Enriques diagrams, arbitrarily near points, and Hilbert schemes

Enriques diagrams, arbitrarily near points, and Hilbert schemes

Steven KLEIMAN, Ragni PIENE, and
with Appendix B by Ilya TYOMKIN

Abstract. Given a smooth family F/Y of geometrically irreducible surfaces, we study sequences of *arbitrarily near* T-points of F/Y; they generalize the traditional sequences of infinitely near points of a single smooth surface. We distinguish a special sort of these new sequences, the *strict* sequences. To each strict sequence, we associate an ordered unweighted Enriques diagram. We prove that the various sequences with a fixed diagram form a functor, and we represent it by a smooth Y-scheme.

We equip this Y-scheme with a free action of the automorphism group of the diagram. We equip the diagram with weights, take the subgroup of those automorphisms preserving the weights, and form the corresponding quotient scheme. Our main theorem constructs a canonical universally injective map from this quotient scheme to the Hilbert scheme of F/Y; further, this map is an embedding in characteristic 0. However, in every positive characteristic, we give an example, in Appendix B, where the map is purely inseparable.

1. Introduction

In the authors' paper [**15**], Proposition (3.6) on p. 225 concerns the locus $H(\mathbf{D})$ that sits in the Hilbert scheme of a smooth irreducible complex surface and that parameterizes the complete ideals $\mathcal{I}$ with a given minimal Enriques diagram $\mathbf{D}$. The latter is an abstract combinatorial structure associated to a sequence of arbitrarily near points rendering $\mathcal{I}$ invertible. The proposition asserts that $H(\mathbf{D})$ is smooth and equidimensional.

The proposition was justified intuitively, then given an ad hoc proof in [**15**]. The intuitive justification was not developed into a formal proof, as this proof is surprisingly long and complicated. However, the proof yields more: it shows $H(\mathbf{D})$ is irreducible; it works for nonminimal $\mathbf{D}$; and it works for *families* of surfaces. Further, it works to a great extent when the characteristic is positive or mixed, but then it only shows $H(\mathbf{D})$ has a finite and universally injective covering by a smooth cover; this covering need not be birational, as examples in Appendix B show.

Originally, the authors planned to develop that formal proof in a paper that also dealt with other loose ends, notably, the details of the enumeration of curves

2000 *Mathematics Subject Classification.* Primary 14N10; Secondary 14C20, 14H40, 14K05.
Key words and phrases. Enriques diagrams, arbitrarily near points, Hilbert schemes, property of exchange.

with eight nodes. However, there is so much material involved that it makes more sense to divide it up. Thus the formal proof alone is developed in the present paper; the result itself is asserted in Corollary 5.8. Here, in more detail, is a description of this paper's contents.

Fix a smooth family of geometrically irreducible surfaces F/Y and an integer $n \geq 0$. Given a Y-scheme T, by a *sequence of arbitrarily near T-points of F/Y*, we mean an $(n+1)$-tuple $(t_0, \dots, t_n)$ where t_0 is a T-point of $F_T^{(0)} := F \times_Y T$ and where t_i, for $i \geq 1$, is a T-point of the blowup $F_T^{(i)}$ of $F_T^{(i-1)}$ at t_{i-1}. (If each t_i is, in fact, a T-point of the exceptional divisor $E_T^{(i)}$ of $F_T^{(i)}$, then $(t_0, \dots, t_n)$ is a sequence of infinitely near points in the traditional sense.) The sequences of arbitrarily near T-points form a functor in T, and it is representable by a smooth Y-scheme $F^{(n)}$, according to Proposition 3.4 below; this result is due, in essence, to Harbourne [**11**, Prp. I.2, p. 104].

We say that the sequence $(t_0, \dots, t_n)$ is *strict* if, for each i, j with $1 \leq j \leq i$, the image $T^{(i)} \subset F_T^{(i)}$ of t_i is either (a) disjoint from, or (b) contained in, the strict transform of the exceptional divisor $E_T^{(j)}$ of $F_T^{(j)}$. If (b) obtains, then we say that t_i is *proximate* to t_j and we write $t_i \succ t_j$.

To each strict sequence, we associate, in Section 3, an unweighted Enriques diagram $\mathbf{U}$ and an ordering $\theta\colon \mathbf{U} \xrightarrow{\sim} \{0, \dots, n\}$. Effectively, $\mathbf{U}$ is just a graph whose vertices are the t_i. There is a directed edge from t_j to t_i provided that $j+1 \leq i$ and that the map from $F_T^{(i)}$ to $F_T^{(j+1)}$ is an isomorphism in a neighborhood of $T^{(i)}$ and embeds $T^{(i)}$ in $E_T^{(j+1)}$. In addition, $\mathbf{U}$ inherits the binary relation of proximity. Finally, θ is defined by $\theta(t_i) := i$. This material is discussed in more detail in Section 2. In particular, to aid in passing from $(t_0, \dots, t_n)$ to $(\mathbf{U}, \theta)$, we develop a new combinatorial notion, which we call a *proximity structure*.

Different strict sequences often give rise to isomorphic pairs $(\mathbf{U}, \theta)$. If we fix a pair, then the corresponding sequences form a functor, and it is representable by a subscheme $F(\mathbf{U}, \theta)$ of $F^{(n)}$, which is Y-smooth with irreducible geometric fibers of a certain dimension. This statement is asserted by Theorem 3.10, which was inspired by Roé's Proposition 2.6 in [**25**].

Given another ordering θ', in Section 4 we construct a natural isomorphism

$$\Phi_{\theta,\theta'}\colon F(\mathbf{U}, \theta) \xrightarrow{\sim} F(\mathbf{U}, \theta').$$

It is easy to describe $\Phi_{\theta,\theta'}$ on geometric points. A geometric point of $F(\mathbf{U}, \theta)$ corresponds to a certain sequence of local rings in the function field of the appropriate geometric fiber of F/Y. Then $\theta' \circ \theta^{-1}$ yields a suitable permutation of these local rings, and so a geometric point of $F(\mathbf{U}, \theta')$. However, it is harder to work with arbitrary T-points. Most of the work is carried out in the proofs of Lemmas 4.1 and 4.2, and the work is completed in the proof of Proposition 4.3.

We easily derive two corollaries. Corollary 4.4 asserts that $\mathrm{Aut}(\mathbf{U})$ acts freely on $F(\mathbf{U}, \theta)$; namely, $\gamma \in \mathrm{Aut}(\mathbf{U})$ acts as $\Phi_{\theta,\theta'}$ where $\theta' := \theta \circ \gamma$. Corollary 4.5 asserts that $\Psi\colon F(\mathbf{U}, \theta)/\mathrm{Aut}(\mathbf{U})$ is Y-smooth with irreducible geometric fibers.

A different treatment of $F(\mathbf{U}, \theta)$ is given by A.-K. Liu in [**20**]. In Section 3 on pp. 400–401, he constructs $F^{(n)}$. In Subsection 4.3.1 on pp. 412–414, he discusses his version of an Enriques diagram, which he calls an "admissible graph." In Subsections 4.3.2, 4.4.1, and 4.4.2 on pp. 414–427, he constructs $F(\mathbf{U}, \theta)$, and proves it is smooth. In Subsection 4.5 on pp. 428–433, he constructs the action of $\mathrm{Aut}(\mathbf{U})$ on

$F(\mathbf{U}, \theta)$. Of course, he uses different notation; also, he doesn't represent functors. But, like the present authors, he was greatly inspired by Vainsencher's approach in [**28**] to enumerating the singular curves in a linear system on a smooth surface.

Our main result is Theorem 5.7. It concerns the Enriques diagram $\mathbf{D}$ obtained by equipping the vertices $V \in \mathbf{U}$ with weights m_V satisfying the *Proximity Inequality,* $m_V \geq \sum_{W \succ V} m_W$. We discuss the theory of such $\mathbf{D}$ in Section 2. Note that $\mathrm{Aut}(\mathbf{D}) \subset \mathrm{Aut}(\mathbf{U})$. Set $d := \sum_V \binom{m_V+1}{2}$. Theorem 5.7 asserts the existence of a universally injective map from the quotient to the Hilbert scheme

$$\Psi\colon F(\mathbf{U}, \theta)/\mathrm{Aut}(\mathbf{D}) \to \mathrm{Hilb}^d_{F/Y}\,.$$

Proposition 5.4 implies that Ψ factors into a finite map followed by an open embedding. So Ψ is an embedding in characteristic 0. However, in any positive characteristic, Ψ can be ramified everywhere; examples are given in Appendix B, whose content is due to Tyomkin. Nevertheless, according to Proposition 5.9, in the important case where every vertex of $\mathbf{D}$ is a root, Ψ is an embedding in any characteristic. Further, adding a nonroot does not necessarily mean there is a characteristic in which Ψ ramifies, as other examples in Appendix B show.

We construct Ψ via a relative version of the standard construction of the complete ideals on a smooth surface over a field, which grew out of Zariski's work in 1938; the standard theory is reviewed in Subsection 5.1. Now, a T-point of $F(\mathbf{U}, \theta)$ represents a sequence of blowing-ups $F_T^{(i)} \to F_T^{(i-1)}$ for $1 \leq i \leq n+1$. On the final blowup $F_T^{(n+1)}$, for each i, we form the preimage of the ith center $T^{(i)}$. This preimage is a divisor; we multiply it by $m_{\theta^{-1}(i)}$, and we sum over i. We get an effective divisor. We take its ideal, and push down to F_T. The result is an ideal, and it defines the desired T-flat subscheme of F_T. The flatness holds and the formation of the subscheme commutes with base change owing to the generalized property of exchange proved in Appendix A. Appendix A is of independent interest.

It is not hard to see that Ψ is injective on geometric points, and that its image is the subset $H(\mathbf{D}) \subset \mathrm{Hilb}^d_{F/Y}$ parameterizing complete ideals with diagram $\mathbf{D}$ on the fibers of F/Y. To prove that Ψ induces a finite map onto $H(\mathbf{D})$, we use a sort of valuative criterion; the work appears in Lemma 5.2 and Proposition 5.4. An immediate corollary, Corollary 5.5, asserts that $H(\mathbf{D})$ is locally closed. This result was proved for complex analytic varieties by Lossen [**21**, Prp. 2.19, p. 35] and for excellent schemes by Nobile and Villamayor [**23**, Thm. 2.6, p. 250]. Their proofs are rather different from each other and from ours.

In [**26**] and [**27**], Russell studies sets somewhat similar to the $H(\mathbf{D})$. They parameterize isomorphism classes of finite subschemes of F supported at one point.

In short, Section 2 treats weighted and unweighted Enriques diagrams and proximity structures. Section 3 treats sequences of arbitrarily near T-points. To certain ones, the *strict* sequences, we associate an unweighted Enriques diagram $\mathbf{U}$ and an ordering θ. Fixing $\mathbf{U}$ and θ, we obtain a functor, which we represent by a smooth Y-scheme $F(\mathbf{U}, \theta)$. Section 4 treats the variance in θ. We produce a free action on $F(\mathbf{U}, \theta)$ of $\mathrm{Aut}(\mathbf{U})$. Section 5 treats the Enriques diagram $\mathbf{D}$ obtained by equipping $\mathbf{U}$ with suitable weights. We construct a map Ψ from $F(\mathbf{U}, \theta)/\mathrm{Aut}(\mathbf{D})$ to $\mathrm{Hilb}_{F/Y}$, whose image is the locus $H(\mathbf{D})$ of complete ideals. We prove $H(\mathbf{D})$ is locally closed. Our main theorem asserts that Ψ is universally injective, and in fact, in characteristic 0, an embedding. Appendix A treats the generalized property of exchange used in constructing Ψ. Finally, Tyomkin's Appendix B treats a few

examples: in some, Ψ is ramified; in others, there's a nonroot, yet Ψ is unramified.

2. Enriques diagrams

In 1915, Enriques [**4**, IV.I, pp. 350–51] explained a way to represent the equisingularity type of a plane curve singularity by means of a directed graph: each vertex represents an arbitrarily near point, and each edge connects a vertex representing a point to a vertex representing a point in its first-order neighborhood; furthermore, the graph is equipped with a binary relation representing the "proximity" of arbitrarily near points. These graphs have, for a long time, been called *Enriques diagrams*, and in 2000, they were given a modern treatment by Casas in [**2**, Sec. 3.9, pp. 98–102].

Based in part on a preliminary edition of Casas' monograph, a more axiomatic treatment was given by the authors in [**16**, § 2], and this treatment is elaborated on here in Subsection 2.1. In this treatment, the vertices are weighted, and the number of vertices is minimized. When the diagram arises from a curve, the vertices correspond to the "essential points" as defined by Greuel et al. [**5**, Sec. 2.2], and the weights are the multiplicities of the points on the strict transforms. Casas' treatment is similar: the Proximity Inequality is always an equality, and the leaves, or extremal vertices, are of weight 1; so the rest of the weights are determined.

At times, it is convenient to work with unweighted diagrams. For this reason, Roé [**25**, §1], inspired by Casas, defined an "Enriques diagram" to be an unweighted graph, and he imposed five conditions, which are equivalent to our Laws of Proximity and of Succession. Yet another description of unweighted Enriques diagrams is developed below in Subsection 2.3 and Proposition 2.4 under the name of "proximity structure." This description facilitates the formal assignment, in Subsection 2.7, of an Enriques diagram to a plane curve singularity. Similarly, the description facilitates the assignment in Section 3 of the Enriques diagram associated to a strict sequence of arbitrarily near points.

At times, it is convenient to order the elements of the set underlying an Enriques diagram or underlying a proximity structure. This subject is developed in Subsections 2.2 and 2.3 and in Corollary 2.5. It plays a key role in the later sections.

Finally, in Subsection 2.6, we discuss several useful numerical characters. Three were introduced in [**15**, Sct. 2, p. 214], and are recalled here. Proposition 2.8 describes the change in one of the three when a singularity is blown up; this result is needed in [**17**]

2.1 (Enriques diagrams)**.** First, recall some general notions. In a directed graph, a vertex V is considered to be one of its own predecessors and one of its own successors. Its other predecessors and successors W are said to be *proper*. If there are no loops, then W is said to be *remote*, or *distant*, if there is a distinct third vertex lying between V and W; otherwise, then W is said to be *immediate*.

A *tree* is a directed graph with no loops; by definition, it has a single initial vertex, or *root,* and every other vertex has a unique immediate predecessor. A final vertex is called a *leaf*. A disjoint union of trees is called a *forest*.

Next, from [**16**, § 2], recall the definition of a minimal Enriques diagram. It is a finite forest $\mathbf{D}$ with additional structure. Namely, each vertex V is assigned a weight m_V, which is an integer at least 1. Also, the forest is equipped with a binary relation; if one vertex V is related to another U, then we say that V is *proximate* to U, and write $V \succ U$. If U is a remote predecessor of V, then we call V a *satellite*

of U; if not, then we say V is *free*. Thus a root is free, and a leaf can be either free or a satellite.

Elaborating on [**16**], call $\mathbf{D}$ an *Enriques diagram* if $\mathbf{D}$ obeys these three laws:

(Law of Proximity) *A root is proximate to no vertex. If a vertex is not a root, then it is proximate to its immediate predecessor and to at most one other vertex; the latter must be a remote predecessor. If one vertex is proximate to a second, and if a distinct third lies between the two, then it too is proximate to the second.*

(Proximity Inequality) *For each vertex* V,

$$m_V \geq \textstyle\sum_{W \succ V} m_W.$$

(Law of Succession) *A vertex may have any number of* free *immediate successors, but at most two immediate successors may be satellites, and they must be satellites of different vertices.*

Notice that, by themselves, the Law of Proximity and the Proximity Inequality imply that a vertex V has at most m_V immediate successors; so, although this property is included in the statement of the Law of Succession in [**16**, § 2], it is omitted here.

Recovering the notion in [**15**], call an Enriques diagram $\mathbf{D}$ *minimal* if $\mathbf{D}$ obeys the following fourth law:

(Law of Minimality) *There are only finitely many vertices, and every leaf of weight* 1 *is a satellite.*

In [**15**], the Law of Minimality did not include the present finiteness restriction; rather, it was imposed at the outset.

2.2 (Unweighted diagrams)**.** In [**25**, §1], Roé defines an Enriques diagram to be an *unweighted* finite forest that is equipped with a binary relation, called "proximity," that is required to satisfy five conditions. It is not hard to see that his conditions are equivalent to our Laws of Proximity and Succession. Let us call this combinatorial structure an *unweighted Enriques diagram*.

Let $\mathbf{U}$ be any directed graph on $n+1$ vertices. By an *ordering* of $\mathbf{U}$, let us mean a bijective mapping

$$\theta\colon \mathbf{U} \xrightarrow{\sim} \{0, \dots, n\}$$

such that, if one vertex V precedes another W, then $\theta(V) \leq \theta(W)$. Let us call the pair $(\mathbf{U}, \theta)$ an *ordered directed graph*.

An ordering θ need not be unique. Furthermore, if one exists, then plainly $\mathbf{U}$ has no loops. Conversely, if $\mathbf{U}$ has no loops — if it is a forest — then $\mathbf{U}$ has at least one ordering. Indeed, then $\mathbf{U}$ has a leaf L. Let $\mathbf{T}$ be the complement of L in $\mathbf{U}$. Then $\mathbf{T}$ inherits the structure of a forest. So, by induction on n, we may assume that $\mathbf{T}$ has an ordering. Extend it to $\mathbf{U}$ by mapping L to n.

Associated to any ordered unweighted Enriques diagram $(\mathbf{U}, \theta)$ is its *proximity matrix* (p_{ij}), which is the $n+1$ by $n+1$ lower triangular matrix defined by

$$p_{ij} := \begin{cases} 1, & \text{if } i = j; \\ -1, & \text{if } \theta^{-1}i \text{ is proximate to } \theta^{-1}j; \\ 0, & \text{otherwise.} \end{cases}$$

The transpose was introduced by Du Val in 1936, and he named it the "proximity matrix" in 1940; Lipman [**19**, p. 298] and others have followed suit. The definition

here is the one used by Roé [**25**] and Casas [**2**, p. 139].

Note that $(\mathbf{U}, \theta)$ is determined up to unique isomorphism by (p_{ij}).

2.3 (Proximity structure)**.** Let $\mathbf{U}$ be a finite set equipped with a binary relation. Call $\mathbf{U}$ a *proximity structure,* its elements *vertices,* and the relation *proximity* if the following three laws are obeyed:

(P1) *No vertex is proximate to itself; no two vertices are each proximate to the other.*

(P2) *Every vertex is proximate to at most two others; if to two, then one of the two is proximate to the other.*

(P3) *Given two vertices, at most one other is proximate to them both.*

A proximity structure supports a natural structure of directed graph. Indeed, construct an edge proceeding from one vertex V to another W whenever either W is proximate only to V or W is proximate both to V and U but V is proximate to U (rather than U to V). Of course, this graph may have loops; for example, witness a triangle with each vertex proximate to the one clockwise before it, and witness a pentagon with each vertex proximate to the two clockwise before it.

Let us say that a proximity structure is *ordered* if its vertices are numbered, say $V_0, \dots, V_n$, such that, if V_i is proximate to V_j, then $i > j$.

Proposition 2.4. *The unweighted Enriques diagrams sit in natural bijective correspondence with the proximity structures whose associated graphs have no loops.*

PROOF. First, take an unweighted Enriques diagram, and let's check that its proximity relation obeys Laws (P1) to (P3).

A vertex is proximate only to a proper successor; so no vertex is proximate to itself. And, if two vertices were proximate to one another, then each would succeed the other; so there would be a loop. Thus (P1) holds.

A root is proximate to no vertex. Every other vertex W is proximate to its immediate predecessor V and to at most one other vertex U, which must be a remote predecessor. Since an immediate predecessor is unique in a forest, V must lie between W and U; whence, V must be proximate to U. Thus (P2) holds.

Suppose two vertices W and X are each proximate to two others U and V. Say V is the immediate predecessor of W. Then U is a remote predecessor of W; so U precedes V. Hence V is also the immediate predecessor of X, and W is also a remote predecessor of X. Thus both W and X are immediate successors of V, and both are satellites of W; so the Law of Succession is violated. Thus (P3) holds.

Conversely, take a proximity structure whose associated graph has no loops. Plainly, a root is proximate to no vertex. Suppose a vertex W is not a root. Then W has an immediate predecessor V. Plainly, W is proximate to V. Plainly, W is proximate to at most one other vertex U, and if so, then V is proximate to U. Since U cannot also be proximate to V by (P1), it follows that V is the only immediate predecessor to W.

Every vertex is, therefore, preceded by a unique root. Plainly the connected component of each root is a tree. Thus the graph is a finite forest.

Returning to U, V, and W, we must show that U precedes W. Now, V is proximate to U. So V is not a root. Hence V has an immediate predecessor V'. If $V' = U$, then stop. If not, then V' is proximate to U owing to the definition of the associated graph, since V is proximate to U. Hence, similarly, V' has an immediate predecessor V''. If $V'' = U$, then stop. If not, then repeat the process.

