathcal{F}_c for some constant c , and we are in a matroid environment, our auction \mathcal{B} has a competitive ratio of

$$\frac{Rev(\mathcal{B}, \mathbf{F})}{Rev(OPT, \mathbf{F})} \geq \frac{1}{2} \cdot \psi(c) > 0$$

where $\psi(x)$ is a positive increasing function defined in section 7.

We remark that, when the distributions F_1, \dots, F_n are unknown to the seller, most of the known auctions with a constant competitive ratio are digital goods auctions. There are few competitive auctions for general downward closed settings that we are aware of, and all of them assume some similarity between bidders.³ In contrast, our auction \mathcal{B} obtains a constant competitive ratio even when all buyers have distinct distributions.

1.2 Maximin Optimality and our digital auction \mathbb{A}

Our digital auction \mathbb{A} is competitive with the optimal auction, where the competitive ratio depends on $\frac{\mu_i}{\sigma_i}$. Examining the competitive ratio is a meaningful way to give revenue guarantees when the full distribution is not known. However, there can be a multiplicity of different auctions which achieve a constant competitive ratio, and it may be difficult to decide which one is the “best” among all parametric auctions.

We restrict ourselves to a simple class of digital auctions, called *posted price mechanisms*. In these mechanisms, each player i is given a take-it-or-leave-it price that does not depend on the other players’ bids. We will show that our auction \mathbb{A} is one such mechanism.

Ideally, the “best” posted price mechanism A^* should satisfy $Rev(A^*, \mathbf{F}) \geq Rev(A, \mathbf{F})$ for all distributions \mathbf{F} and all other posted price mechanisms A . Indeed, if the seller knows the distribution \mathbf{F} , an optimal posted price mechanism exists: it is Myerson’s optimal digital auction. Unfortunately, it is unlikely that such a mechanism exists in the parametric case.⁴ Any definition of optimality for parametric auctions needs to take into account the uncertainty that the seller has over the distribution \mathbf{F} . Since this is a *worst-case uncertainty*, we give a new definition of optimality for parametric auctions which is based on maximizing the worst case revenue.

Definition 1. Let \mathcal{C} be a class of mechanisms. A parametric mechanism A^* is maximin optimal for the class \mathcal{C} if $A^* \in \mathcal{C}$ and for all vectors $\boldsymbol{\mu}, \boldsymbol{\sigma}$ we have

$$A^* = \operatorname{argmax}_{A \in \mathcal{C}} \min_{\mathbf{F}: \mathbb{E}[\mathbf{F}] = \boldsymbol{\mu}, \operatorname{Var}(\mathbf{F}) = \boldsymbol{\sigma}^2} Rev(A, \mathbf{F}).$$

²A distribution is regular if its associated virtual valuation function is increasing.

³For example, Dhangwatnotai, Roughgarden, and Yan [11] assume that for any bidder i with a valuation v_i drawn from a distribution F , there exist another bidder j whose valuation v_j is also drawn from the same distribution F . See the related work section for more details.

⁴Informally, consider any price vector (p_1, \dots, p_n) and a posted price mechanism A that uses p_1, \dots, p_n as reserve prices. If the p_1, \dots, p_n are within a certain range (that depends on the given means and standard deviations), there will exist a distribution \mathbf{F} on which auction A is optimal. However, for a different vector (p'_1, \dots, p'_n) in the same range, there will exist a different distribution \mathbf{F}' on which a different auction A' will be optimal.

We highlight that, A^* is the unique mechanism in \mathcal{C} maximizing worst-case revenue.

We show in section 6 that our digital auction \mathbb{A} is maximin optimal for the class of posted price mechanisms.

1.3 Our Techniques

We make extensive use of Chebyshev-type inequalities. These bounds are frequently used in robust optimization, but to the best of our knowledge they have not been applied to auction theory or mechanism design before. While we focus on matroid and downward closed environments in this paper, we believe that these techniques can be more widely applied in other areas of mechanism design.

2 Preliminaries

Bayesian Valuations. We assume that valuations are drawn from a probability distribution. The i^{th} buyer's valuation is a random variable V_i over some domain $D_i \subset \mathbb{R}_+$. V_i 's cumulative distribution function is $F_i : D_i \rightarrow [0, 1]$, where $F_i(x) = \Pr[V_i \leq x]$. We denote by $\mathbf{V} = (V_1, \dots, V_n)$ the vector of valuations, by $D = D_1 \times \dots \times D_n$ its domain, and by $\mathbf{F} = F_1 \times \dots \times F_n$ its joint distribution function. We denote by $\mathbf{v} = (v_1, \dots, v_n) \in D$ the vector of realized valuations. When we want to emphasize player i 's valuation, we write the vector $\mathbf{v} = (v_1, \dots, v_n)$ as $\mathbf{v} = (v_i, v_{-i})$.

Auctions An auction is given by a pair (A, P) where $A : D \times \Delta D \rightarrow [0, 1]^n$ is an allocation rule and $P : D \times \Delta D \rightarrow \mathbb{R}_+^n$ is a payment rule. If the auctioneer faces a bid vector $\mathbf{v} = (v_1, \dots, v_n)$, then he sells to player i with probability $A_i(\mathbf{v})$, and charges her a price $P_i(\mathbf{v})$ when the item is sold. Each player can only be sold one copy of the good.

We emphasize that $A(\mathbf{v}, \mathbf{F})$ not only depends on the valuations v_1, \dots, v_n but also on the distribution \mathbf{F} , but will sometimes write $A_i(\mathbf{v}, \mathbf{F}) = A_i(\mathbf{v})$ when \mathbf{F} is clear from context. Furthermore, we will often take v_{-i} as fixed and write $A_i(\mathbf{v}, \mathbf{F}) = A_i(v_i)$.

Downward Closed, Matroid and Digital Good Environments. We study single dimensional environments, where each domain D_i is a subset of \mathbb{R} . An *environment* is a collection of sets \mathcal{I} , where each $S \in \mathcal{I}$ is a subset of the set of players $\{1, \dots, n\}$. In a *downward closed environment*, the collection \mathcal{I} must satisfy the downward closure condition: if $S \in \mathcal{I}$ and $R \subset S$, then R must also be in the collection \mathcal{I} . In a *matroid environment*, the collection \mathcal{I} must be a matroid (see [24] for a definition). In a *digital goods environment*, \mathcal{I} is the collection of all subsets of $\{1, \dots, n\}$. An auction for an environment \mathcal{I} can only allocate goods to a set $S \in \mathcal{I}$.

Truthfulness and Monotonicity If player i obtains the good with probability A_i and pays a price P_i , her utility is $v_i \cdot A_i - P_i$. A buyer with valuation v_i can attempt to increase her utility by lying, and reporting a bid $v'_i \neq v_i$. An auction is *truthful* if players have no incentive to misreport their true valuation. That is, for every player i , for every valuation vector \mathbf{v} , and every $v'_i \neq v_i$, we have $v_i \cdot A_i(\mathbf{v}) - P_i(\mathbf{v}) \geq v_i \cdot A_i(\mathbf{v}') - P_i(\mathbf{v}')$, where $\mathbf{v}' = (v'_i, v_{-i})$. It is well known [21, 1] that the auction (A, P) is truthful if and only if $A_i(v_i, v_{-i})$ is monotonic in v_i and $P_i(v_i, v_{-i}) = A_i(v_i, v_{-i})v_i - \int_0^{v_i} A_i(z_i; v_{-i})dz_i$. An important corollary is that, for any monotonic allocation rule $A(\cdot)$, there exists a unique payment rule $P(\cdot)$ that makes the auction (A, P) truthful. Thus, it suffices to specify a monotonic allocation rule $A(\cdot)$ to specify a truthful auction.