Eventually, you must stop since the number of vertices is finite. Thus U precedes W. Furthermore, every vertex between U and W is proximate to U. Thus the Law of Proximity holds.

Continuing with U, V, and W, suppose that W' is a second immediate successor of V and that W' is also proximate to a vertex U'. Then $U' \neq U$ since at most one vertex can be proximate to both V and U by (P3).

Finally, suppose that W'' is a third immediate successor of V and that W'' is also proximate to a vertex U''. Then $U'' \neq U$ and $U'' \neq U'$ by what we just proved. But V is proximate to each of U, U', and U''. So (P2) is violated. Thus the Law of Succession holds, and the proof is complete. □

Corollary 2.5. *The ordered unweighted Enriques diagrams sit in natural bijective correspondence with the ordered proximity structures.*

Proof. Given an unweighted Enriques diagram, its proximity relation obeys Laws (P1) to (P3) by the proof of Proposition 2.4. And, if one vertex V is proximate to another W, then W precedes V. So $\theta(W) < \theta(V)$ for any ordering θ. Hence, if V is numbered $\theta(V)$ for every V, then the proximity structure is ordered.

Conversely, take an ordered proximity structure. The associated directed graph is, plainly, ordered too, and so has no loops. And, the Laws of Proximity and Succession hold by the proof of Proposition 2.4. Thus the corollary holds. □

2.6 (Numerical characters)**.** In [**15**, Sct. 2, p. 214], a number of numerical characters were introduced, and three of them are useful in the present work.

The first character makes sense for any unweighted Enriques diagram $\mathbf{U}$, although it was not defined in this generality before; namely, the *dimension* $\dim(\mathbf{U})$ is the number of roots plus the number of free vertices in $\mathbf{U}$, including roots. Of course, the definition makes sense for a weighted Enriques diagram $\mathbf{D}$; namely, the *dimension* $\dim(\mathbf{D})$ is simply the dimension of the underlying unweighted diagram.

The second and third characters make sense only for a weighted Enriques diagram $\mathbf{D}$; namely, the *degree* and *codimension* are defined by the formulas

$$\deg(\mathbf{D}) := \textstyle\sum_{V\in\mathbf{D}} \binom{m_V+1}{2};$$
$$\operatorname{cod}(\mathbf{D}) := \deg(\mathbf{D}) - \dim(\mathbf{D}).$$

It is useful to introduce a new character, the *type* of a vertex V of $\mathbf{U}$ or of $\mathbf{V}$. It is defined by the formula

$$\operatorname{type}(V) := \begin{cases} 0, & \text{if } V \text{ is a satellite;} \\ 1, & \text{if } V \text{ is a free vertex, but not a root;} \\ 2, & \text{if } V \text{ is a root.} \end{cases}$$

The type appears in the following two formulas:

$$\dim(\mathbf{A}) = \textstyle\sum_{V\in\mathbf{A}} \operatorname{type}(V); \tag{2.6.1}$$
$$\operatorname{cod}(\mathbf{A}) = \textstyle\sum_{V\in\mathbf{A}} \left[\binom{m_V+1}{2} - \operatorname{type}(V)\right]. \tag{2.6.2}$$

Formula 2.6.2 is useful because every summand is nonnegative in general and positive when $\mathbf{A}$ is a minimal Enriques diagram.

2.7 (The diagram of a curve)**.** Let C be a reduced curve lying on a smooth surface over an algebraically closed ground field; the surface need not be complete. In [**15**, Sec. 2, p. 213] and again in [**16**, Sec. 2, p. 72], we stated that, to C, we can

associate a minimal Enriques diagram **D**. (It represents the equisingularity type of C; this aspect of the theory is treated in [**2**, p. 99] and [**5**, pp. 543–4].) Here is more explanation about the construction of **D**.

First, form the configuration of all arbitrarily near points of the surface lying on all the branches of the curve through all its singular points. Say that one arbitrarily near point is *proximate* to a second if the first lies above the second and on the strict transform of the exceptional divisor of the blowup centered at the second. Then Laws (P1) to (P3) hold because three strict transforms never meet and, if two meet, then they meet once and transversely. Plainly, there are no loops. Hence, by Proposition 2.4, this configuration is an unweighted Enriques diagram.

Second, weight each arbitrarily near point with its multiplicity as a point on the strict transform of the curve. By the theorem of strong embedded resolution, all but finitely many arbitrarily near points are of multiplicity 1, and are proximate only to their immediate predecessors; prune off all the infinite unbroken successions of such points, leaving finitely many points. Then the Law of Minimality holds.

Finally, the Proximity Inequality holds for this well-known reason: the multiplicity of a point P' on a strict transform C' can be computed as an intersection number m on the blowup at P' of the surface containing C'; namely, m is the intersection number of the exceptional divisor and the strict transform of C'; the desired inequality results now from Noether's formula for m in terms of multiplicities of arbitrarily near points. (In [**2**, p. 83], the inequality is an equality, because no pruning is done.) Therefore, this weighted configuration is a minimal Enriques diagram. It is **D**.

Notice that, if K is any algebraically closed extension field of the ground field, then the curve C_K also has diagram **D**.

Proposition 2.8. *Let C be a reduced curve lying on a smooth surface over an algebraically closed field. Let **D** be the minimal Enriques diagram of C, and $P \in C$ a singular point of multiplicity m. Form the blowup of the surface at P, the exceptional divisor E, the proper transform C' of C, and the union $C'' := C' \cup E$. Let **D′** be the diagram of C', and **D″** that of C''. Then*

$$\operatorname{cod}(\mathbf{D}) - \operatorname{cod}(\mathbf{D}') \geq \tbinom{m+1}{2} - 2 \text{ and } \operatorname{cod}(\mathbf{D}) - \operatorname{cod}(\mathbf{D}'') = \tbinom{m}{2} - 2;$$

equality holds in the first relation if and only if P is an ordinary m-fold point.

PROOF. We obtain **D′** from **D** by deleting the root R corresponding to P and also all the vertices T that are of weight 1, proximate to R, and such that all successors of T are also (of weight 1 and) proximate to R (and so deleted too). Note that an immediate successor of R is free; if it is deleted, then it has weight 1, and if it is not deleted, then it becomes a root of **D′**. Also, by the Law of Proximity, an undeleted satellite of R becomes a free vertex of **D′**.

Let σ be the total number of satellites of R, and ρ the number of undeleted immediate successors. Then it follows from the Formula (2.6.2) that

$$\operatorname{cod}(\mathbf{D}) - \operatorname{cod}(\mathbf{D}') = \tbinom{m+1}{2} - 2 + \sigma + \rho.$$

Thus the asserted inequality holds, and it is an equality if and only if $\sigma = 0$ and $\rho = 0$. So it is an equality if P is an ordinary m-fold point.

Conversely, suppose $\sigma = 0$ and $\rho = 0$. Then R has no immediate successor V of weight 1 for the following reason. Otherwise, any immediate successor W of V is proximate to V by the Law of Proximity. So W has weight 1 by the Proximity

Inequality. Hence, by recursion, we conclude that V is succeeded by a leaf L of weight 1. So, by the Law of Minimality, L is a satellite. But $\sigma = 0$. Hence V does not exist. But $\rho = 0$. Hence R has no successors whatsoever. So P is an ordinary m-fold point.

Furthermore, we obtain $\mathbf{D}''$ from $\mathbf{D}$ by deleting R and by adding 1 to the weight of each T proximate to R. So a satellite of R becomes a free vertex of $\mathbf{D}''$, and an immediate successor of R becomes a root of $\mathbf{D}''$. In addition, for each smooth branch of C that is transverse at P to all the other branches, we adjoin an isolated vertex (root) of weight 2.

The number of adjoined vertices is $m - \sum_{T\succ R} m_T$. So, by Formula (2.6.2),

$$\begin{aligned}\operatorname{cod}(\mathbf{D}) - \operatorname{cod}(\mathbf{D}'') = \tbinom{m+1}{2} - 2 + \textstyle\sum_{T\succ R}\big[\tbinom{m_T+1}{2} - \operatorname{type}(T)\big] \\ - \textstyle\sum_{T\succ R}\big[\tbinom{m_T+2}{2} - (\operatorname{type}(T)+1)\big] - \big[m - \textstyle\sum_{T\succ R} m_T\big].\end{aligned}$$

The right hand side reduces to $\binom{m}{2} - 2$. So the asserted equality holds. □

3. Infinitely near points

Fix a smooth family of geometrically irreducible surfaces $\pi\colon F \to Y$. In this section, we study sequences of arbitrarily near T-points of F/Y. They are defined in Definition 3.3. Then Proposition 3.4 asserts that they form a representable functor. In essence, this result is due to Harbourne [**11**, Prp. I.2, p. 104], who identified the functor of points of the iterated blow-up that was introduced in [**14**, Sct. 4.1, p. 36] and is recalled in Definition 3.1.

In the second half of this section, we study a special kind of sequence of arbitrarily near T-points, the *strict* sequence, which is defined in Definition 3.5. To each strict sequence is associated a natural ordered unweighted Enriques diagram owing to Propositions 3.8 and 2.4. Finally, Theorem 3.10 asserts that the strict sequences with given diagram $(\mathbf{U}, \theta)$ form a functor, which is representable by a Y-smooth scheme with irreducible geometric fibers of dimension $\dim(\mathbf{U})$. This theorem was inspired by Roé's Proposition 2.6 in [**25**].

Definition 3.1. By induction on $i \geq 0$, let us define more families

$$\pi^{(i)}\colon F^{(i)} \to F^{(i-1)},$$

which are like $\pi\colon F \to Y$. Set $\pi^{(0)} := \pi$. Now, suppose $\pi^{(i)}$ has been defined. Form the fibered product of $F^{(i)}$ with itself over $F^{(i-1)}$, and blow up along the diagonal $\Delta^{(i)}$. Take the composition of the blowup map and the second projection to be $\pi^{(i+1)}$.

In addition, for $i \geq 1$, let $\varphi^{(i)}\colon F^{(i)} \to F^{(i-1)}$ be the composition of the blowup map and the first projection, and let $E^{(i)}$ be the exceptional divisor. Finally, set $\varphi^{(0)} := \pi$; so $\varphi^{(0)} = \pi^{(0)}$.

Lemma 3.2. *Both $\pi^{(i)}$ and $\varphi^{(i)}$ are smooth, and have geometrically irreducible fibers of dimension 2. Moreover, $E^{(i)}$ is equal, as a polarized scheme, to the bundle $\mathbb{P}(\Omega^1_{\pi^{(i-1)}})$ over $F^{(i-1)}$, where $\Omega^1_{\pi^{(i-1)}}$ is the sheaf of relative differentials.*

PROOF. The first assertion holds for $i = 0$ by hypothesis. Suppose it holds for i. Consider the fibered product formed in Definition 3.1. Then both projections are smooth, and have geometrically irreducible fibers of dimension 2; also, the diagonal $\Delta^{(i)}$ is smooth over both factors. It follows that the first assertion holds for $i+1$.

The second assertion holds because $\Omega^1_{\pi^{(i-1)}}$ is the conormal sheaf of $\Delta^{(i)}$. □

Definition 3.3. Let T be a Y-scheme. Given a sequence of blowups

$$F_T^{(n+1)} \xrightarrow{\varphi_T^{(n+1)}} F_T^{(n)} \to \cdots \to F_T' \xrightarrow{\varphi_T^{(1)}} F_T := F \times_Y T$$

whose ith center $T^{(i)} \subset F_T^{(i)}$ is the image of a section t_i of $F_T^{(i)}/T$ for $0 \le i \le n$, call $(t_0, \dots, t_n)$ a *sequence of arbitrarily near T-points* of F/Y.

For $1 \le i \le n+1$, denote the exceptional divisor in $F_T^{(i)}$ by $E_T^{(i)}$.

The following result is a version of Harbourne's Proposition I.2 in [**11**, p. 104].

Proposition 3.4 (Harbourne)**.** *As T varies, the sequences $(t_0, \dots, t_n)$ of arbitrarily near T-points of F/Y form a functor, which is represented by $F^{(n)}/Y$.*

Given $(t_0, \dots, t_n)$ and i, say $(t_0, \dots, t_i)$ is represented by $\tau_i \colon T \to F^{(i)}$. Then $\pi^{(i)}\tau_i = \tau_{i-1}$ where τ_{-1} is the structure map. Also, $F_T^{(i+1)} = F^{(i+1)} \times_{F^{(i)}} T$ where $F^{(i+1)} \to F^{(i)}$ is $\pi^{(i+1)}$; correspondingly, $t_i = (\tau_i, 1)$ and $E_T^{(i+1)} = E^{(i+1)} \times_{F^{(i)}} T$; moreover, $T^{(i)}$ is the scheme-theoretic image of $E_T^{(i+1)}$ under $\varphi_T^{(i+1)} \colon F_T^{(i+1)} \to F_T^{(i)}$. Finally, $\varphi_T^{(i+1)}$ is induced by $\varphi^{(i+1)}$, and $F_T^{(i+1)} \to T$ is induced by $\pi^{(i+1)}$.

PROOF. First, observe that, given a section of any smooth map $a \colon A \to B$, blowing up A along the section's image, C say, commutes with changing the base B. Indeed, let $\mathcal{I}$ be the ideal of C, and for each $m \ge 0$, consider the exact sequence

$$0 \to \mathcal{I}^{m+1} \to \mathcal{I}^m \to \mathcal{I}^m\big/\mathcal{I}^{m+1} \to 0.$$

Since a is smooth, $\mathcal{I}^m/\mathcal{I}^{m+1}$ is a locally free $\mathcal{O}_C$-module, so B-flat. Hence forming the sequence commutes with changing B. However, the blowup of A is just $\operatorname{Proj} \bigoplus_m \mathcal{I}^m$. Hence forming it commutes too.

Second, observe in addition that C is the scheme-theoretic image of the exceptional divisor, E say, of this blowup. Indeed, this image is the closed subscheme of C whose ideal is the kernel of the comorphism of the map $E \to C$. However, this comorphism is an isomorphism, because $E = \mathbb{P}(I/I^2)$ since a is smooth.

The first observation implies that the sequences $(t_0, \dots, t_n)$ form a functor, because, given any Y-map $T' \to T$, each induced map

$$F_T^{(i+1)} \times_T T' \to F_T^{(i)} \times_T T'$$

is therefore the blowing-up along the image of the induced section of $F_T^{(i)} \times_T T'/T'$.

To prove this functor is representable by $F^{(n)}/Y$, we must set up a functorial bijection between the sequences $(t_0, \dots, t_n)$ and the Y-maps $\tau_n \colon T \to F^{(n)}$. Of course, n is arbitrary. So $(t_0, \dots, t_i)$ then determines a Y-map $\tau_i \colon T \to F^{(i)}$, and correspondingly we want the remaining assertions of the proposition to hold as well.

So given $(t_0, \dots, t_n)$, let us construct appropriate Y-maps $\tau_i \colon T \to F^{(i)}$ for $-1 \le i \le n$. We proceed by induction on i. Necessarily, $\tau_{-1} \colon T \to Y$ is the structure map, and correspondingly, $F_T^{(0)} = F^{(0)} \times_{F^{(-1)}} T$ owing to the definitions.

Suppose we've constructed τ_{i-1}. Then $F_T^{(i)} = F^{(i)} \times_{F^{(i-1)}} T$. Set $\tau_i := p_1 t_i$ where $p_1 \colon F_T^{(i)} \to F^{(i)}$ is the projection. Then $\tau_{i-1} = \pi^{(i)}\tau_i$. Also, $t_i = (\tau_i, 1)$; so t_i is the pullback, under the map $(1, \tau_i)$, of the diagonal map of $F^{(i)}/F^{(i-1)}$. Therefore, owing to the first observation, $F_T^{(i+1)} = F^{(i+1)} \times_{(F^{(i)} \times F^{(i)})} F_T^{(i)}$ where $F_T^{(i)} \to F^{(i)} \times_{F^{(i-1)}} F^{(i)}$ is equal to $1 \times \tau_i$. Hence $F_T^{(i+1)} = F^{(i+1)} \times_{F^{(i)}} T$ where $F^{(i+1)} \to F^{(i)}$ is $\pi^{(i+1)}$. It follows formally that $E_T^{(i+1)} = E^{(i+1)} \times_{F^{(i)}} T$, that

$F_T^{(i+1)} \to F_T^{(i)}$ is induced by $\varphi^{(i+1)}$, and that $F_T^{(i+1)} \to T$ is induced by $\pi^{(i+1)}$.

By the second observation above, $T^{(i)}$ is the scheme-theoretic image of $E_T^{(i+1)}$.

Conversely, given a map $\tau_n \colon T \to F^{(n)}$, set $\tau_{i-1} := \pi^{(i)} \cdots \pi^{(n)} \tau_n$ for $0 \le i \le n$; so $\tau_{i-1} \colon T \to F^{(i-1)}$. Set $F_T^{(i)} := F^{(i)} \times_{F^{(i-1)}} T$ where the map $F^{(i)} \to F^{(i-1)}$ is $\pi^{(i)}$ for $0 \le i \le n+1$. Then τ_i defines a section t_i of $F_T^{(i)}/T$. Furthermore, blowing up its image yields the map $F_T^{(i+1)} \to F_T^{(i)}$ induced by $\varphi^{(i+1)}$, because, as noted above, forming the blowup along $\Delta^{(i)}$ commutes with changing the base via $1 \times \tau_i$. Thus $(t_0, \dots, t_n)$ is a sequence of arbitrarily near T-points of F/Y.

Plainly, for each T, we have set up the bijection we sought, and it is functorial in T. Since we have checked all the remaining assertions of the proposition, the proof is now complete. □

Definition 3.5. Given a sequence $(t_0, \dots, t_n)$ of arbitrarily near T-points of F/Y, let us call it *strict* if, for $0 \le i \le n$, the image $T^{(i)}$ of t_i satisfies the following i conditions, defined by induction on i. There are, of course, no conditions on $T^{(0)}$. Fix i, and suppose, for $0 \le j < i$, the conditions on $T^{(j)}$ are defined and satisfied.

The i conditions on $T^{(i)}$ involve the natural embeddings

$$e_T^{(j,i)} \colon E_T^{(j)} \hookrightarrow F_T^{(i)} \quad \text{for } 1 \le j \le i,$$

which we assume defined by induction; see the next paragraph. (The image $e_T^{(j,i)} E_T^{(j)}$ can be regarded as the "strict transform" of $E_T^{(j)}$ on $F_T^{(i)}$.) The jth condition requires $e_T^{(j,i)} E_T^{(j)}$ either (a) to be disjoint from $T^{(i)}$ or (b) to contain $T^{(i)}$ as a subscheme.

Define $e_T^{(i+1,\,i+1)}$ to be the inclusion. Now, for $1 \le j \le i$, we have assumed that $e_T^{(j,i)}$ is defined, and required that its image satisfy either (a) or (b). If (a) is satisfied, then the blowing-up $F_T^{(i+1)} \to F_T^{(i)}$ is an isomorphism on a neighborhood of $e_T^{(j,\,i)} E_T^{(j)}$, namely, the complement of $T^{(i)}$; so then $e_T^{(j,\,i)}$ lifts naturally to an embedding $e_T^{(j,\,i+1)}$. If (b) is satisfied, then $T^{(i)}$ is a relative effective divisor on the T-scheme $e^{(j,i)} E_T^{(j)}$, because $E_T^{(j)}$ and $T^{(i)}$ are flat over T, and the latter's fibers are effective divisors on the former's fibers, which are $\mathbb{P}^1$s; hence, then blowing up $e_T^{(j,i)} E_T^{(j)}$ along $T^{(i)}$ yields an isomorphism. But the blowup of $e_T^{(j,i)} E_T^{(j)}$ embeds naturally in $F_T^{(i)}$. Thus, again, $e_T^{(j,\,i)}$ lifts naturally.

Definition 3.6. Given a strict sequence $(t_0, \dots, t_n)$ of arbitrarily near T-points of F/Y, say that t_i is *proximate* to t_j if $j < i$ and $e^{(j+1,\,i)} E_T^{(j+1)}$ contains $T^{(i)}$.

Lemma 3.7. *Let $(t_0, \dots, t_n)$ be a strict sequence of arbitrarily near T-points of F/Y. Fix $n+1 \ge i \ge j \ge k \ge 1$. Then $\varphi_T^{(j+1)} \cdots \varphi_T^{(i)} e_T^{(k,i)} = e_T^{(k,j)}$, and $T^{(j-1)}$ is the scheme-theoretic image of $e_T^{(j,i)} E_T^{(j)}$ under $\varphi_T^{(j)} \cdots \varphi_T^{(i)}$. Set*

$$Z_T^{(i)} := e_T^{(k,i)} E_T^{(k)} \bigcap e_T^{(j,i)} E_T^{(j)}.$$

If $j > k$ and $Z_T^{(i)} \ne \emptyset$, then $\varphi_T^{(j)} \cdots \varphi_T^{(i)}$ induces an isomorphism $Z_T^{(i)} \xrightarrow{\sim} T^{(j-1)}$, and t_{j-1} is proximate to t_{k-1}; moreover, then $Z_T^{(i)}$ meets no $e_T^{(l,i)} E_T^{(l)}$ for $l \ne j, k$.

PROOF. The formula $\varphi_T^{(j+1)} \cdots \varphi_T^{(i)} e_T^{(k,i)} = e_T^{(k,j)}$ is trivial if $i = j$. It holds by construction if $i = j+1$. Finally, it follows by induction if $i > j+1$. With $k := j$, this formula implies that $E_T^{(j)}$ is the scheme-theoretic image of $e_T^{(j,i)} E_T^{(j)}$ under

$\varphi_T^{(j+1)} \cdots \varphi_T^{(i)}$; whence, Proposition 3.4 implies that $T^{(j-1)}$ is the scheme-theoretic image of $e_T^{(j,i)} E_T^{(j)}$ under $\varphi_T^{(j)} \cdots \varphi_T^{(i)}$.

Suppose $j > k$ and $Z_T^{(i)} \neq \emptyset$. Now, for any l such that $i \geq l \geq j$, both $e_T^{(k,l)} E_T^{(k)}$ and $e_T^{(j,l)} E_T^{(j)}$ are relative effective divisors on $F_T^{(l)}/T$, because they're flat and divisors on the fibers. Hence, on either of $e_T^{(k,l)} E_T^{(k)}$ and $e_T^{(j,l)} E_T^{(j)}$, their intersection $Z_T^{(l)}$ is a relative effective divisor, since each fiber of $Z_T^{(l)}$ is correspondingly a divisor. In fact, each nonempty fiber of $Z_T^{(l)}$ is a reduced point on a $\mathbb{P}^1$.

Since $\varphi_T^{(j+1)} \cdots \varphi_T^{(i)} e_T^{(j,i)} = e_T^{(j,j)}$ and since $e_T^{(j,j)}$ is the inclusion of $E_T^{(j)}$, which is the exceptional divisor of the blowing-up $\varphi^{(j)} \colon F^{(j)} \to F^{(j-1)}$ along $T^{(j-1)}$, the map $\varphi_T^{(j)} \cdots \varphi_T^{(i)}$ induces a proper map $e \colon Z_T^{(i)} \to T^{(j-1)}$. Since the fibers of e are isomorphisms, e is a closed embedding. So since $Z_T^{(i)}$ and $T^{(j-1)}$ are T-flat, e is an isomorphism onto an open and closed subscheme.

Since $\varphi_T^{(j)} \cdots \varphi_T^{(i)} e_T^{(k,i)} = e_T^{(k,\,j-1)}$, it follows that $e_T^{(k,\,j-1)} E_T^{(k)}$ contains a nonempty subscheme of $T^{(j-1)}$. So since $(t_0, \dots, t_n)$ is strict, $e_T^{(k,\,j-1)} E_T^{(k)}$ contains all of $T^{(j-1)}$ as a subscheme. Thus t_{j-1} is proximate to t_{k-1}.

It follows that $\varphi_T^{(j)}$ induces a surjection $Z_T^{(j)} \twoheadrightarrow T^{(j-1)}$. If $i = j$, then this surjection is just e, and so e is an isomorphism, as desired.

Suppose $i > j$. Then $Z_T^{(j)} \bigcap T^{(j)} = \emptyset$. Indeed, suppose not. Then both $e_T^{(k,j)} E_T^{(k)}$ and $E_T^{(j)}$ meet $T^{(j)}$. So since $(t_0, \dots, t_n)$ is strict, $Z_T^{(j)}$ contains $T^{(j)}$ as a closed subscheme. Both these schemes are T-flat, and their fibers are reduced points; hence, they coincide. It follows that $e_T^{(k,\,j+1)} E_T^{(k)}$ and $e_T^{(j,\,j+1)} E_T^{(j)}$ are disjoint on $F^{(j+1)}$. But these subschemes intersect in $Z_T^{(j+1)}$. And $Z_T^{(j+1)} \neq \emptyset$ since $Z_T^{(i)} \neq \emptyset$ and $Z_T^{(i)}$ maps into $Z_T^{(j+1)}$. We have a contradiction, so $Z_T^{(j)} \bigcap T^{(j)} = \emptyset$.

Therefore, $\varphi_T^{(j+1)}$ induces an isomorphism $Z_T^{(j+1)} \xrightarrow{\sim} Z_T^{(j)}$. Similarly, $\varphi_T^{(l+1)}$ induces an isomorphism $Z_T^{(l+1)} \xrightarrow{\sim} Z_T^{(l)}$ for $l = j, \dots, i-1$. Hence $\varphi_T^{(j)} \cdots \varphi_T^{(i)}$ induces an isomorphism $Z_T^{(i)} \xrightarrow{\sim} T^{(j-1)}$.

Finally, suppose $Z_T^{(i)}$ meets $e_T^{(l,i)} E_T^{(l)}$ for $l \neq j, k$, and let's find a contradiction. If $l < j$, then interchange l and j. Then, by the above, $T^{(j-1)}$ lies in both $e_T^{(k,\,j-1)} E_T^{(k)}$ and $e_T^{(l,\,j-1)} E_T^{(l)}$. Therefore, $T^{(j-1)}$ is equal to their intersection, because $T^{(j-1)}$ is flat and its fibers are equal to those of the intersection. It follows that $e_T^{(k,\,j)} E_T^{(k)}$ and $e_T^{(l,\,j)} E_T^{(l)}$ are disjoint on $F^{(j)}$. But both these subschemes contain the image of $Z_T^{(i)}$, which is nonempty. We have a contradiction, as desired. The proof is now complete. □

Proposition 3.8. *Let $(t_0, \dots, t_n)$ be a strict sequence of arbitrarily near T-points of F/Y. Equip the abstract ordered set of t_i with the relation of proximity of* Definition 3.6. *Then this set becomes an ordered proximity structure.*

PROOF. Law (P1) holds trivially.

As to (P2), suppose t_i is proximate to t_j and to t_k with $j > k$. Then $T^{(i)}$ lies in $e_T^{(k+1,i)} E_T^{(k+1)} \bigcap e_T^{(j+1,i)} E_T^{(j+1)}$. So Lemma 3.7 implies t_j is proximate to t_k. Furthermore, the lemma implies the intersection meets no $e_T^{(l+1,i)} E_T^{(l+1)}$ for $l \neq j, k$. So t_i is proximate to no third vertex t_l. Thus (P2) holds.

As to (P3), suppose t_i and t_j are each proximate to both t_k and t_l where

$i > j > k > l$. Given $p > k$, set $Z^{(p)} := e_T^{(l+1,p)} E_T^{(l+1)} \bigcap e_T^{(k+1,p)} E_T^{(k+1)}$. Then $T^{(i)} \subseteq Z^{(i)}$. Now, $Z^{(i)}$ is T-flat with reduced points as fibers by Lemma 3.7. But $T^{(i)}$ is a similar T-scheme. Hence $T^{(i)} = Z^{(i)}$. Similarly, $T^{(j)} := Z^{(j)}$.

Lemma 3.7 yields $\varphi_T^{(j+1)} \cdots \varphi_T^{(i)} e_T^{(m,i)} = e_T^{(m,j)}$ for $m = k, l$. So $\varphi_T^{(j+1)} \cdots \varphi_T^{(i)}$ carries $T^{(i)}$ into $T^{(j)}$. Now, this map is proper, and both $T^{(i)}$ and $T^{(j)}$ are T-flat with reduced points as fibers; hence, $T^{(i)} \xrightarrow{\sim} T^{(j)}$. It follows that

$$\varphi_T^{(j+2)} \cdots \varphi_T^{(i)} T^{(i)} \subseteq Z^{(j+1)} \subset \left(\varphi_T^{(j+1)}\right)^{-1} T^{(j)} = E_T^{(j+1)}.$$

Hence $Z^{(j+1)}$ meets $E_T^{(j+1)}$, contrary to Lemma 3.7. Thus (P3) holds. □

Definition 3.9. Let's say that a strict sequence of arbitrarily near T-points of F/Y has *diagram* $(\mathbf{U}, \theta)$ if $(\mathbf{U}, \theta)$ is isomorphic to the ordered unweighted Enriques diagram coming from Propositions 3.8 and 2.4.

The following result was inspired by Roé's Proposition 2.6 in [**25**].

Theorem 3.10. *Fix an ordered unweighted Enriques diagram $(\mathbf{U}, \theta)$ on $n+1$ vertices. Then the strict sequences of arbitrarily near T-points of F/Y with diagram $(\mathbf{U}, \theta)$ form a functor; it is representable by a subscheme $F(\mathbf{U}, \theta)$ of $F^{(n)}$, which is Y-smooth with irreducible geometric fibers of dimension $\dim(\mathbf{U})$.*

PROOF. If a strict sequence of arbitrarily near T-points has diagram $(\mathbf{U}, \theta)$, then, for any map $T' \to T$, the induced sequence of arbitrarily near T'-points plainly also has diagram $(\mathbf{U}, \theta)$. So the sequences with diagram $(\mathbf{U}, \theta)$ form a subfunctor of the functor of all sequences, which is representable by $F^{(n)}/Y$ by Proposition 3.4.

Suppose $n = 0$. Then $\mathbf{U}$ has just one vertex. So the two functors coincide, and both are representable by F, which is Y-smooth with irreducible geometric fibers of dimension 2. However, $2 = \dim(\mathbf{U})$. Thus the theorem holds when $n = 0$.

Suppose $n \geq 1$. Set $L := \theta^{-1} n$. Then L is a leaf. Set $\mathbf{T} := \mathbf{U} - L$. Then $\mathbf{T}$ inherits the structure of an unweighted Enriques diagram, and it is ordered by the restriction $\theta|\mathbf{T}$. By induction on n, assume the theorem holds for $(\mathbf{T}, \theta|\mathbf{T})$.

Set $G := F(\mathbf{T}, \theta|\mathbf{T}) \subset F^{(n-1)}$ and $H := \pi_n^{-1} G \subset F^{(n)}$. Then H represents the functor of sequences $(t_0, \ldots, t_n)$ of arbitrarily near T-points such that $(t_0, \ldots, t_{n-1})$ has diagram $(\mathbf{T}, \theta|\mathbf{T})$ since $\pi^{(i)} \tau_i = \tau_{i-1}$ by Proposition 3.4. Moreover, H is G-smooth with irreducible geometric fibers of dimension 2 by Lemma 3.2. And G is Y-smooth with irreducible geometric fibers of dimension $\dim(\mathbf{T})$ as the theorem holds for $(\mathbf{T}, \theta|\mathbf{T})$. Thus H is Y-smooth with irreducible geometric fibers of dimension $\dim(\mathbf{T}) + 2$.

Let $(h_0, \ldots, h_n)$ be the universal sequence of arbitrarily near H-points, and $H^{(i)} \subset F_H^{(i)}$ the image of h_i. We must prove that H has a largest subscheme S over which $(h_0, \ldots, h_n)$ restricts to a sequence with diagram $(\mathbf{U}, \theta)$; we must also prove that S is Y-smooth with irreducible geometric fibers of dimension $\dim(\mathbf{U})$.

But, $(h_0, \ldots, h_{n-1})$ has diagram $(\mathbf{T}, \theta|\mathbf{T})$. So $H^{(i)}$ satisfies the i conditions of Definition 3.5 for $i = 0, \ldots, n-1$. Hence S is defined simply by the n conditions on $H^{(n)}$: the jth requires $e_H^{(j,n)} E_H^{(j)}$ either (a) to be disjoint from $H^{(n)}$ or (b) to contain it as a subscheme; (b) applies if L is proximate to $\theta^{-1}(j-1)$, and (a) if not, according to Definition 3.6. Let P be the set of j for which (b) applies. Set

$$S := h_n^{-1}\left(\bigcap_{j \in P} e_H^{(j,n)} E_H^{(j)} - \bigcup_{j \notin P} e_H^{(j,n)} E_H^{(j)}\right)$$

Plainly, S is the desired largest subscheme of H.

It remains to analyze the geometry of S. First of all, $F_G^{(n)} = F^{(n)} \times_{F^{(n-1)}} G$ by Proposition 3.4; so $F_G^{(n)} = H$ since $H := \pi_n^{-1}G$. Also, $F_H^{(n)} = F^{(n)} \times_{F^{(n-1)}} H$ and $h_n = (\zeta_n, 1)$ where $\zeta_n \colon H \hookrightarrow F$, again by Proposition 3.4. Hence

$$F_H^{(n)} = F_G^{(n)} \times_G H = H \times_G H \text{ and } h_n = (1, 1).$$

Plainly, forming $e_T^{(j,n)}$ is functorial in T; whence, $e_H^{(j,n)} E_H^{(j)} = (e_G^{(j,n)} E_G^{(j)}) \times_G H$. Hence, $h_n^{-1} e_H^{(j,n)} E_H^{(j)} = e_G^{(j,n)} E_G^{(j)}$. Therefore,

$$S := \bigcap_{j \in P} e_G^{(j,n)} E_G^{(j)} - \bigcup_{j \notin P} e_G^{(j,n)} E_G^{(j)}.$$

There are three cases to analyze, depending on type(L). In any case,

$$\dim(\mathbf{T}) + \text{type}(L) = \dim(\mathbf{U})$$

owing to Formula 2.6.1. Furthermore, each $e_G^{(j,n)}$ is an embedding. So $e_G^{(j,n)} E_G^{(j)}$ has the form $\mathbb{P}(\Omega)$ for some locally free sheaf Ω of rank 2 on G by Lemma 3.2 and Proposition 3.4. Hence $e_G^{(j,n)} E_G^{(j)}$ is Y-smooth with irreducible geometric fibers of dimension $\dim(\mathbf{T}) + 1$.

Suppose type$(L) = 2$. Then L is a root. So P is empty, and by convention, the intersection $\bigcap_{j \in P} e_H^{(j,n)} E_H^{(j)}$ is all of H. So S is open in H, and maps onto Y. Hence S is Y-smooth with irreducible geometric fibers of dimension $\dim(H/Y)$, and

$$\dim(H/Y) = \dim(\mathbf{T}) + 2 = \dim(\mathbf{U}).$$

Thus the theorem holds in this case.

Suppose type$(L) = 1$. Then L is a free vertex, but not a root. So L has an immediate predecessor, M say. Set $m := \theta(M)$. Then $P = \{m\}$. So S is open in $e_G^{(m,n)} E_G^{(m)}$, and maps onto Y. Hence S is Y-smooth with irreducible geometric fibers of dimension $\dim(e_G^{(m,n)} E_G^{(m)}/Y)$, and

$$\dim(e_G^{(m,n)} E_G^{(m)}/Y) = \dim(\mathbf{T}) + 1 = \dim(\mathbf{U}).$$

Thus the theorem holds in this case too.

Finally, suppose type$(L) = 0$. Then L is a satellite. So L is proximate to two vertices: an immediate predecessor, M say, and a remote predecessor, R say. Set $m := \theta(M)$ and $r := \theta(R)$. Then $P = \{r, m\}$. Set $Z := e_G^{(r,n)} E_G^{(r)} \bigcap e_G^{(m,n)} E_G^{(m)}$. Then $Z \xrightarrow{\sim} G$ and Z meets no $e_G^{(j,n)} E_G^{(m)}$ with $j \notin P$ owing to Lemma 3.7, because $(h_0, \ldots, h_{n-1})$ is strict with diagram $(\mathbf{T}, \theta|\mathbf{T})$. Hence $S = Z$. Therefore, S is Y-smooth with irreducible geometric fibers of dimension $\dim(G/Y)$, and

$$\dim(G/Y) = \dim(\mathbf{T}) + 0 = \dim(\mathbf{U}).$$

Thus the theorem holds in this case too, and the proof is complete. □

4. Isomorphism and enlargement

Fix a smooth family of geometrically irreducible surfaces $\pi \colon F \to Y$. In this section, we study the scheme $F(\mathbf{U}, \theta)$ introduced in Theorem 3.10. First, we work out the effect of replacing the ordering θ by another one θ'. Then we develop, in our context, much of Roé's Subsections 2.1–2.3 in [**25**]; specifically, we study a certain closed subset $E(\mathbf{U}, \theta) \subset F^{(n)}$ containing $F(\mathbf{U}, \theta)$ set-theoretically. Notably, we

prove that, if the sets $F(\mathbf{U}', \theta')$ and $E(\mathbf{U}, \theta)$ meet, then $E(\mathbf{U}', \theta')$ lies in $E(\mathbf{U}, \theta)$; furthermore, $E(\mathbf{U}', \theta') = E(\mathbf{U}, \theta)$ if and only if $(\mathbf{U}, \theta) \cong (\mathbf{U}', \theta')$.

Proposition 4.3 below asserts that there is a natural isomorphism $\Phi_{\theta,\theta'}$ from $F(\mathbf{U}, \theta)$ to $F(\mathbf{U}, \theta')$. On geometric points, $\Phi_{\theta,\theta'}$ is given as follows. A geometric point with field K represents a sequence of arbitrarily near K-points $(t_0, \dots, t_n)$ of F/Y. To give t_i is the same as giving the local ring A_i of the surface $F_K^{(i)}$ at the K-point $T^{(i)}$, the image of t_i. Set $\alpha := \theta' \circ \theta^{-1}$. Then $\alpha i > \alpha j$ if t_i is proximate to t_j. So there is a unique sequence $(\tilde{t}_0, \dots, \tilde{t}_n)$ whose local rings $\widetilde{A}_j$ satisfy $A_i = \widetilde{A}_{\alpha i}$ in the function field of F_K. The sequences $(t_0, \dots, t_n)$ and $(\tilde{t}_0, \dots, \tilde{t}_n)$ correspond under $\Phi_{\theta,\theta'}$.

To construct $\Phi_{\theta,\theta'}$, we must work with a sequence $(t_0, \dots, t_n)$ of T-points for an arbitrary T. To do so, instead of the A_i, we use the transforms $e_T^{(i+1,n+1)} E_T^{(i+1)}$. The notation becomes more involved, and it is harder to construct $(\tilde{t}_0, \dots, \tilde{t}_n)$. We proceed by induction on n: we omit t_n, apply induction, and "reinsert" t_n as $\tilde{t}_{\alpha n}$. Most of the work is done in Lemma 4.2; the reinsertion is justified by Lemma 4.1.

Lemma 4.1. *Let $(\tilde{t}_0, \dots, \tilde{t}_{n-1})$ be a strict sequence of arbitrarily near T-points of F/Y, say with blowups $\widetilde{F}_T^{(i)}$ and so on. Fix l, and let $T^{(l)} \subset \widetilde{F}_T^{(l)}$ be the image of a section t_l of $\widetilde{F}_T^{(l)}/T$. Set $t_i := \tilde{t}_i$ for $0 \le i < l$, and assume the sequence $(t_0, \dots, t_l)$ is strict. Set $T_l := \widetilde{T}^{(l)}$ and $T_i := \widetilde{\varphi}_T^{(l+1)} \cdots \widetilde{\varphi}_T^{(i)} \widetilde{T}^{(i)}$ for $l < i < n$, and assume $T^{(l)}$ and the T_i are disjoint. Then $(t_0, \dots, t_l)$ extends uniquely to a strict sequence $(t_0, \dots, t_n)$, say with blowups $F_T^{(i)}$ and so on, such that t_l is a leaf and $F_T^{(l+1)} \times_{F_T^{(l)}} \widetilde{F}_T^{(i-1)} = F_T^{(i)}$ for $l < i \le n$. Furthermore, the diagram of $(t_0, \dots, t_n)$ induces that of $(\tilde{t}_0, \dots, \tilde{t}_{n-1})$.*

PROOF. Set $F_T^{(l)} := \widetilde{F}_T^{(l)}$; let $F_T^{(l+1)}$ be the blowup of $F_T^{(l)}$ with center $T^{(l)}$, and $E_T^{(l+1)}$ be its exceptional divisor. For $l < i \le n$, set $F_T^{(i+1)} := F_T^{(l+1)} \times_{F_T^{(l)}} \widetilde{F}_T^{(i)}$ and $T^{(i)} := F_T^{(l+1)} \times_{F_T^{(l)}} \widetilde{T}^{(i-1)}$. Now, $T^{(l)}$ and T_i are disjoint for $l \le i < n$. So $F_T^{(i+1)}$ is the blowup of $F_T^{(i)}$ with center $T^{(i)}$. Also, $T^{(i)}$ is the image of a section t_i of $F_T^{(i)}/T$. Moreover, since $(t_0, \dots, t_l)$ and $(\tilde{t}_0, \dots, \tilde{t}_{n-1})$ are strict sequences, it follows that $(t_0, \dots, t_n)$ is a strict sequence too. Furthermore, t_l is a leaf, and the diagram of $(t_0, \dots, t_n)$ induces that of $(\tilde{t}_0, \dots, \tilde{t}_{n-1})$. Plainly, $(t_0, \dots, t_n)$ is unique. □

Lemma 4.2. *Let α be a permutation of $\{0, \dots, n\}$. Let $(t_0, \dots, t_n)$ be a strict sequence of arbitrarily near T-points of F/Y. Assume that, if t_i is proximate to t_j, then $\alpha i > \alpha j$. Then there is a unique strict sequence $(\tilde{t}_0, \dots, \tilde{t}_n)$, say with blowups $\widetilde{F}_T^{(i)}$, exceptional divisors $\widetilde{E}_T^{(i)}$, and so on, such that $F_T^{(n+1)} = \widetilde{F}_T^{(n+1)}$ and $e_T^{(i,n+1)} E_T^{(i)} = \tilde{e}_T^{(\alpha' i, n+1)} \widetilde{E}_T^{(\alpha' i)}$ with $\alpha' i := \alpha(i-1)+1$ for $1 \le i \le n+1$; furthermore, t_i is proximate to t_j if and only if $\tilde{t}_{\alpha i}$ is proximate to $\tilde{t}_{\alpha j}$.*

PROOF. Assume $(\tilde{t}_0, \dots, \tilde{t}_n)$ exists. Let's prove, by induction on j, that both the sequence $(\tilde{t}_0, \dots, \tilde{t}_j)$ and the map $F_T^{(n+1)} \to \widetilde{F}_T^{(j+1)}$ are determined by the equality $F_T^{(n+1)} = \widetilde{F}_T^{(n+1)}$ and the $n+1$ equalities $e_T^{(i,n+1)} E_T^{(i)} = \tilde{e}_T^{(\alpha' i, n+1)} \widetilde{E}_T^{(\alpha' i)}$ where $1 \le i \le n+1$. If $j = -1$, there's nothing to prove. So suppose $j \ge 0$. Then $\widetilde{T}^{(j+1)}$ is determined as the scheme-theoretic image of $\tilde{e}_T^{(j+2,n+1)} \widetilde{E}_T^{(j+2)}$ by Lemma 3.7. So $\tilde{t}_{j+1}$ is determined. But then $\widetilde{F}_T^{(j+2)}$ is determined as the blowup

of $\widetilde{T}^{(j+1)}$. And $F_T^{(n+1)} \to \widetilde{F}_T^{(j+2)}$ is determined, because the preimage of $\widetilde{T}^{(j+1)}$ in $F_T^{(n+1)}$ is a divisor. Thus $(\tilde{t}_0, \dots, \tilde{t}_n)$ is unique.