Deterministic Auctions. The value $A_i(\mathbf{v})$ is the probability that player i obtains the good given that the bid vector is \mathbf{v} . We focus on deterministic allocations, where $A_i(\mathbf{v}) \in \{0, 1\}$. If an allocation $A_i(\mathbf{v})$ is deterministic and truthful, then monotonicity implies that, for every v_{-i} , there exists a reserve price $p^*(v_{-i})$ such that the auction sells to player i when $v_i > p^*(v_{-i})$. The payment that makes this allocation truthful is charging player i a price of $p^*(v_{-i})$ dollars if she wins.

Monotone Hazard Rate and Regularity Given a differentiable cumulative distribution function F_i , let $f_i(v) = \frac{d}{dv} F_i(v)$ be its induced density function. The function $h_i(v) = \frac{f_i(v)}{1-F_i(v)}$ is called the *hazard rate* of F_i . The distribution F_i has a *monotone hazard rate* if $h_i(v)$ is increasing. The distribution F_i is called *regular* if the virtual valuation function $\phi_i(v) = v - \frac{1}{h_i(v)}$ is increasing. An immediate consequence is that any distribution with a monotone hazard rate is regular.

Revenue We denote by $Rev((A, P), \mathbf{F}) = \int_D \sum_{i=1}^n A_i(\mathbf{v}) P_i(\mathbf{v}) d\mathbf{F}(\mathbf{v})$ the expected revenue obtained by an auction (A, P) when valuations follow distribution F , and by $Rev_i((A, P), \mathbf{F}) = \int_D A_i(\mathbf{v}) P_i(\mathbf{v}) d\mathbf{F}(\mathbf{v})$ the expected revenue obtained from player i .

Distribution Parameters. Player i 's valuation is a random variable V_i with mean $E[V_i] = \mu_i, E[(V_i - \mu_i)^2] = \sigma_i^2$. We will write $\boldsymbol{\mu} = (\mu_1, \dots, \mu_n)$ and $\boldsymbol{\sigma} = (\sigma_1, \dots, \sigma_n)$.

Parametric Auctions Informally, an auction (A, P) is *parametric* if its allocation and payment functions can be computed from the valuation vector \mathbf{v} and the $\boldsymbol{\mu}, \boldsymbol{\sigma}$ parameters. More formally, we have the following definition.

Definition 2. A *parametric auction* is a pair of functions (A, P)

1. $A : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow [0, 1]^n$
2. $P : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$
3. $A_i(\mathbf{v}, \boldsymbol{\mu}, \boldsymbol{\sigma})$ is the probability that player i wins a copy of the good when bids are \mathbf{v} and the mean and standard deviation vectors of the distribution are $\boldsymbol{\mu}, \boldsymbol{\sigma}$.
4. $P_i(\mathbf{v}, \boldsymbol{\mu}, \boldsymbol{\sigma})$ is the price that player i has to pay when bids are \mathbf{v} and the mean and standard deviation vectors of the distribution are $\boldsymbol{\mu}, \boldsymbol{\sigma}$.

For convenience of notation, whenever $\boldsymbol{\mu}$ and $\boldsymbol{\sigma}$ are clear we will write $A(\mathbf{v})$ and $P(\mathbf{v})$.

Posted Price Mechanisms A posted price mechanism for digital goods is a digital auction where player i is offered a take-it-or-leave-it price $p_i(\mathbf{F})$ that does not depend on the players' bids. Player i gets a copy of the good if and only if $v_i > P_i(\mathbf{F})$. The optimal digital auction when \mathbf{F} is known is a posted price mechanism. Player i is given a price $p_i^* = \operatorname{argmax}_{p_i} p_i \cdot (1 - F_i(p_i))$. All of the parametric digital auctions that we construct are posted price mechanisms, where P_i only depends on the parameters μ_i, σ_i of player i 's distribution.

3 Further Related Work

3.1 Detail-Free Mechanisms

The Wilson doctrine [28] states that a good mechanism should require as little knowledge about the valuations of the players as possible. Our paper follows the spirit of the Wilson doctrine, removing the assumption that the seller knows the distribution of buyer valuations and replacing it by the strictly weaker assumption that the seller knows only the first and second moments of these distributions.

Baliga and Vohra [2] and Segal [26] have proposed explicit detail-free bayesian auctions where the seller does not need to know the distribution of valuations. However, these auctions are competitive with the optimal auction only when the buyers' valuations are identically and independently distributed from the same (unknown) distribution F .

Prior-Free auctions. Goldberg, Hartline, Karlin, Saks and Wright [17] consider auctions where the valuations are not necessarily drawn from a distribution. To measure the performance of the auctions, they introduce a revenue benchmark $\mathcal{F}_2(v_1, \dots, v_n)$,⁵ and construct auctions that always achieve a constant fraction of this benchmark, for all valuation vectors. As made explicit by Hartline and Roughgarden [18], any auction that achieves a constant fraction of \mathcal{F}_2 on all valuation vectors, will also achieve a constant fraction of the optimal auction’s revenue, as long as valuations are identically and independently distributed.

Our results are incomparable to the existing prior-free auctions. We assume that the valuations are drawn from a distribution, and that the seller has some limited information about this distribution. However, all the previous prior-free auctions are competitive with the optimal auction only when the valuations are independently and identically distributed. In contrast, our results apply in the more general setting where buyers’ valuations can be drawn from distinct distributions.

Prior Independent Auctions An auction is prior-independent if the distributions F_1, \dots, F_n are assumed to exist, but the seller does not know what they are. In this setting, Dhangwatnotai, Roughgarden and Yan [11] construct an auction that has a constant competitive ratio, as long as bidders satisfy a symmetry condition which they call non-singularity. Their analysis requires that, for every bidder i , there exist another bidder j whose valuation is drawn from the same distribution as i ’s valuation. Our result is again, incomparable to theirs. By assuming that the seller knows the mean and variance of each F_i , we can give approximately optimal auctions even when the bidders are *arbitrarily asymmetric*. That is, our auctions do not require any two distributions to be identical in order to guarantee a constant fraction of Myerson’s revenue.

We are aware that the mechanisms in [11] can be transformed into competitive auctions with asymmetric bidders where the auctioneer gets one sample from each distribution F_1, \dots, F_n . These samples are “extra information”, in the same way that the mean and variance of the distributions are extra information. We remark that our assumptions are incomparable: the seller may know μ_i and σ_i without having access to samples from the distribution F_i , and knowing one sample does not give a good estimate of the mean or variance.

3.2 Empirical Estimation of Auctions

One of the main motivations of our work is the application of optimal auctions in practice. This requires estimating the distribution of bidder values. Ostrovsky and Schwarz [23] study the effect of reserve prices on ad auctions. They derive their reserve prices by estimating means and standard deviations of bidder values using previous auction data, and then assuming that valuations are drawn from a log-normal distribution with the estimated mean and standard deviation.

Our work proposes a more conservative way to set reserve prices. Instead of finding these prices by assuming that valuations are drawn from a specific distribution family (in the above case, a log-normal distribution), our auction sets the lowest reserve price that is compatible with any distribution with mean μ and standard deviation σ . We are very interested in doing an empirical analysis of auctions using this more conservative way of setting reserve prices.

3.3 Simultaneous Welfare and Revenue

Daskalakis and Pierrakos [9] study the VCG auction with arbitrary reserve prices, and show that there is a way to set reserve prices in order to simultaneously approximate optimal social welfare and revenue. We use these auctions for matroid and downward closed environments, and our analysis relies on a reduction to auctions with only one player, a technique that was used in [9]. We highlight two differences with this paper. First, the auction in [9] uses knowledge of the distribution, while ours uses only knowledge of the parameters μ, σ . Second, their auction guarantees a constant fraction of the optimal revenue for all regular distributions,

⁵Do not confuse the revenue benchmark \mathcal{F}_2 with our distribution classes \mathcal{F}_c . In our results, \mathcal{F}_c will always refer to a class of distributions.

while ours guarantees a constant fraction of the optimal revenue for all c -informative distributions, where c is a constant. We remark that there are regular distributions which are not c -informative for any constant c , and that there are c -informative distributions which are not regular. Thus, the two classes are incomparable.

3.4 The maximin approach to robust mechanism design

We defined a maximin optimal mechanism as one that maximizes revenue in the worst case over the choice of the distribution \mathbf{F} , as long as \mathbf{F} has the pre-specified mean $\boldsymbol{\mu}$ and standard deviation $\boldsymbol{\sigma}$.

This is a new definition, but it is not without precedent. It follows Wald’s maximin model for decision making under non-bayesian uncertainty [27]. In this model, a decision maker has to maximize a function $f(a, s)$, that depends on her action a and an unknown state of the world s . Wald’s model suggests that the player take an action $a^* = \max_a \min_s f(a, s)$ that maximizes the worst case payoff over all possible states of the world.

As an example of this concept in mechanism design, Chung and Ely [8] study the problem of an auctioneer who knows the distribution of player valuations, but where the players can have arbitrary beliefs about each other. They show that a dominant strategy truthful auction will guarantee the maximum “worst-case revenue” in equilibrium, where the worst case is taken over the choice of players’ beliefs.

4 Digital Parametric Auctions with Arbitrary Independent Distributions

We now construct a digital auction \mathbb{A} which is competitive with the optimal auction. The competitive ratio will depend on the ratio $\frac{\mu_i}{\sigma_i}$.

Our parametric digital auction \mathbb{A}

$\mathbb{A}(\mathbf{v}, \boldsymbol{\mu}, \boldsymbol{\sigma})$

- 1 Find $k_i = \operatorname{argmax}_t [(\mu_i - \sigma_i t) \cdot \frac{t^2}{1+t^2}]$.
- 2 For each player i , set the reserve price $r_i = \mu_i - \sigma_i k_i$.
- 3 Sell a copy of the good to player i if and only if $v_i > r_i$.

Theorem 1. *For any distribution \mathbf{F} with mean $\boldsymbol{\mu}$ and standard deviation $\boldsymbol{\sigma}$, we have*

$$\frac{\operatorname{Rev}_i(\mathbb{A}, \mathbf{F})}{\operatorname{Rev}_i(\operatorname{OPT}, \mathbf{F})} \geq \left(1 - \frac{3}{2} \frac{\sigma}{\mu} k_i\right) = \left(\frac{1}{2} \frac{\sigma}{\mu} k_i^3\right).$$

Proof.

We prove this via a series of lemmas. First, we characterize k_i in terms of $\frac{\mu_i}{\sigma_i}$.

Lemma 1. *Let $r_i = \mu_i - \sigma_i k_i$ be player i ’s reserve price in auction \mathbb{A} . We have that k_i is the unique real solution to the cubic equation $\frac{\mu_i}{\sigma_i} = \frac{1}{2}(3k + k^3)$*

Proof of Lemma 1. The value k_i is obtained by maximizing the differentiable function $(\mu_i - \sigma_i k) \cdot \frac{k^2}{1+k^2}$ over $k \geq 0$. Note that finding k_i is equivalent to finding the value k maximizing $\ln(\mu_i - \sigma_i k) + 2 \ln k - \ln(1 + k^2)$. Taking derivatives of this function, we obtain that k_i satisfies the equation

$$-\frac{\sigma_i}{\mu_i - \sigma_i k_i} + \frac{2}{k_i} - \frac{2k_i}{1 + k_i^2} = 0.$$

Multiplying the denominators out, we get

$$-\sigma_i k_i (1 + k_i^2) + 2(1 + k_i^2)(\mu_i - \sigma_i k_i) - 2k_i^2(\mu_i - \sigma_i k) = 0.$$

Canceling out some terms and rearranging gives

$$\begin{aligned} -\sigma_i k_i(1 + k_i^2) + 2(\mu_i - \sigma_i k_i) &= 0 \\ 2\mu_i &= 3\sigma_i k_i + \sigma_i k_i^3 \\ \frac{\mu_i}{\sigma_i} &= \frac{1}{2}(3k_i + k_i^3), \end{aligned}$$

which is what we wanted to show.

Note that this cubic equation has exactly one real root because the function $\frac{1}{2}(3k_i + k_i^3) - \frac{\mu_i}{\sigma_i}$ is strictly increasing. Indeed, its derivative is $\frac{3}{2} + \frac{3}{2}k_i^2 > 0$. This completes the proof of Lemma 1. \blacksquare

With this characterization of k_i in hand, we can give a lower bound on the expected payment from player i .

Lemma 2. *Let \mathbf{F} be a distribution with mean μ and standard deviation σ . The expected revenue that auction \mathbb{A} obtains from player i is at least $\mu_i - \frac{3}{2}\sigma_i k_i = \frac{1}{2}\sigma_i k_i^3$.*

Proof of Lemma 2. The expected revenue obtained from player i is $(\mu_i - \sigma_i k_i) \cdot (1 - F_i(\mu_i - \sigma_i k_i))$. However, we do not know the value $1 - F_i(\mu_i - \sigma_i k_i)$. We need to give a lower bound. To do this, we use the following one-sided version of Chebyshev's inequality.

(Cantelli's Inequality) For every real-valued distribution with mean μ and variance σ^2 , we have

$$1 - F(\mu - \sigma k) \geq \frac{k^2}{1 + k^2}$$

From Cantelli's inequality, we obtain a bound of $(\mu_i - \sigma_i k_i) \cdot \frac{k_i^2}{1 + k_i^2}$ on the revenue collected from player i . From Lemma 1, we know that k_i satisfies $\frac{\mu_i}{\sigma_i} = \frac{1}{2}(3k_i + k_i^3)$. Multiply both sides of the equation by σ_i to obtain $\mu_i = \frac{1}{2}\sigma_i(3k_i + k_i^3)$. Now we can write $\mu_i - \sigma_i k_i = \frac{1}{2}\sigma_i(k_i + k_i^3) = \frac{1}{2}\sigma_i k_i(1 + k_i^2)$. The lower bound on the expected auction revenue becomes

$$(\mu_i - \sigma_i k_i) \frac{k_i^2}{1 + k_i^2} = \frac{1}{2}\sigma_i k_i(1 + k_i^2) \frac{k_i^2}{1 + k_i^2} = \frac{1}{2}\sigma_i k_i^3.$$

Using again the fact that $\mu_i - \sigma_i k_i = \frac{1}{2}\sigma_i(k_i + k_i^3)$, we can write this revenue bound as $\mu_i - \frac{3}{2}\sigma_i k_i$. This completes the proof of Lemma 2. \blacksquare

We remark that if $\sigma_i = 0$, then the player's valuation is μ_i with probability 1. Thus, the expected revenue from player i is μ_i .

We have given a lower bound on the revenue that \mathbb{A} obtains from each player i . We can also give an upper bound on the revenue that the optimal auction obtains from player i by noting that a truthful, individually rational auction⁶ will always charge a price p_i lower than the player's valuation v_i . Thus, the expected revenue of the auction satisfies $\mathbb{E}[p_i] \leq \mathbb{E}[v_i] = \mu_i$. Using this together with the above lemma, we can bound the player- i competitive ratio of auction \mathbb{A} as follows

$$\frac{Rev_i(\mathbb{A}, \mathbf{F})}{Rev_i(OPT, \mathbf{F})} \geq \frac{\mu_i - \frac{3}{2}\sigma_i k_i}{\mu_i} = 1 - \frac{3}{2} \frac{\sigma_i}{\mu_i} k_i.$$

This completes the proof of the theorem. *Q.E.D.*

⁶An auction is individually rational if no player gets negative utility from participating. If a player does not buy a copy of the good, her price is zero. If she buys a copy of the good, her price p_i is less than her value v_i .