To prove $(\tilde{t}_0, \dots, \tilde{t}_n)$ exists, let's proceed by induction on n. Assume $n = 0$. Then $\alpha = 1$. So plainly $\tilde{t}_0$ exists; just take $\tilde{t}_0 := t_0$.

So assume $n \geq 1$. Set $l := \alpha n$. Define a permutation β of $\{0, \dots, n-1\}$ by $\beta i := \alpha i$ if $\alpha i < l$ and $\beta i := \alpha i - 1$ if $\alpha i > l$.

Suppose t_i is proximate to t_j with $i < n$, and let us check that $\beta i > \beta j$. The hypothesis yields $\alpha i > \alpha j$. So if either $\alpha i < l$ or $\alpha j > l$, then $\beta i > \beta j$. Now, $\alpha i \neq l$ since $i < n$ and $l := \alpha n$. Similarly, $\alpha j \neq l$ since $j < i$ as t_i is proximate to t_j. But if $\alpha i > l$, then $\beta i := \alpha i - 1 \geq l$, and if $\alpha j < l$, then $\beta j := \alpha j < l$. Thus $\beta i > \beta j$.

Since $(t_0, \dots, t_{n-1})$ is strict, induction applies: there exists a strict sequence $(\hat{t}_0, \dots, \hat{t}_{n-1})$, say with blowups $\widehat{F}_T^{(i)}$ and so forth, such that $F_T^{(n)} = \widehat{F}_T^{(n)}$ and $e_T^{(i,n)} E_T^{(i)} = \hat{e}_T^{(\beta' i, n)} \widehat{E}_T^{(\beta' i)}$ with $\beta' i := \beta(i-1) + 1$ for $1 \leq i \leq n$; furthermore, t_i is proximate to t_j if and only if $\tilde{t}_{\beta i}$ is proximate to $\tilde{t}_{\beta j}$. Set $\tilde{t}_i := \hat{t}_i$ for $0 \leq i < l$.

Set $\tilde{t}_l := \widehat{\varphi}_T^{(l+1)} \cdots \widehat{\varphi}_T^{(n)} t_n$ and $\widetilde{T}^{(l)} := \widehat{\varphi}_T^{(l+1)} \cdots \widehat{\varphi}_T^{(n)} T^{(n)}$. Then $\tilde{t}_l$ is a section of $\widehat{F}_T^{(l)}/T$, and $\widetilde{T}^{(l)}$ is its image. Note that, if $T^{(n)}$ meets $\hat{e}_T^{(j,n)} \widehat{E}_T^{(j)}$ with $1 \leq j \leq n$, then $T^{(n)}$ is contained in $\hat{e}_T^{(j,n)} \widehat{E}_T^{(j)}$, because $\hat{e}_T^{(j,n)} \widehat{E}_T^{(j)} = e_T^{(i,n)} E_T^{(i)}$ for $i := \beta'^{-1} j$ and because $(t_0, \dots, t_n)$ is strict. Furthermore, if so, then $l > j$, because t_n is proximate to t_i, and so $\alpha n > \alpha i$, or $l > \beta i = j$; moreover, then $\widetilde{T}^{(l)}$ is contained in $\hat{e}_T^{(j,l)} \widehat{E}_T^{(j)}$, because the latter is equal to $\widehat{\varphi}_T^{(l+1)} \cdots \widehat{\varphi}_T^{(n)} \hat{e}_T^{(j,n)} \widehat{E}_T^{(j)}$ since $l > j$.

Suppose $\widetilde{T}^{(l)}$ meets $\hat{e}_T^{(k,l)} \widehat{E}_T^{(k)}$. Then $T^{(n)}$ meets $(\widehat{\varphi}_T^{(l+1)} \cdots \widehat{\varphi}_T^{(n)})^{-1} \hat{e}_T^{(k,l)} \widehat{E}_T^{(k)}$. So $T^{(n)}$ meets one of the latter's components, which is a $\hat{e}_T^{(j,n)} \widehat{E}_T^{(j)}$ for some j. Hence $T^{(l)} \subset \hat{e}_T^{(j,l)} \widehat{E}_T^{(j)}$, as was noted above. Now, the map $\hat{e}_T^{(j,n)} \widehat{E}_T^{(j)} \to \widehat{F}_T^{(l)}$ factors through $\widehat{E}_T^{(k)}$, and its image is $\hat{e}_T^{(j,l)} \widehat{E}_T^{(j)}$, as was noted above. So $\hat{e}_T^{(j,l)} \widehat{E}_T^{(j)}$ is contained in $\hat{e}_T^{(k,l)} \widehat{E}_T^{(k)}$; whence, the two coincide, since they are flat and coincide on the fibers over T. Thus $\widetilde{T}^{(l)}$ is contained in $\hat{e}_T^{(k,l)} \widehat{E}_T^{(k)}$. Hence, since $(\tilde{t}_0, \dots, \tilde{t}_{l-1})$ is strict, so is $(\tilde{t}_0, \dots, \tilde{t}_l)$. Furthermore, $T^{(n)}$ is contained in $\hat{e}_T^{(k,n)} \widehat{E}_T^{(k)}$. Thus if $\tilde{t}_l$ is proximate to $\tilde{t}_k$, then t_n is proximate to t_i for $i := \beta'^{-1} k$. Moreover, the converse follows from what was noted above.

Set $T_l := \widetilde{T}^{(l)}$ and $T_i := \widetilde{\varphi}_T^{(l+1)} \cdots \widetilde{\varphi}_T^{(i)} \widetilde{T}^{(i)}$ for $l < i < n$. Then $\widetilde{T}^{(l)}$ meets no $\widetilde{T}_i$, because, otherwise, $T^{(n)}$ would meet $(\widehat{\varphi}_T^{(l+1)} \cdots \widehat{\varphi}_T^{(n)})^{-1} \widetilde{T}_{i+1}$, and so $T^{(n)}$ would meet some $\hat{e}_T^{(j,n)} \widehat{E}_T^{(j)}$ with $l < j$, contrary to the note above. So Lemma 4.1 implies $(\tilde{t}_0, \dots, \tilde{t}_l)$ extends to a strict sequence $(\tilde{t}_0, \dots, \tilde{t}_n)$ such that $\tilde{t}_l$ is a leaf and $\widetilde{F}_T^{(l+1)} \times_{\widetilde{F}_T^{(l)}} \widehat{F}_T^{(i)} = \widetilde{F}_T^{(i+1)}$ for $l < i \leq n$; furthermore, the diagram of $(\tilde{t}_0, \dots, \tilde{t}_n)$ induces that of $(\hat{t}_0, \dots, \hat{t}_{n-1})$.

Therefore, t_i is proximate to t_j if and only if $\tilde{t}_{\alpha i}$ is proximate to $\tilde{t}_{\alpha j}$ for $0 \leq i \leq n$, because t_i is proximate to t_j if and only if $t_{\beta i}$ is proximate to $t_{\beta j}$ for $0 \leq i < n$ and because t_n is proximate to t_j if and only if $\tilde{t}_l$ is proximate to $\tilde{t}_k$ for $k := \beta' j$.

Recall from above that $F_T^{(n)} = \widehat{F}_T^{(n)}$ and $\widetilde{F}_T^{(l+1)} \times_{\widetilde{F}_T^{(l)}} \widehat{F}_T^{(n)} = \widetilde{F}_T^{(n+1)}$. But this product is equal to the blowup of $\widehat{F}_T^{(n)}$ along $T^{(n)}$ since $\widetilde{T}^{(l)}$ meets no $\widehat{T}_i$. And the blowup of $F_T^{(n)}$ along $T^{(n)}$ is $F_T^{(n+1)}$. Thus $F_T^{(n+1)} = \widetilde{F}_T^{(n+1)}$.

Recall $e_T^{(i,n)} E_T^{(i)} = \hat{e}_T^{(\beta' i, n)} \widehat{E}_T^{(\beta' i)}$ for $1 \leq i \leq n$. Hence, $e_T^{(i,n+1)} E_T^{(i)}$ is equal to the image of a natural embedding of $\hat{e}_T^{(\beta' i, n)} \widehat{E}_T^{(\beta' i)}$ in $\widetilde{F}_T^{(n+1)}$. In turn, this image is

equal to $\widetilde{e}_T^{(\alpha' i,n)}\widetilde{E}_T^{(\alpha' i)}$ since $\widetilde{T}^{(l)}$ meets no $\widehat{T}_i$. Similarly, $E_T^{(n+1)} = \tilde{e}_T^{(l+1,n+1)}\widetilde{E}_T^{(l+1)}$. Thus $e_T^{(i,n+1)}E_T^{(i)} = \tilde{e}_T^{(\alpha' i,n+1)}\widetilde{E}_T^{(\alpha' i)}$ for $1 \le i \le n+1$. □

Proposition 4.3. *Fix an unweighted Enriques diagram* $\mathbf{U}$. *Then, given two orderings* θ *and* θ', *there exists a natural isomorphism*

$$\Phi_{\theta,\theta'}\colon F(\mathbf{U},\theta) \xrightarrow{\sim} F(\mathbf{U},\theta').$$

Furthermore, $\Phi_{\theta,\theta} = 1$, *and* $\Phi_{\theta',\theta''} \circ \Phi_{\theta,\theta'} = \Phi_{\theta,\theta''}$ *for any third ordering* θ''.

PROOF. Say $\mathbf{U}$ has $n+1$ vertices. Set $\alpha := \theta' \circ \theta^{-1}$. Then α is a permutation of $\{0,\dots,n\}$.

Each T-point of $F(\mathbf{U},\theta)$ corresponds to a strict sequence $(t_0,\dots,t_n)$ owing to Theorem 3.10. For each i, say t_i corresponds to the vertex V_i of $\mathbf{U}$. Then $\theta(V_i) = i$, and if t_i is proximate to t_j, then V_i is proximate to V_j. So $\theta'(V_i) > \theta'(V_j)$ since θ' is an ordering. Hence $\alpha i > \alpha j$.

Therefore, by Lemma 4.2, there is a unique strict sequence $(\tilde{t}_0,\dots,\tilde{t}_n)$ such that t_i is proximate to t_j if and only if $\tilde{t}_{\alpha i}$ is proximate to $\tilde{t}_{\alpha j}$. Plainly $(\tilde{t}_0,\dots,\tilde{t}_n)$ has $(\mathbf{U},\theta')$ as its diagram. Hence $(\tilde{t}_0,\dots,\tilde{t}_n)$ corresponds to a T-point of $F(\mathbf{U},\theta')$ owing to Theorem 3.10.

Due to uniqueness, sending $(t_0,\dots,t_n)$ to $(\tilde{t}_0,\dots,\tilde{t}_n)$ gives a well-defined map of functors. It is represented by a map $\Phi_{\theta,\theta'}\colon F(\mathbf{U},\theta) \to F(\mathbf{U},\theta')$. Again due to uniqueness, $\Phi_{\theta,\theta} = 1$ and $\Phi_{\theta',\theta''} \circ \Phi_{\theta,\theta'} = \Phi_{\theta,\theta''}$ for any θ''. So $\Phi_{\theta',\theta} \circ \Phi_{\theta,\theta'} = 1$ and $\Phi_{\theta,\theta'} \circ \Phi_{\theta',\theta} = 1$. Thus $\Phi_{\theta,\theta'}$ is an isomorphism, and the proposition is proved. □

Corollary 4.4. *Fix an ordered unweighted Enriques diagram* $(\mathbf{U},\theta)$. *Then there is a natural free right action of* $\mathrm{Aut}(\mathbf{U})$ *on* $F(\mathbf{U},\theta)$; *namely,* $\gamma \in \mathrm{Aut}(\mathbf{U})$ *acts as* $\Phi_{\theta,\theta'}$ *where* $\theta' := \theta \circ \gamma$.

PROOF. Let $V \in \mathbf{U}$ be a vertex that precedes another W. Then $\gamma(V)$ precedes $\gamma(W)$ because $\gamma \in \mathrm{Aut}(\mathbf{U})$. Since θ is an ordering, $\theta(\gamma(V)) \le \theta(\gamma(W))$. Hence $\theta'(V) \le \theta'(W)$. Thus θ' is an ordering.

So there is a natural isomorphism $\Phi_{\theta,\theta'}\colon F(\mathbf{U},\theta) \xrightarrow{\sim} F(\mathbf{U},\theta')$ by Proposition 4.3. Now, γ induces an isomorphism of ordered unweighted Enriques diagrams from $(\mathbf{U},\theta')$ to $(\mathbf{U},\theta)$; hence, $F(\mathbf{U},\theta')$ and $F(\mathbf{U},\theta)$ are the same subscheme of $F^{(n)}$, and $\Phi_{\theta,\theta'}$ is an automorphism of $F(\mathbf{U},\theta)$.

Note that, if $\gamma = 1$, then $\theta' = \theta$; moreover, $\Phi_{\theta,\theta} = 1$.

Given $\delta \in \mathrm{Aut}(\mathbf{U})$, set $\theta'' := \theta' \circ \delta$ and $\theta^* := \theta \circ \delta$. Then γ also induces an isomorphism from $(\mathbf{U},\theta'')$ to $(\mathbf{U},\theta^*)$, and so $\Phi_{\theta',\theta''}$ and Φ_{θ,θ^*} coincide. Now, $\Phi_{\theta',\theta''} \circ \Phi_{\theta,\theta'} = \Phi_{\theta,\theta''}$. Thus $\mathrm{Aut}(\mathbf{U})$ acts on $F(\mathbf{U},\theta)$, but it acts on the right because θ'' is equal to $\theta \circ (\gamma\delta)$, not to $\theta \circ (\delta\gamma)$.

Suppose γ has a fixed T-point. Then the T-point is fixed under $\Phi_{\theta,\theta'}$. Now, we defined $\Phi_{\theta,\theta'}$ by applying Lemma 4.2 with $\alpha := \theta' \circ \theta^{-1}$. And the lemma asserts that α is determined by its action on the $e_T^{(i,n+1)}E_T^{(i)}$. But this action is trivial because the T-point is fixed. Hence $\alpha = 1$. But $\alpha = \theta \circ \gamma \circ \theta^{-1}$. Therefore, $\gamma = 1$. Thus the action of $\mathrm{Aut}(\mathbf{U})$ is free, and the corollary is proved. □

Corollary 4.5. *Fix an ordered unweighted Enriques diagram* $(\mathbf{U},\theta)$, *and let* $G \subset \mathrm{Aut}(\mathbf{U})$ *be a subgroup. Then the quotient* $F(\mathbf{U},\theta)/G$ *is* Y*-smooth with irreducible geometric fibers of dimension* $\dim(\mathbf{U})$.

PROOF. The action of G on $F(\mathbf{U}, \theta)$ is free by Corollary 4.4. So G defines a finite flat equivalence relation on $F(\mathbf{U}, \theta)$. Therefore, the quotient exists, and the map $F(\mathbf{U}, \theta) \to F(\mathbf{U}, \theta)/G$ is faithfully flat. Now, $F(\mathbf{U}, \theta)$ is Y-smooth with irreducible geometric fibers of dimension $\dim(\mathbf{U})$ by Theorem 3.10; so $F(\mathbf{U}, \theta)/G$ is too. □

Definition 4.6. For $1 \le i \le j$, set $E^{(i,i)} := E^{(i)}$ and

$$E^{(i,j)} := (\varphi^{(i+1)} \cdots \varphi^{(j)})^{-1} E^{(i)} \text{ if } i < j.$$

Given an ordered unweighted Enriques diagram $(\mathbf{U}, \theta)$ on $n+1$ vertices, say with proximity matrix (p_{ij}), let $E(\mathbf{U}, \theta) \subset F^{(n)}$ be the set of scheme points $\mathbf{t}$ such that, on the fiber $F_{\mathbf{t}}^{(n+1)}$, for $1 \le k \le n$, the divisors $\sum_{i=k}^{n+1} p_{ik} E_{\mathbf{t}}^{(i,n+1)}$ are effective.

Proposition 4.7. *Let $(\mathbf{U}, \theta)$ be an ordered unweighted Enriques diagram. Then $E(\mathbf{U}, \theta)$ is closed and contains $F(\mathbf{U}, \theta)$ set-theoretically.*

PROOF. Say $\mathbf{U}$ has $n+1$ vertices. Fix $\mathbf{t} \in F^{(n)}$ and $1 \le k \le n$. If $\mathbf{t} \in F(\mathbf{U}, \theta)$, then, as is easy to see by induction on j for $k \le j \le n$, the divisor $\sum_{i=k}^{j+1} p_{ik} E_{\mathbf{t}}^{(i,j+1)}$ is equal to the strict transform on $F_{\mathbf{t}}^{(j+1)}$ of $E_{\mathbf{t}}^{(k)}$, in other words, to $e_T^{(k,j+1)} E_T^{(k)}$ where $T := \operatorname{Spec} \kappa(\mathbf{t})$. Hence $E(\mathbf{U}, \theta)$ contains $F(\mathbf{U}, \theta)$.

Set $\widetilde{E}^{(k)} := \sum_{i=k}^{n+1} p_{ik} E^{(i,n+1)}$. Then $\mathrm{h}^0\big(F_{\mathbf{t}}^{(n+1)}, \mathcal{O}(\widetilde{E}_{\mathbf{t}}^{(k)})\big) \le 1$ for any $\mathbf{t}$, and equality holds if and only if $\mathbf{t} \in F(\mathbf{U}, \theta)$, as the following essentially standard argument shows. Plainly, it suffices to show that, if $\widetilde{E}_{\mathbf{t}}^{(k)}$ is linearly equivalent to an effective divisor D, then $\widetilde{E}_{\mathbf{t}}^{(k)} = D$.

Let H be the preimage on $F_{\mathbf{t}}^{(n+1)}$ of an ample divisor on $F_{\mathbf{t}}$. Then the intersection number $\widetilde{E}_{\mathbf{t}}^{(k)} \cdot H$ vanishes by the projection formula because each component of $\widetilde{E}_{\mathbf{t}}^{(k)}$ maps to a point in $F_{\mathbf{t}}$. So $D \cdot H$ vanishes too. Hence each component of D must also map to a point in $F_{\mathbf{t}}$ because D is effective and H is ample. Hence D is some linear combination of the $E_{\mathbf{t}}^{(i,n+1)}$ because they form a basis of the group of divisors whose components each map to a point. Furthermore, the combining coefficients must be the p_{ik} because these coefficients are given by the intersection numbers with the $E_{\mathbf{t}}^{(i,n+1)}$. Thus $\widetilde{E}_{\mathbf{t}}^{(k)} = D$.