4.1 A constant competitive Ratio

Theorem 1 tells us that $Rev_i(\mathbb{A}, \mathbf{F}) \geq (1 - \frac{3}{2} \frac{\sigma_i}{\mu_i} k_i) Rev_i(OPT, \mathbf{F})$ for every player i . Thus, we can bound the revenue over all players as

$$\sum_{i=1}^n Rev_i(\mathbb{A}, \mathbf{F}) \geq \sum_{i=1}^n (1 - \frac{3}{2} \frac{\sigma_i}{\mu_i} k_i) Rev_i(OPT, \mathbf{F}) \geq \min_i (1 - \frac{3}{2} \frac{\sigma_i}{\mu_i} k_i) \sum_{i=1}^n Rev_i(OPT, \mathbf{F}).$$

This gives us the competitive ratio

$$\frac{Rev(\mathbb{A}, \mathbf{F})}{Rev(OPT, \mathbf{F})} \geq \min_i (1 - \frac{3}{2} \frac{\sigma_i}{\mu_i} k_i).$$

One key informal observation is that the competitive ratio $\min_i (1 - \frac{3}{2} \frac{\sigma_i}{\mu_i} k_i)$ is constant whenever $\frac{\mu_i}{\sigma_i}$ is bounded below by a constant for all players. Thus, for a very large class of distributions, our parametric auction obtains a constant fraction of the revenue. We remark that, unlike previous auctions, we make no assumptions on players having identical distributions.

More formally, make the following definitions:

1. The function $k(a)$ maps a to the unique real root of the cubic equation $\frac{1}{2}(k^3 + 3k) = a$.
2. The function $\rho(a) \stackrel{\text{def}}{=} 1 - \frac{3}{2} \frac{1}{a} k(a)$. We can also write $\rho(a) = \frac{1}{2} \frac{1}{a} k^3(a)$ by our definition of $k(a)$. (See lemma 1).
3. For any constant $c > 0$, the class of distributions $\mathcal{F}_c \stackrel{\text{def}}{=} \{F : \mathbb{E}[F] = \mu, Var(F) = \sigma^2, \frac{\mu}{\sigma} > c\}$.

We can now show the following theorem.

Theorem 2. *For any constant $c > 0$, let $\mathbf{F} = F_1 \times \dots \times F_n$ be a distribution where each F_i is in the class \mathcal{F}_c . Then*

$$\frac{Rev(\mathbb{A}, \mathbf{F})}{Rev(OPT, \mathbf{F})} \geq \rho(c).$$

Proof. Theorem 1 tells us that $\frac{Rev(\mathbb{A}, \mathbf{F})}{Rev(OPT, \mathbf{F})} \geq \min_i \rho(\frac{\mu_i}{\sigma_i})$. The fact that $F_i \in \mathcal{F}_c$ implies that $\frac{\mu_i}{\sigma_i} > c$. If we show that the function $\rho(\cdot)$ is increasing, we can conclude that $\rho(\frac{\mu_i}{\sigma_i}) > \rho(c)$ for all i , which gives us

$$\min_i \rho(\frac{\mu_i}{\sigma_i}) > \rho(c).$$

Now we show that $\rho(a)$ is an increasing function of a when $a \geq 0$. Since $\rho(a) \stackrel{\text{def}}{=} 1 - \frac{3}{2} \frac{1}{a} k(a)$, it's derivative is $\rho'(a) = \frac{3}{2} \frac{1}{a^2} k(a) - \frac{3}{2} \frac{1}{a} k'(a)$.

The function $k(a)$ was defined implicitly as $2a = k^3 + 3k$. Implicit differentiation gives us $2 = 3k^2 \cdot k'(a) + 3k'(a)$, which we can rewrite as $k'(a) = \frac{2}{3} \frac{1}{1+k^2(a)}$. Plugging this into our expression for $\rho'(a)$ we obtain

$$\rho'(a) = \frac{3}{2} \frac{1}{a^2} k(a) - \frac{1}{a} \cdot \frac{1}{1+k^2(a)}.$$

We can multiply the above equality by $a^2 \cdot (1+k^2(a))$ without changing the sign of $\rho'(a)$. To prove $\rho'(a) \geq 0$, it suffices to show

$$\begin{aligned} a^2 \cdot (1+k^2(a)) \rho'(a) &\geq 0 \\ \frac{3}{2} k(a)(k^2(a) + 1) - a &\geq 0 \\ k^3(a) + \frac{1}{2} k^3(a) + \frac{3}{2} k(a) - a &\geq 0 \end{aligned}$$

$$k^3(a) + 0 \geq 0.$$

Since $k(0) = 0$ and k is an increasing function, we have $k^3(a) \geq 0$ when $a \geq 0$. This shows that $\rho(\cdot)$ is an increasing function when $a > 0$, and thus that $\rho(\frac{\mu_i}{\sigma_i}) > \rho(c)$ when $F_i \in \mathcal{F}_c$. We can conclude that when all $F_1, \dots, F_n \in \mathcal{F}_c$, the competitive ratio of \mathbb{A} satisfies the bound

$$\frac{Rev(\mathbb{A}, \mathbf{F})}{Rev(OPT, \mathbf{F})} \geq \min_i \rho(\frac{\mu_i}{\sigma_i}) > \rho(c).$$

This completes the proof of the theorem. *Q.E.D.*

4.2 The competitive ratio of \mathbb{A} when the distributions have a monotone hazard rate.

As mentioned in the introduction, the class \mathcal{F}_c is a very large class of distributions. In particular, any distribution F with a monotone hazard rate belongs to the class \mathcal{F}_1 . Barlow, Marshall and Proschan [4] noted that for a random variable X with monotone hazard rate, $E[X^2] \leq 2E[X]^2$. Since $Var(X) = E[X^2] - E[X]^2$, this gives us $Var(X) \leq E[X]^2$. Taking square roots, we obtain $\sigma \leq \mu$ or, equivalently, $\frac{\mu}{\sigma} > 1$.

Theorem 2 tells us that the auction \mathbb{A} has a competitive ratio greater than $\rho(1)$ when the distributions F_1, \dots, F_n are in \mathcal{F}_1 . In particular, this is true when the distributions have a monotone hazard rate. We can use this to conclude the following corollary.

Corollary 1. *When each F_i has a monotone hazard rate, the competitive ratio of \mathbb{A} is bounded below by*

$$\frac{Rev(\mathbb{A}, \mathbf{F})}{Rev(OPT, \mathbf{F})} \geq \rho(1) > 10.5\%.$$

Proof. The fact that $\frac{Rev(\mathbb{A}, \mathbf{F})}{Rev(OPT, \mathbf{F})} \geq \rho(1)$ is immediate from Theorem 2 and the fact that a distribution with monotone hazard rate satisfies $\frac{\mu}{\sigma} > 1$. We need to show via a computation that $\rho(1) > 10.5\%$. Recall that $\rho(1) = 1 - \frac{3}{2} \frac{1}{1} \cdot k(1)$. Solving the cubic equation $k^3 + 3k = 2$ gives us $k(1) = \sqrt[3]{1 + \sqrt{2}} - \frac{1}{\sqrt[3]{1 + \sqrt{2}}} \cong 0.596$. Plugging this into the formula for $\rho(1)$, we obtain $\rho(1) > 10.5\%$. ■

5 A Better Digital Auction for Distributions with Monotone Hazard Rate

In the previous section, we presented a digital auction \mathbb{A} and showed that it obtained a constant competitive ratio for a wide class of distribution, including all distributions with an increasing hazard rate.