Thus $E(\mathbf{U}, \theta)$ is the set of $\mathbf{t} \in F^{(n)}$ such that $\mathrm{h}^0\big(F_{\mathbf{t}}^{(n+1)}, \mathcal{O}(\widetilde{E}_{\mathbf{t}}^{(k)})\big) \ge 1$ for all k. Hence $E(\mathbf{U}, \theta)$ is closed by semi-continuity [**7**, Thm. (7.7.5), p. 67]. □

Proposition 4.8. *Let $(\mathbf{U}, \theta)$ and $(\mathbf{U}', \theta')$ be two ordered unweighted Enriques diagrams on $n+1$ vertices, and let $\mathbf{P}$ and $\mathbf{P}'$ be their proximity matrices. Then the following conditions are equivalent:*

(1) *The sets $F(\mathbf{U}', \theta')$ and $E(\mathbf{U}, \theta)$ meet.*
(2) *The set $E(\mathbf{U}', \theta')$ is contained in the set $E(\mathbf{U}, \theta)$.*
(3) *The matrix $\mathbf{P}'^{-1}\mathbf{P}$ only has nonnegative entries.*

Furthermore, $E(\mathbf{U}', \theta') = E(\mathbf{U}, \theta)$ if and only if $(\mathbf{U}, \theta) \cong (\mathbf{U}', \theta')$.

PROOF. Fix $\mathbf{t} \in F^{(n)}$, and define two sequences of divisors on $F_{\mathbf{t}}^{(n+1)}$ by these matrix equations:

$$(\widetilde{E}_{\mathbf{t}}^{(1)}, \dots, \widetilde{E}_{\mathbf{t}}^{(n+1)}) = (E_{\mathbf{t}}^{(1,n+1)}, \dots, E_{\mathbf{t}}^{(n+1,n+1)})\mathbf{P};$$
$$(\widetilde{E}_{\mathbf{t}}^{(1)\prime}, \dots, \widetilde{E}_{\mathbf{t}}^{(n+1)\prime}) = (E_{\mathbf{t}}^{(1,n+1)}, \dots, E_{\mathbf{t}}^{(n+1,n+1)})\mathbf{P}'.$$

These two equations imply the following one:

$$(\widetilde{E}_{\mathbf{t}}^{(1)}, \dots, \widetilde{E}_{\mathbf{t}}^{(n+1)}) = (\widetilde{E}_{\mathbf{t}}^{(1)\prime}, \dots, \widetilde{E}_{\mathbf{t}}^{(n+1)\prime})\mathbf{P}'^{-1}\mathbf{P}; \tag{4.8.1}$$

in other words, $\widetilde{E}_{\mathbf{t}}^{(j)} = \sum_{k=1}^{n+1} q_{kj}\widetilde{E}_{\mathbf{t}}^{(k)\prime}$ where say $(q_{kj}) := \mathbf{P}'^{-1}\mathbf{P}$.

Suppose $\mathbf{t} \in F(\mathbf{U}', \theta')$. Then $\widetilde{E}_{\mathbf{t}}^{(k)\prime}$ is the proper transform on $F_{\mathbf{t}}^{(n+1)}$ of $E_{\mathbf{t}}^{(k)}$, as we noted at the beginning of the proof of Proposition 4.7. So the $\widetilde{E}_{\mathbf{t}}^{(k)\prime}$ form a basis of the group of divisors whose components each map to a point in $F_{\mathbf{t}}$. Hence, by (4.8.1), if $\widetilde{E}_{\mathbf{t}}^{(j)}$ is effective, then $q_{kj} \geq 0$ for all k. Thus (1) implies (3).

Suppose $\mathbf{t} \in E(\mathbf{U}', \theta')$. Then $\widetilde{E}_{\mathbf{t}}^{(k)\prime}$ is effective. Suppose too $q_{kj} \geq 0$ for all k, j. Then $\widetilde{E}_{\mathbf{t}}^{(j)}$ is effective for all j by (4.8.1). So $\mathbf{t} \in E(\mathbf{U}, \theta)$. Thus (3) implies (2).

By Proposition 4.7, $E(\mathbf{U}, \theta)$ contains $F(\mathbf{U}, \theta)$. By Theorem 3.10, $F(\mathbf{U}, \theta)$ is nonempty. Thus (2) implies (1). So (1), (2), and (3) are equivalent.

Furthermore, suppose $E(\mathbf{U}', \theta') = E(\mathbf{U}, \theta)$. Then both $\mathbf{P}'^{-1}\mathbf{P}$ and $\mathbf{P}^{-1}\mathbf{P}'$ have nonnegative entries since (2) implies (3). But each matrix is the inverse of the other, and both are lower triangular. Hence both are the identity. So $\mathbf{P}' = \mathbf{P}$; whence, $(\mathbf{U}, \theta) \cong (\mathbf{U}', \theta')$. The converse is obvious. Thus the proposition is proved. □

5. The Hilbert scheme

Fix a smooth family of geometrically irreducible surfaces $\pi\colon F \to Y$. In this section, we prove our main result, Theorem 5.7. It asserts that, given an Enriques diagram $\mathbf{D}$ and an ordering θ, there exists a natural map Ψ from the quotient $F(\mathbf{D}, \theta)/\mathrm{Aut}(\mathbf{D})$ into the Hilbert scheme $\mathrm{Hilb}^d_{F/Y}$ with $d := \deg \mathbf{D}$ and with $F(\mathbf{D}, \theta) := F(\mathbf{U}, \theta)$ where $\mathbf{U}$ is the unweighted diagram underlying $\mathbf{D}$.

The quotient $F(\mathbf{D}, \theta)/\mathrm{Aut}(\mathbf{D})$ parameterizes the strict sequences of arbitrarily near points of F/Y with diagram $(\mathbf{U}, \theta)$, up to automorphism of $\mathbf{D}$. The image of Ψ parameterizes the (geometrically) complete ideals of F/Y with diagram $\mathbf{D}$. The map Ψ is universally injective. In fact, Ψ is an embedding in characteristic 0. However, in positive characteristic, Ψ can be purely inseparable; Appendix B discusses examples found by Tyomkin.

We close this section with Proposition 5.9, which addresses the important special case where every vertex of $\mathbf{D}$ is a root; here, Ψ is an embedding in any characteristic. Further, other examples in Appendix B show that Ψ can remain an embedding even after a nonroot is added.

5.1 (Geometrically complete ideals)**.** Let K be a field, $(t_0, \dots, t_n)$ a sequence of arbitrarily near K-points of F/Y. Since $\mathrm{Spec}(K)$ consists of a single reduced point, the sequence is strict. Let $(\mathbf{U}, \theta)$ be its diagram in the sense of Definition 3.9.

Suppose $\mathbf{U}$ underlies an Enriques diagram $\mathbf{D}$, say with weights m_V for $V \in \mathbf{U}$. Using the divisors $E_K^{(i,n+1)}$ on $F_K^{(n+1)}$ of Definition 4.6, set

$$E_K := \textstyle\sum_V m_V E_K^{(\theta(V)+1,\, n+1)} \text{ and } \mathcal{L}_K := \mathcal{O}_{F_K^{(n+1)}}(-E_K).$$

Given $V \in \mathbf{U}$, set $j := \theta(V)$ and $D_V := e_K^{(j+1,n+1)} E_K^{(j+1)}$. Inspired by Lipman's remark [**19**, p. 306], let's compute the intersection number $-(E_K \cdot D_V)$, that is, $\deg(\mathcal{L}|D_V)$. Plainly, $(E_K^{(j+1,\, n+1)} \cdot D_V) = -1$. And, for $W \neq V$, plainly $(E_K^{(\theta(W)+1,\, n+1)} \cdot D_V)$ is equal to 1 if $W \succ V$, and to 0 if not. Hence $-(E_K \cdot D_V)$ is equal to $m_V - \sum_{W \succ V} m_W$, which is at least 0 by the Proximity Inequality.

Set $\varphi_K := \varphi_K^{(1)} \cdots \varphi_K^{(n+1)}$, and form $\mathcal{I} := \varphi_{K*}\mathcal{L}_K$ on F_K. Then $\mathcal{I}$ is a complete

ideal, one that is integrally closed; also, $\mathcal{I}\mathcal{O}_{F_K^{(n+1)}} = \mathcal{L}_K$ and $R^q\varphi_{K*}\mathcal{L}_K = 0$ for $q \geq 1$. These three statements hold since $(E_K \cdot D_V) \leq 0$ for all V and, as is well known, $R^q\varphi_{K*}\mathcal{O}_{F_K^{(n+1)}} = 0$ for $q \geq 1$; see Lipman's discussion [**18**, §18, p. 238] and his Part (ii) of [**18**, Thm. (12.1), p. 220]; also see Deligne's Théorème 2.13 [**3**, p. 22]. Furthermore,

$$\dim_K \mathrm{H}^0(\mathcal{O}_{F_K}/\mathcal{I}) = d \quad \text{where } d := \deg \mathbf{D}.$$

This formula is a modern version of Enriques' formula [**4**, Vol. II, p. 426]; it was proved in different ways independently by Hoskin [**12**, 5.2, p. 85], Deligne [**3**, 2.13, p. 22], and Casas [**1**, 6.1, p. 438]; Hoskin and Deligne worked in greater generality, Casas worked over $\mathbb{C}$.

The m_V are determined by $\mathcal{I}$ because the divisors $E_K^{(i,n+1)}$ are numerically independent; their intersection numbers with divisors are defined because they are complete. The m_V may be found as follows. Let $\mathcal{P}$ be the ideal of the image $T^{(0)}$ of t_0, which is a K-point of F_K. Let m be the largest integer such that $\mathcal{P}^m \supset \mathcal{I}$. Then $m = m_V$ where $V := \theta^{-1}(0)$, since $\mathcal{P}\mathcal{O}_{F_K^{(n+1)}} = \mathcal{O}_{F_K^{(n+1)}}(-E_K^{(1,n+1)})$. Note in passing that $\mathcal{P}$ is a minimal prime of $\mathcal{I}$ since $m_V \geq 1$.

The remaining m_W can be found by recursion. Indeed, on $F_K^{(1)}$, form the ideal $\mathcal{I}' := \mathcal{I}\mathcal{O}(m_V E^{(1)})$. Then $\mathcal{I}'$ is the direct image from $F_K^{(n+1)}$ of $\mathcal{O}(-E'_K)$ where $E'_K := \sum_{W \neq V} m_W E_K^{(\theta(W)+1,\, n+1)}$. Hence $\mathcal{I}'$ is the complete ideal associated to the sequence $(t_1, \ldots, t_n)$ of arbitrarily near K-points of $F^{(1)}/Y$ and to the ordered Enriques diagram $(\mathbf{D}', \theta')$ where $\mathbf{D}' := \mathbf{D} - V$ and $\theta'(W) := \theta(W) - 1$.

The ideal $\mathcal{I}$ determines the diagram $\mathbf{D}$. Indeed, for $0 \leq i \leq n$, let A_i, $\mathbf{m}_i$ be the local ring of the surface $F_K^{(i)}$ at the K-point that is the image of t_i. Then according to Lipman's preliminary discussion in [**19**, p. 294–295], the set $\{A_i\}$ consists precisely of 2-dimensional regular local K-domains whose fraction field is that of F_K and whose maximal ideal contains the stalk of $\mathcal{I}$ at some point of F_K. Furthermore, t_i is proximate to t_j if and only if A_i is contained in the ring of the valuation v_j defined by the formula: $v_j(f) := \max\{m \mid f \in \mathbf{m}_j^m\}$. Finally, if $W := \theta^{-1}(j)$, then the weight m_W is the largest integer m such that $\mathbf{m}_j^m$ contains the appropriate stalk of $\mathcal{I}$.

Let $\mathcal{J}$ be an arbitrary ideal on F_K of finite colength. Let L/K be an arbitrary field extension. If the extended ideal $\mathcal{J}_L$ on F_L is complete, then $\mathcal{J}$ is complete, and the converse holds if L/K is separable; see Nobile and Villamayor's proof of [**23**, Prp. (3.2), p. 251]. Let us say that $\mathcal{J}$ is *geometrically complete* if $\mathcal{J}_L$ on F_L is complete for every L, or equivalently, for some algebraically closed L. In characteristic 0, if $\mathcal{J}$ is complete, then it is geometrically complete.

The extended ideal $\mathcal{I}_L$ on F_L is, plainly, the complete ideal associated to the extension of the sequence $(t_0, \ldots, t_n)$ and to the same ordered Enriques diagram $(\mathbf{D}, \theta)$. Hence $\mathcal{I}$ is geometrically complete.

Suppose that K is algebraically closed. Suppose that $\mathcal{J}$ is complete and that $\dim_K \mathrm{H}^0(\mathcal{O}_{F_K}/\mathcal{J})$ is finite and nonzero. Then $\mathcal{J}$ arises from some sequence $(s_0, \ldots, s_n)$ and some ordered Enriques diagram. Indeed, choose a minimal prime $\mathcal{P}$ of $\mathcal{J}$. Then $K \xrightarrow{\sim} \mathcal{O}_{F_K}/\mathcal{P}$ since K is algebraically closed. Hence $\mathcal{P}$ defines a K-point $S^{(0)}$ of F_K, so a section s_0 of F_K/K. Set $m_0 := \max\{\, m \mid \mathcal{P}^m \supset \mathcal{J} \,\}$.

Let F'_K be the blowup of F_K at $S^{(0)}$, and E'_K the exceptional divisor. Set $\mathcal{J}' := \mathcal{J}\mathcal{O}_{F'_K}(m_0 E'_K)$. Then $\mathcal{J}'$ is complete by Zariski and Samuel's [**29**, Prp. 5,

p. 381]. If $\mathcal{J}' = \mathcal{O}_{F'_K}$, then stop. If not, then repeat the process again and again, obtaining a sequence $(s_0, s_1, \dots)$. Only finitely many repetitions are necessary because, as Lipman [**19**, p. 295] points out, the local ring of $F_K^{(i)}$ at $S^{(i)}$ is dominated by a Rees valuation of $\mathcal{J}$, that is, the valuation associated to an exceptional divisor of the normalized blowup of $\mathcal{J}$. Then $\mathcal{J}'$ arises from the sequence of s_i weighted by the $m_{\theta^{-1}(i)}$ owing to Lipman's [**18**, prp. (6.2), p. 208] and discussion before it.

Lemma 5.2. *Let A be a discrete valuation ring, set $T := \operatorname{Spec} A$, and denote by $\eta \in T$ the generic point and by $y \in T$ the closed point. Fix a map $T \to Y$. Let $\mathbf{D}$ be an Enriques diagram, say with $n+1$ vertices, and $\mathcal{I}$ a coherent ideal on F_T that generates geometrically complete ideals on F_η and F_y, each with diagram $\mathbf{D}$. Let θ be an ordering of $\mathbf{D}$, and $\tilde{\mathbf{t}}$ a $k(\eta)$-point of $F(\mathbf{D}, \theta)$ such that $\mathcal{I}_\eta$ generates an invertible sheaf on $F_\eta^{(n+1)}$. Then $\tilde{\mathbf{t}}$ extends to a T-point $\mathbf{t}$ of $F(\mathbf{D}, \theta)$.*

PROOF. Let θ' be a second ordering. By the construction of the isomorphism $\Phi_{\theta,\theta'}$ in the proof of Proposition 4.3, a T-point of $F(\mathbf{D}, \theta)$ corresponds to the T-point of $F(\mathbf{D}, \theta')$ given by Lemma 4.2 with $\alpha := \theta' \circ \theta^{-1}$. Moreover, the lemma says that $F_T^{(n+1)}$ is unchanged. It follows that, to construct $\mathbf{t}$, we may replace θ by θ'. Thus we may assume that $E(\mathbf{D}, \theta)$ is a minimal element among the various closed subsets $E(\mathbf{D}, \theta')$ of $F^{(n)}$.

Let $R \in \mathbf{D}$ be a root, and temporarily set $i := \theta(R)$. Say $\tilde{\mathbf{t}}$ corresponds to the sequence of blowups $F_\eta^{(j+1)} \to F_\eta^{(j)}$ with centers η_j. The image of η_i in F_T is a $k(\eta)$-point; denote its closure by T_R. Since A is a discrete valuation ring, the structure map is an isomorphism $T_R \xrightarrow{\sim} T$.

Let $Z \subset F_T$ be the subscheme with ideal $\mathcal{I}$. Its fibers Z_η and Z_y are finite, and both have degree $\deg(\mathbf{D})$ since the two ideals are geometrically complete with diagram $\mathbf{D}$ by hypothesis. Since T is reduced, Z is T-flat.

As R varies, the points $(T_R)_\eta$ are exactly the components of Z_η again because its ideal $\mathcal{I}_\eta$ is geometrically complete with diagram $\mathbf{D}$. Hence the several T_R are just the components of Z that meet Z_η. But every component of Z meets Z_η since Z is T-flat. Thus the T_R are the the components of Z.

Since $T_R \xrightarrow{\sim} T$ for each R, the fiber $(T_R)_y$ is a single point, so a component of the discrete set Z_y. The number of T_R is the number of roots of $\mathbf{D}$, which is also the number of points of Z_y. Hence the several T_R are disjoint.

Given R, let m_R be its weight, $\mathcal{P}_R$ the ideal of T_R in F_T. Then $(\mathcal{P}_R^{m_R})_\eta \supset \mathcal{I}_\eta$. Let's see that $\mathcal{P}_R^{m_R} \supset \mathcal{I}$. Indeed, form the image, $\mathcal{M}$ say, of $\mathcal{I}$ in $\mathcal{O}_{F_T}/\mathcal{P}_R^{m_R}$. Then $\mathcal{M}_\eta = 0$. Let $u \in A$ be a uniformizing parameter. Then $\mathcal{M}$ is annihilated by a power of u. Now, $\mathcal{P}_R$ is quasi-regular by [**9**, (17.12.3), p. 83] since $T_R \xrightarrow{\sim} T$ and F_T is T-smooth. Hence $\mathcal{P}_R^j/P_R^{j+1}$ is T-flat for all j by [**9**, (16.9.4), p. 47]. Hence $\mathcal{O}_{F_T}/\mathcal{P}_R^{m_R}$ is T-flat. So u is a nonzerodivisor on $\mathcal{O}_{F_T}/\mathcal{P}_R^{m_R}$. Hence $\mathcal{M} = 0$. Thus $\mathcal{P}_R^{m_R} \supset \mathcal{I}$.

Let n_R be the largest integer such that $(\mathcal{P}_R^{n_R})_y \supset \mathcal{I}_y$. Then $n_R \geq m_R$. Now, $\mathcal{I}_y$ is geometrically complete with diagram $\mathbf{D}$. Hence n_R is the weight of the root corresponding to $(\mathcal{P}_R)_y$. Hence $\sum_R n_R = \sum_R m_R$. But $n_R \geq m_R$. Therefore, $n_R = m_R$ for every root R.

Let $\mathbf{D}'$ be the diagram obtained from $\mathbf{D}$ by omitting the roots. Let θ' be the ordering of $\mathbf{D}'$ induced by θ; namely, $\theta'(V) := \theta(V) - r_V$ where r_V denotes the number of roots R of $\mathbf{D}$ such that $\theta(R) < \theta(V)$. Let F'_T be obtained from F_T by

blowing up $\bigcup T_R$, and for each R, let E'_R be the preimage of T_R. Set

$$\mathcal{I}' := \mathcal{IO}_{F'_T}\big(\textstyle\sum_R m_R E'_R\big).$$

Finally, let n' be the number of vertices of $\mathbf{D}'$.

Then $\mathcal{I}'$ generates geometrically complete ideals on F'_η and F'_y, each with diagram $\mathbf{D}'$ owing to the theory of geometrically complete ideals over a field; see Subsection 5.1. (To ensure that the ideals on F'_η and F'_y have the same diagram, it is necessary to omit all the roots of $\mathbf{D}$. Indeed, $\mathbf{D}$ might have two roots with the same multiplicity, but the diagram obtained by omitting one root might differ from that obtained by eliminating the other. Conceivably, the two roots get interchanged under the specialization.)

Plainly, $\tilde{\mathbf{t}}$ induces a $k(\eta)$-point $\tilde{\mathbf{t}}'$ of $F(\mathbf{D}', \theta')$ such that $\mathcal{I}'_\eta$ generates an invertible sheaf on the corresponding $F_\eta^{\prime(n'+1)}$, which is equal to $F_\eta^{(n+1)}$. Hence, by induction on n, we may assume that $\tilde{\mathbf{t}}'$ extends to T-point $\mathbf{t}'$ of $F(\mathbf{D}', \theta')$ such that, on the corresponding scheme $F_T^{\prime(n'+1)}$, the ideal $\mathcal{I}'$ generates an invertible ideal. It remains to show that $\mathbf{t}'$ and the several isomorphisms $T_R \xrightarrow{\sim} T$ yield an extension $\mathbf{t}$ of $\tilde{\mathbf{t}}$.

Proceed by induction on i where $0 \le i \le n$. Suppose we have constructed a sequence $(t_0, \dots, t_{i-1})$ extending the sequence $(\tilde{t}_0, \dots, \tilde{t}_{i-1})$ coming from $\tilde{\mathbf{t}}$; suppose also that, if we blow up $F_T^{(i)}$ along the preimage of $\bigcup_{k\ge i} T_k$, then we get $F_T^{\prime(i')}$ where, for $0 \le j \le n$, we let j' denote j diminished by the number of roots R of $\mathbf{D}$ such that $\theta(R) < j$. Note that the base case $i := 0$ obtains: the sequence $(t_0, \dots, t_{i-1})$ is empty; furthermore, $F_T^{(i)} = F_T$ and $F_T^{\prime(i')} = F'_T$, which is the blowup of F_T along $\bigcup_{k\ge i} T_k$.

Note that $F_T^{(i)} \to F_T$ is an isomorphism off $\bigcup_{k<i} T_k$. Indeed, given $j < i$, let $R' \in \mathbf{D}$ be the root preceding $\theta^{-1}(j)$, and set $k := \theta(R')$. Since θ is an ordering, $k \le j$. Since $(t_0, \dots, t_{i-1})$ extends $(\tilde{t}_0, \dots, \tilde{t}_{i-1})$, the image of $T_\eta^{(j)}$ in F_T is just $(T_k)_\eta$. So $T^{(j)}$ maps into T_k, and $k < i$.

Set $V := \theta^{-1}(i) \in \mathbf{D}$. First suppose V is a root of $\mathbf{D}$. Then $(i+1)' = i'$. Also, T_i is defined, and the isomorphism $T_i \xrightarrow{\sim} T$ provides a section t_i of $F_T^{(i)}$ owing to the preceding note. By the same token, the blowup of $F_T^{(i+1)}$ along the preimage of $\bigcup_{k\ge i+1} T_k$ is equal to the blowup of $F_T^{(i)}$ along the preimage of $\bigcup_{k\ge i} T_k$. But the latter blowup is equal to $F_T^{\prime(i')}$. It follows that t_i does the trick.

Next suppose V is not a root, so $V \in \mathbf{D}'$. Also $\bigcup_{k\ge i} T_k = \bigcup_{k\ge i+1} T_k$. Now, by the induction assumption, $F_T^{\prime(i')}$ is equal to $F_T^{(i)}$ off the preimage of $\bigcup_{k\ge i} T_k$. Take $t_i := t'_i$ where $(t'_0, \dots, t'_i)$ comes from $\mathbf{t}'$. It is not hard to see that t_i does the trick.

It is not immediately obvious that $(t_0, \dots, t_n)$ is strict, even though $(t'_0, \dots, t'_{n'})$ is strict. However, $\mathbf{t}$ is a T-point of $F^{(n)}(T)$ and $\mathbf{t}_\eta$ is a $k(\eta)$-point of $F(\mathbf{D}, \theta)$; furthermore, $\mathbf{t}_y$ is a $k(y)$-point of $F(\mathbf{D}, \phi)$ for some ordering ϕ of $\mathbf{D}$. Since T is irreducible, $\mathbf{t}_y$ is a point of the closure of $F(\mathbf{D}, \theta)$ in $F^{(n)}$, so is a point of $E(\mathbf{D}, \theta)$. Hence $E(\mathbf{D}, \theta)$ contains $E(\mathbf{D}, \phi)$ by Proposition 4.8. But, by the initial reduction, $E(\mathbf{U}, \theta)$ is minimal, so equal to $E(\mathbf{D}, \phi)$. Hence $(\mathbf{D}, \theta) \cong (\mathbf{D}, \phi)$ again by Proposition 4.8. So $\mathbf{t}_y$ is a point of $F(\mathbf{D}, \theta)$. Since T is reduced, $\mathbf{t}$ is therefore a T-point of $F(\mathbf{D}, \theta)$, as desired. □

Definition 5.3. Given an Enriques diagram $\mathbf{D}$, say with $d := \deg \mathbf{D}$, let $H(\mathbf{D}) \subset$

$\mathrm{Hilb}^d_{F/Y}$ denote the subset parameterizing the geometrically complete ideals with diagram $\mathbf{D}$ on the geometric fibers of F/Y; see Subsection 5.1.

Proposition 5.4. *Let $\mathbf{D}$ be an Enriques diagram, set $d := \deg \mathbf{D}$, and choose an ordering θ. Then there exists a natural map $\Upsilon_\theta \colon F(\mathbf{D}, \theta) \to \mathrm{Hilb}^d_{F/Y}$, whose formation commutes with base extension of Y. Its image is $H(\mathbf{D})$, and it factors into a finite map $F(\mathbf{D}, \theta) \to U$ and an open embedding $U \hookrightarrow \mathrm{Hilb}^d_{F/Y}$. Moreover, $\Upsilon_\theta = \Upsilon_{\theta'} \circ \Phi_{\theta,\theta'}$ for any second ordering θ'.*

PROOF. Say $\mathbf{D}$ has $n+1$ vertices V with weights m_V. On $F^{(n+1)}$, set

$$E := \textstyle\sum_V m_V E^{(\theta(V)+1,\, n+1)} \text{ and } \mathcal{L} := \mathcal{O}(-E).$$

Consider the standard short exact sequence:

$$0 \to \mathcal{L} \to \mathcal{O}_{F^{(n+1)}} \to \mathcal{O}_E \to 0.$$

It remains exact on the fibers of $\pi^{(n+1)} \colon F^{(n+1)} \to F^{(n)}$. And $\pi^{(n+1)}$ is flat by Lemma 3.2. Hence $\mathcal{L}$ and $\mathcal{O}_E$ are flat over $F^{(n)}$ owing to the local criterion.

Fix a T-point of $F(\mathbf{D}, \theta) \subset F^{(n)}$. It corresponds to a strict sequence of arbitrarily near T-points of F/Y by Theorem 3.10. Set $\varphi := \varphi_T^{(1)} \cdots \varphi_T^{(n+1)}$. Let $t \in T$. Then $R^i\varphi_{t*}(\mathcal{L}_t) = 0$ and $R^i\varphi_{t*}(\mathcal{O}_{F_t^{(n+1)}}) = 0$ for $i \geq 1$ by [**3**, Thm. 2.13, p. 22]. Therefore, by Lemma A.2, the induced sequence on F_T,

$$0 \to \varphi_*\mathcal{L}_T \to \varphi_*\mathcal{O}_{F_T^{(n+1)}} \to \varphi_*\mathcal{O}_{E_T} \to 0, \tag{5.4.1}$$

is an exact sequence of T-flat sheaves, and forming it commutes with extending T.

The middle term in (5.4.1) is equal to $\mathcal{O}_{F_T}$: the comorphism $\mathcal{O}_{F_T} \to \varphi_*\mathcal{O}_{F_T^{(n+1)}}$ is an isomorphism, since forming it commutes with passing to the fibers of F_T/T, and on the fibers, it is an isomorphism as it is the comorphism of a birational map between smooth varieties. The third term in (5.4.1) is a locally free $\mathcal{O}_T$-module of rank d because its fibers are vector spaces of dimension d owing again to [**3**, Thm. 2.13, p. 22]. Therefore, (5.4.1) defines a T-point of $\mathrm{Hilb}^d_{F/Y}$.

The construction of this T-point is, plainly, functorial in T, and commutes with base extension of Y. Hence it yields a map $\Upsilon_\theta \colon F(\mathbf{D}, \theta) \to \mathrm{Hilb}^d_{F/Y}$, whose formation commutes with extension of Y.

To see that $H(\mathbf{D})$ is the image of Υ_θ, just observe that, in view of Subsection 5.1, if T is the spectrum of an algebraically closed field, then $\varphi_*\mathcal{L}_T$ is a geometrically complete ideal on F_T with diagram $\mathbf{D}$, and every such ideal on F_T is of this form for some choice of T-point of $F(\mathbf{D}, \theta)$.

Let θ' be a second ordering. Then by the construction of $\Phi_{\theta,\theta'}$ in the proof of Proposition 4.3, our T-point of $F(\mathbf{D}, \theta)$ is carried to that of $F(\mathbf{D}, \theta')$ given by Lemma 4.2 with $\alpha := \theta' \circ \theta^{-1}$. Moreover, the lemma says that $F_T^{(n+1)}$ is unchanged and implies that $E^{(\theta(V)+1,\, n+1)} = E^{(\theta'(V)+1,\, n+1)}$ for all V. Hence $\Upsilon_\theta = \Upsilon_{\theta'} \circ \Phi_{\theta,\theta'}$.

By Zariski's Main Theorem in the form of [**8**, Thm. (8.12.6), p. 45], there exists a factorization

$$\Upsilon_\theta \colon F(\mathbf{D}, \theta) \xrightarrow{\Omega} H \xrightarrow{\Theta} \mathrm{Hilb}^d_{F/Y},$$

where Ω is an open embedding and Θ is a finite map. Let W be the image of Ω, so $\Theta(W) = H(\mathbf{D})$. Replace H by the closure of W, and let us prove $W = \Theta^{-1}H(\mathbf{D})$.

Let $v \in \Theta^{-1}H(\mathbf{D})$. Then v is the specialization of a point $w \in W$ since H is the closure of W. And w is the image of a point $\mathbf{w} \in F(\mathbf{D}, \theta)$. Hence, by [**6**, Thm. (7.1.9), p. 141], there is a map $\tau \colon T \to H$ where T is the spectrum of a

discrete valuation ring, such that the closed point $y \in T$ maps to v and the generic point $\eta \in T$ maps to w; also there is a $k(\eta)$-point $\tilde{\mathbf{t}}$ of $F(\mathbf{D}, \theta)$ supported at $\mathbf{w}$.

The map $\Theta \circ \tau$ corresponds to a coherent ideal $\mathcal{I}$ on F_T. Now, both $\Theta(w)$ and $\Theta(v)$ lie in $H(\mathbf{D})$; so $\mathcal{I}$ generates geometrically complete ideals on F_η and F_y, each with diagram $\mathbf{D}$. And $\Upsilon_\theta(\tilde{\mathbf{t}})$ corresponds to $\mathcal{I}_\eta$ on F_η; so $\mathcal{I}_\eta$ generates an invertible sheaf on $F_\eta^{(n+1)}$. Hence, by Lemma 5.2, the $k(\eta)$-point $\tilde{\mathbf{t}}$ extends to T-point $\mathbf{t}$ of $F(\mathbf{D}, \theta)$.

Then $\Upsilon_\theta(\mathbf{t})\colon T \to W$ carries η to w. But H/Y is separated. Hence $\Upsilon_\theta(\mathbf{t}) = \tau$ by the valuative criterion [**6**, Prp. (7.2.3), p. 142]. But $\tau(y) = v$. Hence $v \in W$. Thus $W \supset \Theta^{-1}H(\mathbf{D})$. But $\Theta(W) = H(\mathbf{D})$. Therefore, $W = \Theta^{-1}H(\mathbf{D})$.

But W is open in H, and Θ is finite. So $\Theta(H)$ and $\Theta(H - W)$ are closed in $\mathrm{Hilb}^d_{F/Y}$. Hence $H(\mathbf{D})$ is open in $\Theta(H)$. So there is an open subscheme U of $\mathrm{Hilb}^d_{F/Y}$ such that $U \bigcap \Theta(H) = H(\mathbf{D})$. Furthermore, $W \to U$ is finite, as it is the restriction of Θ. So $F(\mathbf{D}, \theta) \to U$ is finite. The proof is now complete. □

Corollary 5.5. *Let* $\mathbf{D}$ *be an Enriques diagram, and set* $d := \deg \mathbf{D}$. *Then* $H(\mathbf{D})$ *is a locally closed subset of* $\mathrm{Hilb}^d_{F/Y}$.

PROOF. By Proposition 5.4, $H(\mathbf{D})$ is the image of a finite map into an open subscheme U of $\mathrm{Hilb}^d_{F/Y}$. So $H(\mathbf{D})$ is closed in U, so locally closed in $\mathrm{Hilb}^d_{F/Y}$. □

Remark 5.6. Lossen [**21**, Prp. 2.19, p. 35] proved a complex analytic version of Corollary 5.5. Independently, Nobile and Villamayor [**23**, Thm. 2.6, p. 250] proved the corollary assuming $\mathrm{Hilb}^d_{F/Y}$ is reduced and excellent; in fact, they worked with an arbitrary flat family of ideals on a reduced excellent scheme, but of course, any flat family is induced by a map to the Hilbert scheme. All three approaches are rather different.

Theorem 5.7. *Let* $\mathbf{D}$ *be an Enriques diagram, and set* $d := \deg \mathbf{D}$. *Choose an ordering* θ, *and form the map* Υ_θ *of Proposition* 5.4. *Then* Υ_θ *induces a map*

$$\Psi\colon F(\mathbf{D}, \theta)/\mathrm{Aut}(\mathbf{D}) \to \mathrm{Hilb}^d_{F/Y}\,.$$

It is universally injective; in fact, it is an embedding in characteristic 0. *Furthermore,* Ψ *is independent of the choice of* θ, *up to a canonical isomorphism.*

PROOF. By Corollary 4.4, $\mathrm{Aut}(\mathbf{D})$ acts freely. Hence, the quotient map

$$\Pi\colon F(\mathbf{D}, \theta) \to F(\mathbf{D}, \theta)/\mathrm{Aut}(\mathbf{D})$$

is faithfully flat. By Proposition 5.4, the action of $\mathrm{Aut}(\mathbf{D})$ is compatible with Υ_θ, and is compatible with a second choice of ordering θ', up to the isomorphism $\Phi_{\theta,\theta'}$. Hence, by descent theory, Υ_θ induces the desired map Ψ. Plainly, its formation commutes with base change.

Plainly, a map is universally injective if it is injective on geometric points. Furthermore, since Π is surjective, Proposition 5.4 also implies that Ψ too factors into a finite map followed by an open embedding. Now, a finite map is a closed embedding if its comorphism is surjective. Hence, to prove that Ψ is an embedding, it suffices to prove that its fibers over Y are embeddings. Now, forming Ψ commutes with extending Y. Therefore, we may assume Y is the spectrum of an algebraically closed field K.

To prove Ψ is universally injective, plainly we need only prove Ψ is injective on K-points. Since Π is surjective, every K-point of $F(\mathbf{D}, \theta)/\mathrm{Aut}(\mathbf{D})$ is the image

of a K-point of $F(\mathbf{D}, \theta)$. Hence we need only observe that, if two K-points $\mathbf{t}'$ and $\mathbf{t}''$ of $F(\mathbf{D}, \theta)$ have the same image in $\mathrm{Hilb}^d_{F/Y}(K)$ under Υ_θ, then the two differ by an automorphism γ of $\mathbf{D}$. But that image corresponds to a geometrically complete ideal $\mathcal{I}$ on F_K with diagram $\mathbf{D}$. In turn, as explained in Subsection 5.1, $\mathcal{I}$ determines a set $\mathbf{A}$ of 2-dimensional regular local K-domains whose fraction field is that of F_K, and $\mathbf{A}$ has a proximity structure, under which it is isomorphic to $\mathbf{D}$. Say $\mathbf{t}' \in F(\mathbf{A}, \theta')$ and $\mathbf{t}'' \in F(\mathbf{A}, \theta'')$. Then $\theta'^{-1} \circ \theta''$ induces the desired automorphism $\gamma \in \mathrm{Aut}(\mathbf{D})$.

By Corollary 4.5, $F(\mathbf{D}, \theta)/\mathrm{Aut}(\mathbf{D})$ is smooth and irreducible. By Corollary 5.5, $H(\mathbf{D})$ is a locally closed subset of $\mathrm{Hilb}^d_{F/Y}$, so carries an induced reduced structure. And Ψ induces a bijective finite map $\beta\colon F(\mathbf{D}, \theta)/\mathrm{Aut}(\mathbf{D}) \to H(\mathbf{D})$.

Suppose K is of characteristic 0. Then β is birational. If, perchance, $\mathbf{D}$ is minimal in the sense of [**15**, Section 2, p. 213], then $H(\mathbf{D})$ is smooth by the direct, alternative proof of [**15**, Prp. (3.6), p. 225]; hence, β is an isomorphism. In any case, it follows from Proposition 3.3.14 on p. 70 of [**10**] that β is unramified; hence, β is an isomorphism. The proof is now complete. □

Corollary 5.8. *Fix an Enriques diagram* $\mathbf{D}$*, and set* $d := \deg \mathbf{D}$*. Assume the characteristic is* 0*. Then* $H(\mathbf{D}) \subset \mathrm{Hilb}^d_{F/Y}$ *supports a natural structure of* Y*-smooth subscheme with irreducible geometric fibers of dimension* $\dim(\mathbf{D})$*.*

PROOF. By Theorem 5.7, Υ_θ induces an embedding of $F(\mathbf{D}, \theta)/\mathrm{Aut}(\mathbf{D})$ into $\mathrm{Hilb}^d_{F/Y}$. By Proposition 5.4, the image is $H(\mathbf{D})$. And by Corollary 4.5, the source is Y-smooth, and has irreducible geometric fibers of dimension $\dim(\mathbf{D})$. □

Proposition 5.9. *Given positive integers* $r_1, \dots, r_k$*, let* $G(r_i) \subset \mathrm{Hilb}^{r_i}_{F/Y}$ *be the open subscheme over which the universal family is smooth, and let*

$$G(r_1, \dots, r_k) \subset G(r_1) \times_Y \cdots \times_Y G(r_k)$$

be the open subscheme over which, for $i \neq j$*, the fibers of the universal families over* $G(r_i)$ *and* $G(r_j)$ *have empty intersection. Set* $r := \sum r_i$*.*

Given distinct integers $m_1, \dots, m_k \geq 2$*, let* $\mathbf{D}$ *be the the weighted Enriques diagram with* r *vertices, each a root, and an ordering* θ *such that the first* r_1 *vertices are roots of weight* m_1*, the next* r_2 *are of weight* m_2*, and so on. Set* $d := \sum \binom{m_i+1}{2} r_i$*.*

Then $F(\mathbf{D}, \theta)$ *is equal to the complement in the relative direct product* $F^{\times_Y r}$ *of the* $\binom{r}{2}$ *large diagonals, and* $F(\mathbf{D}, \theta)/\mathrm{Aut}(\mathbf{D})$ *is equal to* $G(r_1, \dots, r_k)$*. Further,* Υ_θ *always induces an embedding*

$$\Psi\colon G(r_1, \dots, r_k) \hookrightarrow \mathrm{Hilb}^d_{F/Y};$$

on T*-points,* Ψ *acts by taking a* k*-tuple* $(W_1, \dots, W_k)$ *where* W_i *is a smooth length-*r_i *subscheme of* F_T*, say with ideal* $\mathcal{I}_i$*, to the length-*d *subscheme* W *with ideal* $\prod \mathcal{I}_i^{m_i}$*.*

PROOF. Let $(t_0, \dots, t_{r-1})$ be a strict sequence of arbitrarily near T-points of F/Y with diagram $(\mathbf{D}, \theta)$. Plainly, the t_i are just sections of F_T, and their images are disjoint. So $F(\mathbf{D}, \theta)$ is equal to the asserted complement.

Plainly, $\mathrm{Aut}(\mathbf{D})$ is the product of k groups, the ith being the full symmetric group on the r_i roots in the ith set. So the quotient $F(\mathbf{D}, \theta)/\mathrm{Aut}(\mathbf{D})$ is equal to the open subscheme of $\mathrm{Hilb}^{r_1}_{F/Y} \times_Y \cdots \times_Y \mathrm{Hilb}^{r_k}_{F/Y}$ whose geometric points parameterize the k-tuples whose ith component is an unordered set of r_i geometric points of F such that all r points are distinct; in other words, the quotient is equal to the

asserted open subscheme.

Since each vertex is a root of some weight m_i, plainly Ψ acts on T-points in the asserted way, owing to the following standard general result, which is easily proved by descending induction: let A be a locally Noetherian scheme, $\mathcal{I}$ a regular ideal, $b\colon B \to A$ the blow-up of $\mathcal{I}$, and E the exceptional divisor; let $m \geq 0$ and set $\mathcal{L} := \mathcal{O}_B(-mE)$; then $R^q b_* \mathcal{L} = 0$ for $q \geq 1$ and $b_* \mathcal{L} = \mathcal{I}^m$.

Finally, to prove that Ψ is always an embedding, we may assume that Y is the spectrum of an algebraically closed field K, owing to the proof of Theorem 5.7. By the same token, Ψ is universally injective, and factors into a finite map followed by an open embedding. Hence, we need only show that Ψ is unramified.

Let v be a K-point of $\mathrm{Hilb}^r_{F/Y}$; let $V \subset F$ be the corresponding subscheme, and $\mathcal{I}$ its ideal. Recall the definition of isomorphism from the tangent space at v to the normal space $\mathrm{Hom}(\mathcal{I}, \mathcal{O}_V)$; the definition runs as follows. Let $K[\epsilon]$ be the ring of dual numbers, and set $T := \mathrm{Spec}(K[\epsilon])$. An element of the tangent space corresponds to a T-point of $\mathrm{Hilb}^r_{F/Y}$ supported at v; so it represents a T-flat subscheme $V_\epsilon \subset F_T$ that deforms V. The natural splitting $K[\epsilon] = K \oplus K\epsilon$ induces a splitting $\mathcal{O}_{V_\epsilon} = \mathcal{O}_V \oplus \mathcal{O}_V\epsilon$. Similarly, the ideal $\mathcal{I}_\epsilon$ of V_ϵ splits: $\mathcal{I}_\epsilon = \mathcal{I} \oplus \mathcal{I}\epsilon$. Then the natural map $\mathcal{O}_{F_T} \to \mathcal{O}_{V_\epsilon}$ restricts to a map $\mathcal{I} \to \mathcal{O}_V\epsilon$, which is equal to the desired map $\zeta\colon \mathcal{I} \to \mathcal{O}_V$.