In this section, we give a simpler auction \mathcal{A} which will guarantee an even better competitive ratio, under the assumption that each F_i has an increasing hazard rate. Unlike our auction \mathbb{A} , we cannot give any guarantees on the competitive ratio when the F_i do not have increasing hazard rates.

Our auction will use the following observation from Barlow and Marshall [3]:

Fact 1. *Let F be a distribution over the real numbers with monotone hazard rate. Let μ be the mean of F . Then $1 - F(\mu) \geq \frac{1}{e}$.*

We can now describe our competitive parametric auction

$\mathcal{A}(\mathbf{v}, \boldsymbol{\mu}, \boldsymbol{\sigma})$

- 1 Set a reserve for player i at $r_i = \mu_i$.
- 2 Sell a copy of the good to player i if and only if $v_i > r_i$.

Theorem 3. For all distributions \mathbf{F} with monotone hazard rate, we have

$$\frac{\text{Rev}(\mathbb{A}, \mathbf{F})}{\text{Rev}(\text{OPT}, \mathbf{F})} \geq \frac{1}{e}$$

Proof. The proof is immediate using Barlow and Marshall's observation and the fact that the optimal auction cannot obtain more than μ_i expected revenue from player i .

The expected revenue that \mathbb{A} obtains from player i is $\mu_i \cdot (1 - F(\mu_i))$. By the above observation, this is greater than or equal to $\mu_i \cdot \frac{1}{e}$. Since the optimal auction cannot obtain more than μ_i expected revenue from player i , our theorem is proved. *Q.E.D.*

6 Maximin Optimality and our auction \mathbb{A}

Recall our definition of maximin optimality for parametric auctions.

Definition. Let \mathcal{C} be a class of mechanisms. A parametric auction A^* is maximin optimal for class \mathcal{C} if $A^* \in \mathcal{C}$ and

$$A^* = \operatorname{argmax}_{A \in \mathcal{C}} \min_{\mathbf{F}: \mathbb{E}[\mathbf{F}] = \boldsymbol{\mu}, \text{Var}(\mathbf{F}) = \boldsymbol{\sigma}^2} \text{Rev}(A, \mathbf{F}).$$

Theorem 4. The auction \mathbb{A} is maximin optimal for the class \mathcal{C} of parametric posted price mechanisms.

Proof. First, we note that $\mathbb{A} \in \mathcal{C}$, the class of posted price auctions. Player i 's posted price is $r_i = \mu_i - \sigma_i k_i$. Now we need to show that \mathbb{A} maximizes the worst-case revenue among any such mechanisms.

In Theorem 1, we used Cantelli's inequality to show that

$$\text{Rev}(\mathbb{A}, \mathbf{F}) \geq \sum_{i=1}^n (\mu_i - \sigma_i k_i) \cdot \frac{k_i^2}{1 + k_i^2}.$$

To prove Theorem 4, it suffices to show that for any $\boldsymbol{\mu}, \boldsymbol{\sigma}$ and any parametric posted price mechanism $A(\mathbf{v}, \boldsymbol{\mu}, \boldsymbol{\sigma})$, there exists a distribution \mathbf{F} with mean $\boldsymbol{\mu}$ and standard deviation $\boldsymbol{\sigma}$ such that

$$\text{Rev}(A, \mathbf{F}) < \sum_{i=1}^n (\mu_i - \sigma_i k_i) \cdot \frac{k_i^2}{1 + k_i^2}.$$

Since the digital auction $A(\mathbf{v}, \boldsymbol{\mu}, \boldsymbol{\sigma})$ is a parametric posted price mechanism, it is completely characterized by a vector of reserve prices $(p_1(\boldsymbol{\mu}, \boldsymbol{\sigma}), \dots, p_n(\boldsymbol{\mu}, \boldsymbol{\sigma}))$. The auction sells a copy of the good to player i if and only if $v_i > p_i$. Thus, player i 's expected payment is $p_i \cdot (1 - F_i(p_i))$.

We will prove the theorem by showing that, for each player i , there exists a distribution F_i with mean μ_i and variance σ_i on which player i 's expected payment is less than or equal to $(\mu_i - \sigma_i k_i) \cdot \frac{k_i^2}{1 + k_i^2}$. We split into two cases: $p_i > \mu_i$ and $p_i \leq \mu_i$.

Assume $p_i > \mu_i$ and write $p_i = \mu_i + \frac{\sigma_i}{t_i}$, for some positive t_i . Consider the family of distributions $\{C(k; \mu_i, \sigma_i)\}_{k \in \mathbb{R}}$ where $C(k; \mu_i, \sigma_i)$ takes the values

$$H(k; \mu_i, \sigma_i) = \mu_i + \frac{\sigma_i}{k} \text{ with probability } \frac{k^2}{1 + k^2},$$

$$L(k; \mu_i, \sigma_i) = \mu_i - \sigma_i k \text{ with probability } \frac{1}{1 + k^2}.$$

Let player i 's valuation v_i be drawn from distribution $C(k; \mu_i, \sigma_i)$. When k is smaller than t_i , the auction will sell the good if and only if $v_i = H(k; \mu_i, \sigma_i)$, and it will charge player i the reserve price $p_i = \mu_i + \frac{\sigma_i}{t_i}$. The expected revenue collected from player i in this case is $(\mu_i + \frac{\sigma_i}{t_i}) \cdot \frac{k^2}{1 + k^2}$. Taking the limit as $k \rightarrow 0$, our expected revenue becomes arbitrarily small. In particular, it becomes smaller than $(\mu_i - \sigma_i k_i) \cdot \frac{k_i^2}{1 + k_i^2}$.

Now assume $p_i \leq \mu_i$. Write $p_i = \mu_i - \sigma_i t_i$. Since $A \neq \mathbb{A}$, at least one player i must have $t_i \neq k_i$. We focus on this player. Choose F_i to be the distribution $C(t_i; \mu_i, \sigma_i)$. For this distribution, the valuation v_i will be

$$H(t_i) = \mu_i + \frac{\sigma_i}{t_i} \text{ with probability } \frac{t_i^2}{1+t_i^2},$$

$$L(t_i) = \mu_i - \sigma_i t_i \text{ with probability } \frac{1}{1+t_i^2}.$$

The auction A only sells to player i when $v_i > \mu_i - \sigma_i t_i$.⁷ For this particular distribution, the auction will sell at price $\mu_i - \sigma_i t_i$, but only when the valuation is $H(t_i) = \mu_i + \frac{\sigma_i}{t_i}$. This happens with probability $\frac{t_i^2}{1+t_i^2}$. Thus, the expected revenue collected from player i will be

$$(\mu_i - \sigma_i t_i) \cdot \frac{t_i^2}{1+t_i^2} < (\mu_i - \sigma_i k_i) \cdot \frac{k_i^2}{1+k_i^2},$$

where the last inequality is because k_i is the unique maximum of the function $f(t) = (\mu_i - \sigma_i t) \cdot \frac{t^2}{1+t^2}$, and because we assumed $k_i \neq t_i$. We have shown that, when $p_i < \mu_i$, there exists a distribution F_i for which the expected revenue collected from player i is less than or equal to $(\mu_i - \sigma_i k_i) \cdot \frac{k_i^2}{1+k_i^2}$.