Assume $v \in G(r_1, \ldots, r_k)$. Then V is the union of k sets of reduced K-points of F. The ith set has r_i points; let $\mathcal{I}_i$ be the ideal of its union. Further, Ψ carries V and V_ϵ to the subschemes W and W_ϵ defined by $\mathcal{I}_1^{m_1} \cdots \mathcal{I}_k^{m_k}$ and $\mathcal{I}_{1,\epsilon}^{m_1} \cdots \mathcal{I}_{k,\epsilon}^{m_k}$. So Ψ is unramified at v if the induced map on tangent spaces is injective:

$$\psi\colon T_{G(r_1,\ldots,r_k),v} \hookrightarrow \mathrm{Hom}(\mathcal{I}_1^{m_1} \cdots \mathcal{I}_k^{m_k}, \mathcal{O}_W).$$

Say $v = (v_1, \ldots, v_k)$ with $v_i \in G(r_i)$, and say v_i represents $V_i \subset F$. Then

$$T_{G(r_1,\ldots,r_k),v} = \bigoplus T_{\mathrm{Hilb}^{r_i}_{F/Y},v_i} = \bigoplus \mathrm{Hom}(\mathcal{I}_i, \mathcal{O}_{V_i}).$$

Given any $\zeta \in T_{G(r_1,\ldots,r_k),v}$, its image $\psi(\zeta)$ is equal to the restriction of the canonical map $\mathcal{O}_{F_T} \to \mathcal{O}_{W_\epsilon}$. So ψ splits into a direct sum of local components

$$\psi_x\colon \mathrm{Hom}(\mathcal{I}_{i,x}, \mathcal{O}_{V,x}) \to \mathrm{Hom}(\mathcal{I}_{i,x}^{m_i}, \mathcal{O}_{W,x}) \quad \text{for } x \in V_i \text{ and } i = 1, \ldots, k.$$

It remains to prove that each ψ_x is injective. Fix an x.

Set $\mathcal{I} := \mathcal{I}_i$ and $m := m_i$. Fix generators $\mu, \nu \in \mathcal{I}_x$. Set $a := \zeta_x\mu \in K$ and $b := \zeta_x\nu \in K$. Then $\mathcal{I}_{\epsilon,x}$ is generated by $\mu - a\epsilon$ and $\nu - b\epsilon$; so $\mathcal{I}^m_{\epsilon,x}$ is generated by

$$\mu^m - ma\mu^{m-1}a\epsilon,\ \mu^{m-1}\nu - (m-1)\mu^{m-2}\nu a\epsilon - \mu^{m-1}b\epsilon,\ \ldots,$$
$$\mu\nu^{m-1} - a\nu^{m-1}\epsilon - (m-1)b\mu\nu^{m-2}\epsilon,\ \nu^m - m\nu^{m-1}b\epsilon.$$

Hence, modulo $\mathcal{I}^m_{\epsilon,x}$, the generators $\mu^{m-1}\nu$ and $\mu\nu^{m-1}$ of $\mathcal{I}^m_x$ are congruent to $(m-1)\mu^{m-2}\nu a\epsilon + \mu^{m-1}b\epsilon$ and $a\nu^{m-1}\epsilon + (m-1)b\mu\nu^{m-2}\epsilon$. (They're equal if $m = 2$.)

Form the latter's classes in $\mathcal{O}_{W,x}$. Then, therefore, these classes are the images of those generators under the map $\psi_x\zeta_x$. Hence, in any characteristic, we can recover a and b from the images of $\mu^{m-1}\nu$ and $\mu\nu^{m-1}$. But a and b determine ζ_x. Thus ψ_x is injective, and the proof is complete. □

Appendix A. Generalized property of exchange

This appendix proves two lemmas of general interest, which we need. The first lemma generalizes the property of exchange to a triple (T, f, F) where T is a (locally Noetherian) scheme, $f\colon P \to Q$ is a proper map of T-schemes of finite type, and $\mathcal{F}$ is a T-flat coherent sheaf on P. The original treatment was made by Grothendieck and Dieudonné in [**7**, Sec. 7.7, pp. 65–72], and somewhat surprisingly, deals only with the case of $Q = T$. (Although they replace F by a complex of flat and coherent sheaves bounded below, this extension is minor and we do not need it.)

The first lemma is proved by generalizing the treatment in Section II, 5 of [**22**, pp. 46–55]. Alternatively, as Illusie pointed out in a private conversation, the lemma can be proved using the methods that he developed in [**13**].

The first lemma is used to prove the second. The second is used in the proof of Proposition 5.4, which constructs the map from the scheme of T-points with given Enriques diagram to the Hilbert scheme.

Lemma A.1 (Generalized property of exchange)**.** *Let T be a scheme, $f\colon P \to Q$ a proper map of T-schemes of finite type, and $\mathcal{F}$ a T-flat coherent sheaf on P. Let $q \in Q$ be a point, $t \in T$ its image, and $i \geq 0$ an integer. Assume that, on the fiber Q_t, the base-change map of sheaves*

$$\rho_t^i\colon (R^i f_*\mathcal{F})_t \to R^i f_{t*}\mathcal{F}_t$$

is surjective at q. Then there exists a neighborhood U of q in Q such that, for any T-scheme T', the base-change map of sheaves

$$\rho_{T'}^i\colon (R^i f_*\mathcal{F})_{T'} \to R^i f_{T'*}\mathcal{F}_{T'}$$

is bijective on the open subset $U_{T'}$ of $Q_{T'}$. Furthermore, the map ρ_t^{i-1} is also surjective at q if and only if sheaf $R^i f_*\mathcal{F}$ is T-flat at q.

PROOF. The question is local on Q; so we may assume that $T = \operatorname{Spec} A$ and $Q = \operatorname{Spec} B$ where A is a Noetherian ring and B is a finitely generated A-algebra. Also, we may assume that B is A-flat by expressing B as a quotient of a polynomial ring over A and then replacing B with that ring. For convenience, when given a B-module or a map of B-modules, let us say that it has a certain property at q to mean that it acquires this property on localizing at the prime corresponding to q.

There is a finite complex $K^\bullet$ of A-flat finitely generated B-modules, and on the category of A-algebras C, there is, for every $j \geq 0$, an isomorphism of functors

$$\mathrm{H}^j(K^\bullet \otimes_A C) \xrightarrow{\sim} \mathrm{H}^j(P \otimes_A C,\ \mathcal{F} \otimes_A C).$$

Indeed, this statement results, mutatis mutandis, from the proof of the theorem on page 46 of [**22**].

Let k be the residue field of t. Then there is a natural map of exact sequences

$$\begin{array}{ccccccc} K^{i-1} \otimes k & \to & \mathrm{Z}^i(K^\bullet) \otimes k & \to & \mathrm{H}^i(K^\bullet) \otimes k & \to & 0 \\ \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle z_k^i} & & \downarrow{\scriptstyle h_k^i} & & \\ K^{i-1} \otimes k & \to & \mathrm{Z}^i(K^\bullet \otimes k) & \to & \mathrm{H}^i(K^\bullet \otimes k) & \to & 0. \end{array} \tag{A.1.1}$$

Since ρ_k^i is surjective at q, so is h_k^i. Hence z_k^i is surjective at q.

Consider the following map of exact sequences:

$$\begin{array}{ccccccc} \mathrm{Z}^i(K^\bullet)\otimes k & \to & K^i\otimes k & \to & \mathrm{B}^{i+1}(K^\bullet)\otimes k & \to & 0 \\ \downarrow{\scriptstyle z_k^i} & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle b_k^{i+1}} & & \\ \mathrm{Z}^i(K^\bullet\otimes k) & \to & K^i\otimes k & \to & \mathrm{B}^{i+1}(K^\bullet\otimes k) & \to & 0. \end{array}$$

Now, z_k^i is surjective at q. Hence b_k^{i+1} is bijective at q.

Hence $\mathrm{B}^{i+1}(K^\bullet)\otimes k \to K^{i+1}\otimes k$ is injective at q. Set $L := K^{i+1}/\mathrm{B}^{i+1}(K^\bullet)$. Since K^{i+1} is A-flat, the local criterion of flatness implies that L is A-flat at q. Hence, by the openness of flatness, there is a $g \in B$ outside the prime corresponding to q such that the localization L_g is A-flat. We can replace B by B_g, and so assume L is A-flat.

Let C be any A-algebra. Then the following sequence is exact:

$$0 \to \mathrm{Z}^i(K^\bullet)\otimes C \to K^i\otimes C \to K^{i+1}\otimes C \to L\otimes C \to 0. \tag{A.1.2}$$

It follows that, in the map of exact sequences

$$\begin{array}{ccccccc} K^{i-1}\otimes C & \to & \mathrm{Z}^i(K^\bullet)\otimes C & \to & \mathrm{H}^i(K^\bullet)\otimes C & \to & 0 \\ \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle z_C^i} & & \downarrow{\scriptstyle h_C^i} & & \\ K^{i-1}\otimes C & \to & \mathrm{Z}^i(K^\bullet\otimes C) & \to & \mathrm{H}^i(K^\bullet\otimes C) & \to & 0, \end{array}$$

z_C^i is bijective. Hence h_C^i is bijective. Thus the first assertion holds: ρ_C^i is bijective.

If $\mathrm{H}^i(K^\bullet)$ is A-flat at q, then plainly the sequence

$$0 \to \mathrm{B}^i(K^\bullet)\otimes k \to \mathrm{Z}^i(K^\bullet)\otimes k \to \mathrm{H}^i(K^\bullet)\otimes k \to 0 \tag{A.1.3}$$

is exact. The converse holds too by the local criterion for flatness, because $\mathrm{Z}^i(K^\bullet)$ is A-flat owing to the exactness of (A.1.2) with $C := A$ and to the flatness of L.

Since z_k^i is bijective, (A.1.3) is exact if and only if b_k^i is injective. The latter holds if and only if z_k^{i-1} is surjective, owing to the map of exact sequences

$$\begin{array}{ccccccc} Z^{i-1}(K^\bullet)\otimes k & \to & K^{i-1}\otimes k & \to & \mathrm{B}^i(K^\bullet)\otimes k & \to & 0 \\ \downarrow{\scriptstyle z_k^{i-1}} & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle b_k^i} & & \\ 0 \to Z^{i-1}(K^\bullet\otimes k) & \to & K^{i-1}\otimes k & \to & \mathrm{B}^i(K^\bullet\otimes k) & \to & 0. \end{array}$$

Finally, z_k^{i-1} is surjective if and only if h_k^{i-1} is so, owing to (A.1.1) with $i-1$ in place of i. Putting it all together, we've proved that h_k^{i-1} is surjective if and only if $\mathrm{H}^i(K^\bullet)$ is A-flat at q. In other words, the second assertion holds too. □

Lemma A.2. *Let T be a scheme, $f\colon P \to Q$ a proper map of T-schemes of finite type, and*

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0 \tag{A.2.1}$$

a short exact sequence of T-flat coherent sheaves on P. For each point $t \in T$, let f_t and $\mathcal{F}_t$ and $\mathcal{G}_t$ denote the restrictions to the fiber P_t, and assume that

$$R^i f_{t*}(\mathcal{F}_t) = 0 \text{ and } R^i f_{t*}(\mathcal{G}_t) = 0 \text{ for } i \geq 1. \tag{A.2.2}$$

Then the induced sequence on Q,

$$0 \to f_*\mathcal{F} \to f_*\mathcal{G} \to f_*\mathcal{H} \to 0, \tag{A.2.3}$$

is a short exact sequence of T-flat coherent sheaves, and forming it commutes with

base extension.

PROOF. Since $\mathcal{H}$ is T-flat, the sequence (A.2.1) remains exact after restriction to the fiber P_t for each $t \in T$, and so the restricted sequence induces a long exact sequence of cohomology. Hence, (A.2.2) yields

$$R^i f_{t*}(\mathcal{H}_t) = 0 \text{ for all } i \geq 1.$$

By hypothesis, $\mathcal{F}$, $\mathcal{G}$, $\mathcal{H}$ are T-flat. Hence, by the generalized property of exchange, Lemma A.1, the sheaves $f_*\mathcal{F}$, $f_*\mathcal{G}$, $f_*\mathcal{H}$ are T-flat, and forming them commutes with extending T. By the same token, $R^1 f_*(\mathcal{F}) = 0$; whence, Sequence (A.2.3) is exact. The assertion follows. □

Appendix B. A few examples BY ILYA TYOMKIN

Let F be the affine plane over the spectrum $Y := \mathrm{Spec}(K)$ of an algebraically closed field K of positive characteristic p. In this appendix, we analyze a few simple examples of minimal Enriques diagrams $\mathbf{D}$. Some depend on p, and have an ordering θ for which the universally injective map of Theorem 5.7,

$$\Psi\colon F(\mathbf{D}, \theta)/\mathrm{Aut}(\mathbf{D}) \to \mathrm{Hilb}^d_{F/Y},$$

is purely inseparable. Others are independent of p; they have several vertices, but only one root, yet they have an ordering θ for which Ψ is an embedding. In fact, in every case, θ is unique, and $\mathrm{Aut}(\mathbf{D})$ is trivial.

We take F to be the affine plane just to simplify the presentation. With little modification, everything works for any smooth irreducible surface F.

It is unknown what conditions on an arbitrary Enriques diagram $\mathbf{D}$ serve to guarantee here that Ψ is unramified, so an embedding. Nevertheless, in view of the analysis in this appendix, it is reasonable to make the following guess.

Guess B.1. *If $p > \frac{1}{2}\sum_{V\in\mathbf{D}} m_V$, then Ψ is unramified.*

This guess is sharp in the sense that, if $p \leq \frac{1}{2}\sum_{V\in\mathbf{D}} m_V$, then Ψ may be ramified. For example, consider the plane curve $C : x_2^p = x_1^{p+1}$. In the notation of Definition B.2, the minimal diagram of C is $\mathbf{M}_{p,p}$. It has $p+1$ vertices with $m_V = p, 1, 1, \ldots, 1$. So $p = \frac{1}{2}\sum_{V\in\mathbf{D}} m_V$. And Ψ is ramified by Proposition B.4.

Similarly, consider $C : y(y - x^p) = 0$. Its minimal diagram has p vertices V with $m_V = 2$. So $p = \frac{1}{2}\sum_{V\in\mathbf{D}} m_V$. And Ψ is ramified by an argument similar to the proof of Proposition B.4.

On the other hand, if $\mathbf{D}$ has a single vertex of weight $2p$, then Ψ is unramified by Proposition 5.9, and of course, $p = \frac{1}{2}\sum_{V\in\mathbf{D}} m_V$.

In general, if a branch has tangency of order divisible by p to an exceptional divisor E, then the multiplicity of the root must be at least p and there must be at least p other vertices. So $p \leq \frac{1}{2}\sum_{V\in\mathbf{D}} m_V$. Instead, if, at a point $P \in F$, all the branches have a tangency of order divisible by p to the same smooth curve D, then there must be at least p vertices V with $m_V \geq 2$. So again, $p \leq \frac{1}{2}\sum_{V\in\mathbf{D}} m_V$. Thus, if we guess that Ψ can be ramified in only these two ways, then we arrive at Guess B.1.

Further, although Ψ does not sense first-order deformations either along E or along D, nevertheless after we add a transverse branch at P, then Ψ does sense first-order deformations of the new branch; thus Ψ becomes unramified. This intuition is developed into a rigorous proof for the ordinary tacnode in Proposition B.7, and

a similar procedure works if the tacnode is replaced by an ordinary cusp.

Definition B.2. Fix $m \geq p$. Let $\mathbf{M}_{p,m}$ denote the minimal Enriques diagram of the plane curve singularity with $1 + m - p$ branches whose tangent lines are distinct, whose first branch is $\{ x_2^p = x_1^{p+1} \}$, and whose remaining $m - p$ branches are smooth.

Example B.3. For motivation, consider the following special case. Take $p := 2$ and $m := 2$. Then $\mathbf{M}_{p,m}$ is the minimal Enriques diagram $\mathbf{A}_2$ of the cuspidal curve $C : x_2^2 = x_1^3$. This diagram has three vertices and a unique ordering θ.

Take $F := \mathbb{A}^2_K$ and $T := \operatorname{Spec}(K)$. In $F(\mathbf{A}_2, \theta) \subset F^{(2)}$, form the locus L of sequences (t_0, t_1, t_2) of arbitrarily near T-points of F/K such that t_0 is the constant map from T to the origin. Plainly, the second projection induces an isomorphism $L \xrightarrow{\sim} E'_K$ where E'_K is the exceptional divisor of the blow up F'_K of F at the origin.

The strict transform C' of C is tangent to E'_K with order 2, and C' is given by the equation $s^2 = x_1$ where $s := x_2/x_1$. Notice that this equation is preserved by any first order deformation along E'_K of the point of contact; indeed,

$$(s + b\epsilon)^2 = s^2$$

as $p = 2$ and $\epsilon^2 = 0$. This observation suggests that the restriction of Ψ,

$$(\Psi|L)\colon L \to \operatorname{Hilb}^5_{F/K},$$

is purely inseparable; and indeed, $\Psi|L$ is so, as we check next.

Let $\mathbf{D}'$ be the diagram obtained from $\mathbf{A}_2$ by omitting the root, let θ' be the unique ordering of $\mathbf{D}'$, and consider the corresponding map

$$\Psi'\colon F'(\mathbf{D}', \theta') \to \operatorname{Hilb}^2_{F'_K/K}.$$

Plainly, the projection $(t_0, t_1, t_2) \mapsto (t_1, t_2)$ embeds L into $F'(\mathbf{D}', \theta')$.

So Ψ' induces a map $\Psi'_L\colon L \to \operatorname{Hilb}^2_{F'_K/K}$. It carries (t_0, t_1, t_2) to the subscheme of F'_T with ideal $\mathcal{I}'$ defined by the formula

$$\mathcal{I}' := \bigl(\varphi_T^{(2)}\varphi_T^{(3)}\bigr)_*\mathcal{O}_{F_T^{(3)}}\bigl(-E_T^{(2,3)} - E_T^{(3,3)}\bigr).$$

But $E_T^{(2,3)} + E_T^{(3,3)} \leq E_T^{(1,3)}$. So

$$\mathcal{O}_{F_T^{(3)}}\bigl(-E_T^{(2,3)} - E_T^{(3,3)}\bigr) \supseteq \mathcal{O}_{F_T^{(3)}}\bigl(-E_T^{(1,3)}\bigr).$$

Hence $\mathcal{I}'$ contains the ideal of E'_T. Therefore, Ψ'_L factors through $\operatorname{Hilb}^2_{E'_K/K}$, which is isomorphic to $\operatorname{Sym}^2(L)$. The corresponding map $L \to \operatorname{Sym}^2(L)$ is the diagonal map since $\Psi'_L(t_0, t_1, t_2)$ has the same support as t_1. This diagonal map is purely inseparable as $p = 2$.

Finally, $\Psi'_L\colon L \to \operatorname{Hilb}^2_{E'_K/K}$ is a factor of $\Psi|L$ because $\Psi(t_0, t_1, t_2)$ is the subscheme of F_T with ideal $\bigl(\varphi_T^{(1)}\bigr)_*\mathcal{I}'(2E'_T)$. Thus $\Psi|L$ is, indeed, purely inseparable. In fact, Ψ is purely inseparable by Proposition B.4 below.

Proposition B.4. *Fix $m \geq p$. Set $\mathbf{D} := \mathbf{M}_{p,m}$ and $d := \binom{m+1}{2} + p$. Then $\mathbf{D}$ has a unique ordering θ; also $\operatorname{Aut}(\mathbf{D}) = 1$ and $\deg \mathbf{D} = d$. Take $F = \mathbb{A}^2_K$. Then $\dim F(\mathbf{D}, \theta) = 3$, and $\Upsilon_\theta\colon F(\mathbf{D}, \theta) \to \operatorname{Hilb}^d_{F/Y}$ is purely inseparable; also, $\Psi = \Upsilon_\theta$.*

PROOF. Plainly, $\mathbf{D}$ has $p+1$ vertices, say $V_0, \dots, V_p$ ordered by succession. Then proximity is given by $V_k \succ V_{k-1}$ and $V_k \succ V_0$ for $k > 0$. Further, the weights are given by $m_{V_0} = m$ and $m_{V_k} = 1$ for $k > 0$. Set $\theta(V_k) := k$; plainly, θ is an ordering of $\mathbf{D}$, and is the only one. Also, plainly, $\mathrm{Aut}(\mathbf{D}) = 1$ and $\deg \mathbf{D} = d$.