The above analysis holds for each player individually. However, since A is a digital auction, we can simply consider a product distribution $\mathbf{F} = (F_1, \dots, F_n)$, where each F_i is chosen to limit the amount of revenue that A collects from player i . Adding up over all players, we conclude that there exists a distribution \mathbf{F} such that

$$\text{Rev}(A, \mathbf{F}) \leq \sum_{i=1}^n (\mu_i - \sigma_i k_i) \cdot \frac{k_i^2}{1+k_i^2}.$$

Recalling that our parametric auction \mathbb{A} satisfies $\text{Rev}(\mathbb{A}, \mathbf{F}) \geq \sum_{i=1}^n (\mu_i - \sigma_i k_i) \cdot \frac{k_i^2}{1+k_i^2}$ for all distributions \mathbf{F} with mean $\boldsymbol{\mu}$ and standard deviation $\boldsymbol{\sigma}$, we obtain that \mathbb{A} maximizes worst-case revenue. This concludes the proof of Theorem 4. *Q.E.D.*

7 Parametric Auctions For Matroid and Downward Closed Environments

We present in this section a parametric auction for general single-dimensional domains. Our competitive ratio will be a function that depends on the distribution parameters. When the distributions F_1, \dots, F_n are regular and belong to the class \mathcal{F}_c for some constant $c > 0$, we will obtain a constant competitive ratio for matroid environments. When the distributions F_1, \dots, F_n have a monotone hazard rate, we will obtain a competitive ratio for all downward closed environments. Recall that both matroid and downward closed environments are characterized by a *family of feasible sets* \mathcal{I} , where each $I \in \mathcal{I}$ is a subset of $\{1, \dots, n\}$. The environment specifies that we can only produce a final allocation that is a set of \mathcal{I} .

Our parametric auction will be a VCG auction with reserve prices \mathbf{r} . Because we do not know the distributions F_1, \dots, F_n , we cannot set optimal monopoly reserve prices. Instead, we will need to set the reserve price for player i using only the parameters μ_i, σ_i . Recall the definition of a second-price auction with lazy reserve prices $\mathbf{r} = (r_1, \dots, r_n)$ [11].

⁷For this proof, the strictness of the inequality matters. If we sold when $v_i \geq \mu_i - \sigma_i t_i$, then a similar argument would apply, but we would need to use a distribution $C(t_i + \epsilon; \mu_i, \sigma_i)$ for arbitrarily small values of ϵ .

$VCG - L_r(\mathbf{v})$

- 1 Run the VCG auction to obtain an outcome (x, π) , where x is an allocation in \mathcal{I} that maximizes social welfare, and π is the corresponding vector of VCG prices.
- 2 Sell only to players i in the set $S = \{i : v_i > r_i\}$. That is, if $v_i \leq r_i$, then set player i 's allocation to be $x_i = 0$ and her price to be $\pi_i = 0$.
- 3 Charge every player i in the set S the price $\max(\pi_i, r_i)$.

With this definition in hand, we can now define our single-good auction \mathcal{B} .

$\mathcal{B}(\mathbf{v}, \boldsymbol{\mu}, \boldsymbol{\sigma})$

- 1 Find $k_i = \operatorname{argmax}_t [(\mu_i - \sigma_i t) \cdot \frac{t^2}{1+t^2}]$.
- 2 For each player i , set the reserve price $q_i = \mu_i - \frac{3}{2}\sigma_i k_i$.
- 3 Run the auction $VCG - L_q$ to determine a price and allocation.

Theorem 5. *Let $c > 0$ be a constant and let F_1, \dots, F_n be regular distributions in the class \mathcal{F}_c . There exists a positive increasing function ψ such that, for all matroid environments*

$$\frac{\operatorname{Rev}(\mathcal{B}, \mathbf{F})}{\operatorname{Rev}(\operatorname{OPT}, \mathbf{F})} \geq \frac{1}{2}\psi(c) > 0$$

Proof. If the distribution F_i is known, one way to set the reserve price r_i is to choose $r_i = p_i \stackrel{\text{def}}{=} \operatorname{argmax}_p p \cdot (1 - F_i(p))$. Call p_i the *optimal monopoly price*. Dhangwatnotai, Roughgarden, and Yan [11] study the auction $VCG - L_p$ that sets the optimal monopoly prices as reserve prices. They show that for matroid settings and regular distributions F_1, \dots, F_n , the auction VCG_p obtains a revenue of at least $\frac{1}{2}$ of the optimal auction. We will show that our auction \mathcal{B} obtains a constant fraction of $VCG - L_p$. Combining this with Dhangwatnotai, Roughgarden, and Yan's result, we can conclude that our auction \mathcal{B} obtains a constant fraction of the optimal revenue.

We first prove that $q_i \leq p_i$. Since p_i is the optimal monopoly price, it satisfies $(1 - F_i(p_i)) \cdot p_i \geq (1 - F_i(r)) \cdot r$ for any $r \neq p_i$. In particular, we can plug in $r = \mu_i - \sigma_i k_i$, player i 's reserve price in our digital auction \mathbb{A} . This gives us $(1 - F_i(p_i)) \cdot p_i \geq (\mu_i - \sigma_i k_i) \cdot (1 - F_i(\mu_i - \sigma_i k_i))$. Applying Cantelli's inequality and our results from lemma 2, we obtain

$$p_i \cdot (1 - F_i(p_i)) \geq (\mu_i - \sigma_i k_i) \cdot (1 - F_i(\mu_i - \sigma_i k_i)) \geq (\mu_i - \sigma_i k_i) \frac{k_i^2}{1 + k_i^2} = \mu_i - \frac{3}{2}\sigma_i k_i = q_i.$$

Since $1 - F_i(p_i) < 1$, we have $p_i \geq (1 - F_i(p_i))p_i \geq q_i$. This shows that our parametric reserve price q_i is always less than or equal to the optimal monopoly reserve price p_i .

Now we proceed to show that $VCG - L_q$ is competitive with $VCG - L_p$. Our strategy will be to reduce this problem to proving a lower bound on $\frac{q_i}{p_i}$, and then proving said lower bound. Given a vector of valuations $\mathbf{v} = (v_1, \dots, v_n)$, let $\pi(\mathbf{v})$ be the vector of prices charged by the VCG auction (without reserves). We remark that, since we are using the VCG with *lazy reserve prices*, the VCG prices are the same regardless of whether we use \mathbf{q} or \mathbf{p} as reserve prices.

Let $W \in \mathcal{I}$ be the set of winners in the VCG auction without reserve prices. The revenue of the auction $VCG - L_p$ is

$$\sum_{i \in W} \Pr[v_i \geq p_i] \cdot \max(\pi_i(\mathbf{v}), p_i).$$

The revenue of our parametric auction $VCG - L_q$ is

$$\sum_{i \in W} \Pr[v_i \geq q_i] \cdot \max(\pi_i(\mathbf{v}), q_i).$$

Since $p_i \geq q_i$, we have $Pr[v_i \geq q_i] \geq Pr[v_i \geq p_i]$. Thus, it suffices to give a lower bound for

$$\frac{\max(\pi_i(\mathbf{v}), q_i)}{\max(\pi_i(\mathbf{v}), p_i)} \geq C$$

for some constant C , for all players i and valuation vectors \mathbf{v} . If we can prove such a lower bound, then we can show that $\max(\pi_i(\mathbf{v}), q_i) \geq C \cdot \max(\pi_i(\mathbf{v}), p_i)$, and taking probabilities over \mathbf{F} on both sides we obtain

$$Rev(VCG - L_{\mathbf{q}}, \mathbf{F}) \geq C \cdot Rev(VCG - L_{\mathbf{p}}, \mathbf{F}).$$

We first note that, for all \mathbf{v} , we have $\frac{\max(\pi_i(\mathbf{v}), q_i)}{\max(\pi_i(\mathbf{v}), p_i)} \geq \frac{q_i}{p_i}$. If $\pi_i \geq p_i$, then π_i is also greater than or equal to q_i (because $p_i \geq q_i$) and we have $\frac{\max(\pi_i(\mathbf{v}), q_i)}{\max(\pi_i(\mathbf{v}), p_i)} = 1 \geq \frac{q_i}{p_i}$. If $\pi_i \leq p_i$, then $\max(\pi_i, p_i) = p_i$, and we have $\frac{\max(\pi_i(\mathbf{v}), q_i)}{\max(\pi_i(\mathbf{v}), p_i)} \geq \frac{q_i}{p_i}$.

Now it suffices to give a lower bound $\frac{q_i}{p_i} \geq C$. In particular, we show $\frac{q_i}{p_i} \geq \psi(\frac{\mu_i}{\sigma_i})$ for some increasing function $\psi(\cdot)$.⁸ Recall that q_i is equal by definition to $q_i(\mu_i, \sigma_i) = \mu_i - \frac{3}{2}\sigma_i k_i$, where k_i is the unique real solution to the equation $\frac{\mu_i}{\sigma_i} = \frac{1}{2}(k^3 + 3k)$. To bound $\frac{q_i}{p_i}$, it suffices to give an upper bound on p_i . We do this by separating in two cases.

The first case is when $p_i < \mu_i$. In this case, we have $\frac{q_i}{p_i} \geq 1 - \frac{3}{2}\frac{\sigma_i}{\mu_i} \cdot k_i$. The expression $1 - \frac{3}{2}\frac{\sigma_i}{\mu_i} \cdot k_i$ is the function $\rho(\frac{\mu_i}{\sigma_i})$ defined in section 4. We showed in theorem 2 that this function is increasing with $\frac{\mu_i}{\sigma_i}$.

The second case is when $p_i \geq \mu_i$. Since p_i is the optimal monopoly reserve price, it must be the case that $p_i \cdot (1 - F_i(p_i)) \geq r \cdot (1 - F(r))$ for any price $r \neq p_i$. In particular, we can choose $r_i \stackrel{\text{def}}{=} \mu_i - \sigma_i k_i$. As in our analysis above and in theorem 1, we have $r_i \cdot (1 - F(r_i)) \geq \mu_i - \frac{3}{2}\sigma_i k_i = q_i$. Thus, the optimal monopoly reserve price must satisfy $p_i \cdot (1 - F_i(p_i)) \geq q_i$.

Since $p_i \geq \mu_i$, we can write $p_i = \mu_i + \sigma_i t$ for some non-negative t . Using a right-tailed version of Cantelli's inequality, we can give a bound

$$1 - F_i(\mu_i + \sigma_i t) \leq \frac{1}{1 + t^2}.$$

Combining this with the fact that $p_i \cdot (1 - F_i(p_i)) \geq q_i$, we have the inequality

$$(\mu_i + \sigma_i t) \cdot \frac{1}{1 + t^2} \geq q_i$$

$$\mu_i + \sigma_i t \geq q_i(1 + t^2)$$

$$q_i(1 + t^2) - \sigma_i t - \mu_i \leq 0.$$

Thus, in order for $p_i = \mu_i + \sigma_i t$ to be the optimal monopoly reserve price, the above quadratic polynomial in t must be less than zero. Since the principal coefficient of this polynomial is $q_i > 0$, the polynomial is only negative between its roots. Thus, we must have that t lies in some interval $[z_1, z_2]$ where z_1, z_2 are the roots of $q_i(1 + t^2) - \sigma_i t - \mu_i = 0$, and we can write $p_i = \mu_i + \sigma_i t \leq \mu_i + \sigma_i z_2$.

We now have to write z_2 in terms of μ_i, σ_i . Recall that z_2 is the largest root of the quadratic equation $q_i(1 + t^2) - \sigma_i t - \mu_i = 0$. Divide both sides of this equation by μ_i to obtain

$$\frac{q_i}{\mu_i}(1 + t^2) - \frac{\sigma_i}{\mu_i}t - 1 = 0.$$

Since $q_i = \mu_i - \frac{3}{2}\sigma_i k_i$, we have that $\frac{q_i}{\mu_i} = 1 - \frac{3}{2}\frac{\sigma_i}{\mu_i} k_i \stackrel{\text{def}}{=} \rho(\frac{\mu_i}{\sigma_i})$, our function defined in section 4. Thus, we can write our equation as

$$\rho(\frac{\mu_i}{\sigma_i}) \cdot (1 + t^2) - \frac{\sigma_i}{\mu_i}t - 1 = 0.$$

⁸In section 7.1 we give an explicit constant lower bound for distributions with monotone hazard rate.

For convenience of notation, write $a \stackrel{\text{def}}{=} \frac{\mu_i}{\sigma_i}$. We want to solve $\rho \cdot (1 + t^2) - \frac{1}{a}t - 1 = 0$. Using the quadratic formula, we get

$$\begin{aligned} z_2 &= \frac{1 + \sqrt{1 + 4a^2\rho - 4a^2\rho^2}}{2a\rho} \\ z_2 &= \frac{1}{2a\rho} + \sqrt{\frac{1}{4a^2\rho^2} + \frac{4a^2\rho}{4a^2\rho^2} - \frac{4a^2\rho^2}{4a^2\rho^2}} \\ z_2 &= \frac{1}{2a\rho} + \sqrt{\frac{1}{4a^2\rho^2} + \frac{1}{\rho} - 1}. \end{aligned}$$

Since ρ is a function of $a = \frac{\mu_i}{\sigma_i}$, the root z_2 is also a function of a . Furthermore, the function $z_2(a)$ is decreasing with both ρ and a . Since ρ is an increasing function of a , we conclude that z_2 is decreasing in $a = \frac{\mu_i}{\sigma_i}$. We conclude that

$$p_i = \mu_i + \sigma_i t \leq \mu_i + \sigma_i z_2\left(\frac{\mu_i}{\sigma_i}\right)$$

where $z_2(\cdot)$ is a decreasing function. Combining this with the fact that $\rho\left(\frac{\mu_i}{\sigma_i}\right)$ is by definition equal to $\frac{q_i}{\mu_i}$, we get

$$\frac{q_i}{p_i} = \frac{\frac{q_i}{\mu_i}}{\frac{p_i}{\mu_i}} = \frac{\rho\left(\frac{\mu_i}{\sigma_i}\right)}{1 + \frac{\sigma_i}{\mu_i}t} \geq \frac{\rho\left(\frac{\mu_i}{\sigma_i}\right)}{1 + \frac{\sigma_i}{\mu_i} \cdot z_2\left(\frac{\mu_i}{\sigma_i}\right)}.$$

Define

$$\psi\left(\frac{\mu_i}{\sigma_i}\right) = \frac{\rho\left(\frac{\mu_i}{\sigma_i}\right)}{1 + \frac{\sigma_i}{\mu_i} \cdot z_2\left(\frac{\mu_i}{\sigma_i}\right)}.$$

By our analysis above, the numerator is increasing in $\frac{\mu_i}{\sigma_i}$ and the denominator is decreasing in $\frac{\mu_i}{\sigma_i}$. Thus $\psi(\cdot)$ is an increasing function. Note also that $\psi\left(\frac{\mu_i}{\sigma_i}\right) \leq \rho\left(\frac{\mu_i}{\sigma_i}\right)$, so $\psi(\cdot)$ is also a lower bound on $\frac{q_i}{p_i}$ for the case $p_i \leq \mu_i$.

Since $\psi(\cdot)$ is increasing, and we assumed that the distributions F_1, \dots, F_n are in the class \mathcal{F}_c , we can infer that

$$\frac{q_i}{p_i} > \psi(c).$$

From our analysis above, we conclude that $\text{Rev}(VCG_{\mathbf{q}}, \mathbf{F}) \geq \psi(c) \cdot \text{Rev}(VCG_{\mathbf{p}}, \mathbf{F})$.

Finally, applying Theorem 3.7 from Hartline and Roughgarden [19] we get

$$\frac{\text{Rev}(VCG_{\mathbf{p}}, \mathbf{F})}{\text{Rev}(OPT, \mathbf{F})} \geq \frac{1}{2}.$$

Combining this with our result we obtain

$$\frac{\text{Rev}(VCG_{\mathbf{q}}, \mathbf{F})}{\text{Rev}(OPT, \mathbf{F})} \geq \frac{1}{2} \cdot \psi(c).$$

Since $\mathcal{B} = \text{Rev}(VCG_{\mathbf{q}}, \mathbf{F})$, this proves our theorem. *Q.E.D.*

7.1 An explicit lower bound for distributions with monotone hazard rate.

In theorem 5, we gave a parametric auction for matroid environments with a competitive ratio bounded below by $\frac{1}{2}\psi(c)$ for all regular distributions in \mathcal{F}_c . In this section, we give an explicit bound for all distributions with a monotone hazard rate. Our auction applies to the more general downward closed environments.

Theorem 6. *Let \mathcal{I} be a downward closed environment. When each F_i has a monotone hazard rate, the competitive ratio of our auction \mathcal{B} is bounded below by*

$$\frac{\text{Rev}(\mathcal{B}, \mathbf{F})}{\text{Rev}(OPT, \mathbf{F})} > 0.90\%.$$

Proof. Recall that a key step in the proof of theorem 5 was giving a bound $\frac{q_i}{p_i} \geq \psi\left(\frac{\mu_i}{\sigma_i}\right)$, where q_i is our parametric reserve price and p_i is the monopoly reserve price that the optimal auction would set. This allowed us to conclude that $\frac{Rev(\mathcal{B}, \mathbf{F})}{Rev(OPT, \mathbf{F})} \geq \frac{1}{2}\psi\left(\frac{\mu_i}{\sigma_i}\right)$. By improving the bound on $\frac{q_i}{p_i}$, we can improve the competitive ratio of our auction.

In this proof, we show that for F_i with monotone hazard rate, the ratio $\frac{q_i}{p_i}$ satisfies the bound

$$\frac{q_i}{p_i} \geq 2.47\%.$$

Once we show this, we can apply a result from [11] arguing that $VCG - L_{\mathbf{p}}$ obtains a $\frac{1}{e}$ fraction of the expected revenue of the optimal auction on monotone hazard rate distributions and general downward closed environments. Combined with our lower bound on $\frac{q_i}{p_i}$, we can apply our reasoning from theorem 5 to show

$$\frac{Rev(\mathcal{B}, \mathbf{F})}{Rev(OPT, \mathbf{F})} \geq \frac{1}{e} \cdot 2.47\% \geq 0.90\%.$$

We use two facts about distributions with a monotone hazard rate

1. When F_i has a monotone hazard rate, $\frac{\mu_i}{\sigma_i} \geq 1$.⁹ Thus, we have $F_i \in \mathcal{F}_1$ for each player i .
2. For any distribution F_i with a monotone hazard rate and with mean μ_i , we have $1 - F_i(\mu_i) \geq \frac{1}{e}$. [3]

We now give a bound on $\frac{q_i}{p_i}$. Recall that q_i is equal by definition to $q_i(\mu_i, \sigma_i) = \mu_i - \frac{3}{2}\sigma_i k_i$, where k_i is the unique real solution to the equation $\frac{\mu_i}{\sigma_i} = \frac{1}{2}(k^3 + 3k)$. We now give a bound on p_i by separating into two cases.

We begin with the case where $p_i \leq \mu_i$. In this case, as shown in theorem 5, we have $\frac{q_i}{p_i} \geq 1 - \frac{3}{2}\sigma_i k_i \stackrel{\text{def}}{=} \rho\left(\frac{\mu_i}{\sigma_i}\right)$. We showed in theorem 2 that $\rho(\cdot)$ is an increasing function. Since F_i has a monotone hazard rate, the observation above tells us that $\frac{\mu_i}{\sigma_i} \geq 1$. Thus,

$$\frac{q_i}{p_i} \geq \rho(1) \geq 10.5\%.$$

The complicated case is when $p_i \geq \mu_i$. Since p_i is the optimal monopoly reserve price, it satisfies $p_i \cdot (1 - F_i(p_i)) \geq r \cdot (1 - F_i(r))$ for every $r \neq p_i$. In particular, we can choose $r = \mu_i$ and obtain $p_i \cdot (1 - F_i(p_i)) \geq \mu_i \cdot (1 - F_i(\mu_i)) \geq \frac{\mu_i}{e}$.

Write $p_i = \mu_i + \sigma_i t$ for some $t > 0$. We will give an upper bound on t . A right-tailed version of Cantelli's inequality tells us that

$$1 - F_i(p_i) = 1 - F_i(\mu_i + \sigma_i t) \leq \frac{1}{1 + t^2}.$$

Combining this with the fact that $p_i \cdot (1 - F_i(p_i)) \geq \frac{\mu_i}{e}$, we get

$$\begin{aligned} (\mu_i + \sigma_i t) \frac{1}{1 + t^2} &\geq \frac{\mu_i}{e} \\ \left(1 + \frac{\sigma_i t}{\mu_i}\right) &\geq \frac{1 + t^2}{e}. \end{aligned}$$

Since F_i has a monotone hazard rate, $\frac{\sigma_i}{\mu_i} \leq 1$. This yields $1 + t \geq 1 + \frac{\sigma_i}{\mu_i} t$. Combining this with our above inequality, we get

$$1 + t \geq \frac{1 + t^2}{e}.$$

A calculation shows that t satisfies this inequality only if $t \leq \frac{e}{2} + \frac{1}{2}\sqrt{-4 + 4e + e^2} \leq 3.25$. Thus, we have $p_i \leq \mu_i + 3.25\sigma_i \leq 4.25\mu_i$, where the last inequality is derived by using the fact that $\mu_i \geq \sigma_i$.

⁹Barlow, Marshall and Proschan [4] noted that for a random variable X with monotone hazard rate, $E[X^2] \leq 2E[X]^2$. Since $Var(X) = E[X^2] - E[X]^2$, this gives us $Var(X) \leq E[X]^2$. Taking square roots, we obtain $\sigma \leq \mu$.

We can now give the following bound on $\frac{q_i}{p_i}$:

$$\frac{q_i}{p_i} \geq \frac{\mu_i - \frac{3}{2}\sigma_i k_i}{4.25\mu_i} = \frac{1 - \frac{3}{2}\frac{\sigma_i}{\mu_i} k_i}{4.25} = \frac{\rho(\frac{\mu_i}{\sigma_i})}{4.25} \geq \frac{\rho(1)}{4.25}$$

where the last inequality is because $\rho(\cdot)$ is an increasing function and $\frac{\mu_i}{\sigma_i} \geq 1$. Since $\rho(1) \geq 10.5\%$, we obtain $\frac{q_i}{p_i} \geq 2.47\%$. Using the reasoning from theorem 5, we conclude that

$$\frac{Rev(\mathcal{B}, \mathbf{F})}{Rev(OPT, \mathbf{F})} \geq 0.90\%.$$

Q.E.D.

8 Conclusion and Future Work

We introduced parametric auctions, a new type of auction where the seller only needs to know the parameters μ_i, σ_i of the buyer distributions F_1, \dots, F_n . For digital goods environments, matroid environments, and downward closed environments, we constructed auctions that obtained a constant fraction of the optimal auction revenue for a large class of distributions, including all monotone hazard rate distributions and many irregular distribution. Furthermore, our digital auction \mathbb{A} is the best parametric posted price mechanism in a maximin sense. When an adversary chooses the distribution \mathbf{F} , our auction \mathbb{A} maximizes the worst-case revenue.

Our results make extensive use of moment bounds both for general distributions and for monotone hazard rate distributions. The theory of moment bounds is very well developed, but to the best of our knowledge has never been applied to auctions before. In the future, we are interested in analyzing parametric auctions with multi-dimensional types, extending our work to other settings such as public goods, and evaluating empirically the performance of our parametric auctions. We believe that our model and techniques are very robust, and that they can be widely applied.

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