Theorem 3.10 says that $\dim F(\mathbf{D}, \theta) = \dim \mathbf{D}$, but plainly $\dim \mathbf{D} = 3$. Now, $\Psi = \Upsilon_\theta$ because $\mathrm{Aut}(\mathbf{D}) = 1$. Further, Theorem 5.7 says that Ψ is universally injective. Hence Ψ is purely inseparable, because it is everywhere ramified owing to the following lemma. □

Lemma B.5. *Under the conditions of Proposition* B.4, *let* $t \in F(\mathbf{D}, \theta)$ *be a* K*-point. Then* $\mathrm{Ker}(d_t \Upsilon_\theta)$ *is of dimension* 1.

PROOF. Say t represents the sequence $(t_0, \dots, t_p)$ of arbitrarily near K-points of F/Y. Choose coordinates x_1, x_2 on F such that $t_0 : x_1 = x_2 = 0$ and such that t_1 is the point of intersection of the exceptional divisor E_0 with the proper transform of the x_1-axis. Set $s_0 := x_2/x_1$, set $s_1 := x_1/s_0$, and set $s_k := s_{k-1}/s_0$ for $2 \le k \le p-1$. Then $t_1 : s_0 = x_1 = 0$, and $t_k : s_0 = s_{k-1} = 0$ for $2 \le k \le p$.

Set $z := \Upsilon_\theta(t) \in \mathrm{Hilb}^d_{F/Y}(K)$. Let Z denote the corresponding subscheme, and $\mathcal{I}$ its ideal. Recall from the proof of Proposition 5.4 that $\mathcal{I} = \varphi_{K*}\mathcal{O}(-E_K)$ where $E_K = \sum_{i=0}^p m_{V_i} E^{(i+1,\,p+1)}$. Recall from the proof of Proposition B.4 that $m_{V_0} = m$ and $m_{V_k} = 1$ for $k > 0$ and that $V_k \succ V_0$ for $k > 0$. It follows that

$$E_K = m e_K^{(1,\,p+1)} E^{(1)} + \sum_{k=1}^{p} k(m+1) e_K^{(k+1,\,p+1)} E_K^{(k+1)}.$$

Set $\delta(r) := 0$ if $0 \le r < p$ and $\delta(r) := 1$ if $p \le r \le m$. Set

$$f_r := x_1^{m+1-r-\delta(r)} x_2^r \quad \text{for } 0 \le r \le m.$$

Let's now show that the f_r generate $\mathcal{I}$.

First, note that, for each r and for $1 \le k \le p-1$,

$$f_r = x_1^{m+1-\delta(r)} s_0^r = s_k^{m+1-\delta(r)} s_0^{k(m+1-\delta(r))+r}.$$

Hence, the pullback of f_r vanishes along $e_K^{(1,\,p+1)} E_K^{(1)}$ to order at least m, and along $e_K^{(k+1,\,p+1)} E_K^{(k+1)}$ to order at least $k(m+1)$ for $k \ge 1$, since $r - k\delta(r) \ge 0$. Thus $f_r \in \mathcal{I}$ for each r.

Let $\mathcal{J}$ be the ideal generated by the f_r. Then $\mathcal{J} \subset \mathcal{I}$. Now, $K[x_1, x_2]/\mathcal{J}$ is spanned as a K-vector space by the monomials $x_1^{m+1-r-\delta(r)} x_2^l$ for $0 \le l < r \le m$ and by $x_1^{m+1-p} x_2^l$ for $0 \le l < p$. Hence $\mathcal{J} = \mathcal{I}$ because

$$\dim\bigl(K[x_1, x_2]/\mathcal{J}\bigr) \le \textstyle\sum_{r=0}^m r + p = d = \dim K[x_1, x_2]/\mathcal{I}.$$

Let $K[\epsilon]$ be the ring of dual numbers, and set $T := \mathrm{Spec}(K[\epsilon])$. Let $(t'_0, \dots, t'_p)$ be a strict sequence of arbitrarily near T-points of F/Y lifting $(t_0, \dots, t_p)$. Then there are $a_1, a_2, b \in K$ so that, after setting $x'_1 := x_1 + a_1\epsilon$ and $x'_2 := x_2 + a_2\epsilon$ and setting $s'_0 := x'_2/x'_1 + b\epsilon$ and $s'_1 := x'_1/s'_0$ and $s'_k := s'_{k-1}/s'_0$ for $2 \le k \le p-1$, we have $t'_0 : x'_1 = x'_2 = 0$ and $t'_1 : s'_0 = x'_1 = 0$ and $t'_k : s'_0 = s'_{k-1} = 0$ for $2 \le k \le p$.

Let $t' \in F(\mathbf{D}, \theta)(T)$ represent $(t'_0, \dots, t'_p)$. Set $z' := \Upsilon_\theta(t') \in \mathrm{Hilb}^d_{F/Y}(K)(T)$. Let Z' denote the corresponding subscheme, and $\mathcal{I}'$ its ideal. Let's show that $\mathcal{I}'$ is generated by the following elements:

$$f'_r := (x'_1)^{m+1-r-\delta(r)} (x'_2)^r \quad \text{for } 0 \le r \le m.$$

The f'_r reduce to the f_r, which generate $\mathcal{I}$. Further, $\mathcal{I}'$ reduces to $\mathcal{I}$ as Z' is flat over $K[\epsilon]$. Hence it suffices to prove that $\mathcal{I}'$ contains the f'_r.

Note that $(s'_0 - b\epsilon)^p = (s'_0)^p$ as the characteristic is p. Hence, for each r,

$$f'_r = (x'_1)^{m+1-\delta(r)}(s'_0 - b\epsilon)^r = (s'_k)^{m+1-\delta(r)}(s'_0)^{k(m+1)+(p-k)\delta(r)}(s'_0 - b\epsilon)^{r-p\delta(r)}$$

for $1 \le k \le p-1$. Therefore, the pullback of f'_r vanishes along $e_T^{(1,\,p+1)}E_T^{(1)}$ to order at least m, and along $e_T^{(k+1,\,p+1)}E_T^{(k+1)}$ to order at least $k(m+1)$ for $k \ge 1$ since $(p-k)\delta(r) \ge 0$ and $r - p\delta(r) \ge 0$. Thus $\mathcal{I}'$ contains the f'_r.

Recall that $T_z\mathrm{Hilb}^d_{F/Y}(K) = \mathrm{Hom}(\mathcal{I}, \mathcal{O}_Z)$. Furthermore, it follows from the computations above that

$$d_t\Upsilon_\theta(t')(f'_r) = (m+1-r-\delta(r))x_1^{m-r-\delta(r)}x_2^r a_1 + r x_1^{m+1-r-\delta(r)}x_2^{r-1}a_2$$

for $0 \le r \le m$. Therefore,

$$\ker(d_t\Upsilon_\theta) = \big\{(a_1, a_2, b) \;\big|\; a_1 = a_2 = 0\big\},$$

and we are done. □

Definition B.6. Fix $m \ge 3$. Let $\mathbf{N}_m$ denote the minimal Enriques diagram of the following plane curve singularity: an ordinary tacnode $\{\, x_2(x_2 - x_1^2) = 0 \,\}$ union with $m-2$ smooth branches whose tangent lines are distinct and different from the common tangent line of the two branches of the tacnode.

Proposition B.7. *Fix $m \ge 3$. Set $\mathbf{D} := \mathbf{N}_m$ and $d := \binom{m+1}{2} + 3$. Then $\mathbf{D}$ has a unique ordering θ; also $\mathrm{Aut}(\mathbf{D}) = 1$ and $\deg \mathbf{D} = d$. Take $F = \mathbb{A}^2_K$. Then $\dim F(\mathbf{D}, \theta) = 3$, and $\Upsilon_\theta \colon F(\mathbf{D}, \theta) \to \mathrm{Hilb}^d_{F/K}$ is an embedding; also, $\Psi = \Upsilon_\theta$.*

PROOF. Plainly, $\mathbf{D}$ has 2 vertices, say V_0 and V_1 ordered by succession. Then proximity is given by $V_1 \succ V_0$. Further, the weights are given by $m_{V_0} = m$ and $m_{V_1} = 2$. Set $\theta(V_k) := k$; plainly, θ is an ordering of $\mathbf{D}$, and is the only one. Also, plainly, $\mathrm{Aut}(\mathbf{D}) = 1$ and $\deg \mathbf{D} = d$. Theorem 3.10 says that $\dim F(\mathbf{D}, \theta) = \dim \mathbf{D}$, but plainly $\dim \mathbf{D} = 3$. Now, $\Psi = \Upsilon_\theta$ because $\mathrm{Aut}(\mathbf{D}) = 1$. Further, Theorem 5.7 says that Ψ is universally injective. Hence Ψ is an embedding because it is nowhere ramified owing to the following lemma. □

Lemma B.8. *Under the conditions of Proposition* B.7*, let $t \in F(\mathbf{D}, \theta)$ be a K-point. Then* $\mathrm{Ker}(d_t\Upsilon_\theta) = 0$.

PROOF. Say t represents the sequence (t_0, t_1) of arbitrarily near K-points of F/Y. Choose coordinates $x_1,\ x_2$ on F such that $t_0 : x_1 = x_2 = 0$ and such that t_1 is the point of intersection of the exceptional divisor E_0 with the proper transform of the x_1-axis. Set $s := x_2/x_1$. Then $t_1 : s = x_1 = 0$.

Set $z := \Upsilon_\theta(t) \in \mathrm{Hilb}^d_{F/Y}(K)$. Let Z denote the corresponding subscheme, and $\mathcal{I}$ its ideal. Recall from the proof of Proposition 5.4 that $\mathcal{I} = \varphi_{K*}\mathcal{O}(-E_K)$ where $E_K = \sum_{i=0}^{1} m_{V_i}E^{(i+1,\,2)}$. Recall from the proof of Proposition B.7 that $m_{V_0} = m$ and $m_{V_1} = 2$ and that $V_1 \succ V_0$. It follows that

$$E_K = m e_K^{(1,\,2)}E^{(1)} + (m+2)E_K^{(2)}.$$

Set $\delta(0) := 2$, set $\delta(1) := 1$, and set $\delta(r) := 0$ if $r \ge 2$. Set

$$f_r := x_1^{m-r+\delta(r)}x_2^r \quad \text{for } 0 \le r \le m.$$

Let's now show that the f_r generate $\mathcal{I}$.

First, note that, for each r,

$$f_r = x_1^{m+\delta(r)} s^r.$$

Hence, the pullback of f_r vanishes along $e_K^{(1,2)} E_K^{(1)}$ to order at least m, and along $E_K^{(2)}$ to order at least $m+2$, since $m + r + \delta(r) \geq m + 2$. Thus $f_r \in \mathcal{I}$ for each r.

Let $\mathcal{J}$ be the ideal generated by the f_r. Then $\mathcal{J} \subset \mathcal{I}$. Now, $K[x_1, x_2]/\mathcal{J}$ is spanned as a K-vector space by the monomials $x_1^{m-r+\delta(r)} x_2^l$ for $0 \leq l < r \leq m$ and by x_1^{m-1}, $x_1^{m-1}x_2$, and x_1^{m+1}. Hence $\mathcal{J} = \mathcal{I}$ because

$$\dim\bigl(K[x_1, x_2]/\mathcal{J}\bigr) \leq \textstyle\sum_{r=0}^{m} r + 3 = d = \dim K[x_1, x_2]/\mathcal{I}.$$

Furthermore, the monomials x_1^{m-1} and $x_1^{m-1}x_2$ and x_1^{m+1}, and $x_1^{m-r+\delta(r)} x_2^l$ for $0 \leq l < r \leq m$ form a basis of the K-vector space $K[x_1, x_2]/\mathcal{I}$.

Let $K[\epsilon]$ be the ring of dual numbers, and set $T := \operatorname{Spec}(K[\epsilon])$. Let (t'_0, t'_1) be a strict sequence of arbitrarily near T-points of F/Y lifting (t_0, t_1). Then there are $a_1, a_2, b \in K$ so that, after setting $x'_1 := x_1 + a_1\epsilon$ and $x'_2 := x_2 + a_2\epsilon$ and $s' := x'_2/x'_1 + b\epsilon$, we have $t'_0 : x'_1 = x'_2 = 0$ and $t'_1 : s' = x'_1 = 0$.

Let $t' \in F(\mathbf{D}, \theta)(T)$ represent (t'_0, t'_1). Set $z' := \Upsilon_\theta(t') \in \operatorname{Hilb}^d_{F/Y}(T)$. Let Z' denote the corresponding subscheme, and $\mathcal{I}'$ its ideal. Let's show that $\mathcal{I}'$ is generated by the following elements:

$$f'_r := (x'_1)^{m-r+\delta(r)}(x'_2)^r + rb\epsilon(x'_1)^{m-r+1+\delta(r)}(x'_2)^{r-1} \quad \text{for } 0 \leq r \leq m.$$

The f'_r reduce to the f_r, which generate $\mathcal{I}$. Further, $\mathcal{I}'$ reduces to $\mathcal{I}$ as Z' is flat over $K[\epsilon]$. Hence it suffices to prove that $\mathcal{I}'$ contains the f'_r.

The equation $x'_2/x'_1 = s' - b\epsilon$ yields

$$f'_r = (x'_1)^{m+\delta(r)}(s')^r \quad \text{for } 0 \leq r \leq m.$$

Hence, the pullback of f'_r vanishes along $e_T^{(1,2)} E_T^{(1)}$ to order at least m, and along $E_T^{(2)}$ to order at least $m+2$ since $m + r + \delta(r) \geq m + 2$. Thus $\mathcal{I}'$ contains the f'_r.

Recall that $T_z\operatorname{Hilb}^d_{F/Y}(K) = \operatorname{Hom}(\mathcal{I}, \mathcal{O}_Z)$. Furthermore, it follows from the computations above that

$$d_t\Upsilon_\theta(t')(f'_r) = x_1^{m-r+\delta(r)-1} x_2^{r-1} \bigl((m - r + \delta(r))x_2a_1 + rx_1a_2 + rx_1^2b\bigr)$$

for $0 \leq r \leq m$. In particular, $rx_1^{m-2}x_2^2b \in \mathcal{I}$ yields

$$\begin{aligned}
d_t\Upsilon_\theta(t')(f'_1) &= mx_1^{m-1}x_2a_1 + x_1^m a_2 + x_1^{m+1}b \\
d_t\Upsilon_\theta(t')(f'_0) &= (m+2)x_1^{m+1}a_1, \quad \text{and} \\
d_t\Upsilon_\theta(t')(f'_3) &= (m-3)x_1^{m-4}x_2^3a_1 + 3x_1^{m-3}x_2^2a_2.
\end{aligned}$$

Recall that, in $K[x_1, x_2]/\mathcal{I}$, the monomials

$$x_1^{m-1},\ x_1^{m-1}x_2,\ x_1^{m+1},\ \text{and } x_1^{m-r+\delta(r)}x_2^l \text{ for } 0 \leq l < r \leq m$$

are linearly independent. But $m \geq 3$, so at least one of the coefficients m, $m+2$, and $m-3$ is prime to the characteristic. Thus, $\ker(d_t\Upsilon_\theta) = 0$, and we are done. □

References

[1] Casas, E., *Infinitely near imposed singularities and singularities of polar curves,* Math. Ann. **287** (1990), 429–454. 5.1

[2] Casas, E., "Singularities of plane curves," London Math Society Lecture Note Series **276**, Cambridge Univ. Press, 2000. 2, 2.2, 2.7

[3] Deligne, P. *Exp. X, Intersections sur les surfaces régulières,* in "SGA 7 II, Groupes de Monodromie en géométrie algébrique," Lecture Notes in Math. **340**, Springer-Verlag, 1973, pp. 1–38. 5.1, 5, 5

[4] Enriques, F., and Chisini, O., "Lezioni sulla teoria geometrica delle equazioni e delle funzioni algebriche," Zanichelli, Bologna, 1915. 2, 5.1

[5] Greuel, G.-M., Lossen, C., and Shustin, E., *Plane curves of minimal degree with prescribed singularities*, Invent. Math. **133** (1998), 539–580. 2, 2.7

[6] Grothendieck, A., and Dieudonné, J. "Eléments de Géométrie Algébrique II," Publ. Math. IHES, Vol. 8, 1961. 5

[7] Grothendieck, A., and Dieudonné, J. "Eléments de Géométrie Algébrique III-2," Publ. Math. IHES, Vol. 17, 1963. 4, A

[8] Grothendieck, A., and Dieudonné, J. "Eléments de Géométrie Algébrique IV-3," Publ. Math. IHES, Vol. 28, 1966. 5

[9] Grothendieck, A., and Dieudonné, J. "Eléments de Géométrie Algébrique IV-4," Publ. Math. IHES, Vol. 32, 1967. 5

[10] Gustavsen, T. S., "Topics in deformation and moduli theory for singularities on curves and surfaces," Thesis, U. Oslo, 1998. 5

[11] Harbourne, B., *Iterated blow-ups and moduli for rational surfaces*, in "Algebraic Geometry, Sundance 1986," A. Holme, R. Speiser (Eds.), Lecture Notes in Math **1311** (1988), 101–117. 1, 3, 3

[12] Hoskin, M. A., *Zero-dimensional valuation ideals associated with plane curve branches,* Proc. London Math. Soc. **6** (1956), 70–99. 5.1

[13] Illusie, L., *Conditions de finitude relative,* Exposé III in SGA 6, "Théorie des intersections et théorème de Riemann–Roch," Lecture Notes in Math **225** (1971), 222–273. A

[14] Kleiman, S. L., *Multiple-point formulas I: Iteration,* Acta Math **147** (1981), 13–49. 3

[15] Kleiman, S., and Piene, R., *Enumerating singular curves on surfaces*, in "Algebraic geometry — Hirzebruch 70," Cont. Math. **241** (1999), 209–38 (corrections and revision in math.AG/9903192). 1, 2, 2.1, 2.6, 2.7, 5

[16] Kleiman, S., and Piene, R., *Node polynomials for families: methods and applications,* Math. Nachr. **271** (2004), 1–22. 2, 2.1, 2.7

[17] Kleiman, S., and Piene, R., *Node polynomials for curves on surfaces*, in preparation. 2

[18] Lipman, J., *Rational singularities, with applications to algebraic surfaces and unique factorization*, Publ. Math. IHES, **36** (1969), 195–279. 5.1

[19] Lipman, J., *Proximity inequalities for complete ideals in two-dimensional regular local rings*, Cont. Math, **159** (1994), 293–306. 2.2, 5.1

[20] Liu, A.-K., *Family blowup formula, admissible graphs and the enumeration of singular curves. I,* J. Differential Geom. **56** (2000), no. 3, 381–579. 1

[21] Lossen, C.: "The geometry of equisingular and equianalytical families of curves on a surface," Dr. dissertation, Universität Kaiserslautern, 1998. 1, 5.6

[22] Mumford, D., "Abelian varieties," Tata Institute Studies in Math., Oxford U. Press, 1966. A, A

[23] Nobile, A., and Villamayor, O. E., *Equisingular stratifications associated to families of planar ideals*, J. Alg. **193** (1997), 239–59. 1, 5.1, 5.6

[24] Roé, J. *On the existence of plane curves with imposed multiple points*, J. Pure Appl. Algebra **156** (2001), no. 1, 115–126.

[25] Roé, J. *Varieties of clusters and Enriques diagrams,* Math. Proc. Cambridge Philos. Soc. **137** (2004), no. 1, 69–94. 1, 2, 2.2, 3, 3, 4

[26] Russell, H. *Counting singular plane curves via Hilbert schemes,* Adv. Math. 179 (2003), no. 1, 38–58. 1

[27] Russell, H. *Alignment correspondences*, J. Algebra **278** (2004), no. 2, 553–570. 1

[28] Vainsencher, I.: *Enumeration of n-fold tangent hyperplanes to a surface*, J. Alg. Geom. **4** (1995), 503–26. 1

[29] Zariski, O., and Samuel, P. "Commutative algebra. Vol. II," D. Van Nostrand Co., Inc., 1960.

5.1

DEPARTMENT OF MATHEMATICS, MIT, 77 MASSACHUSETTS AVENUE, CAMBRIDGE, MA 02139, USA
E-mail address: kleiman@math.mit.edu

CMA AND DEPARTMENT OF MATHEMATICS, UNIVERSITY OF OSLO, PO BOX 1053, BLINDERN, NO-0316 OSLO, NORWAY
E-mail address: ragnip@math.uio.no

DEPARTMENT OF MATHEMATICS, BEN GURION UNIVERSITY OF THE NEGEV, P.O.B. 653, BE'ER SHEVA 84105, ISRAEL
E-mail address: tyomkin@cs.bgu.ac.il

Enriques diagrams, arbitrarily near points, and Hilbert schemes

Steven KLEIMAN, Ragni PIENE, and
with Appendix B by Ilya TYOMKIN

Abstract. Given a smooth family F/Y of geometrically irreducible surfaces, we study sequences of *arbitrarily near* T-points of F/Y; they generalize the traditional sequences of infinitely near points of a single smooth surface. We distinguish a special sort of these new sequences, the *strict* sequences. To each strict sequence, we associate an ordered unweighted Enriques diagram. We prove that the various sequences with a fixed diagram form a functor, and we represent it by a smooth Y-scheme.

We equip this Y-scheme with a free action of the automorphism group of the diagram. We equip the diagram with weights, take the subgroup of those automorphisms preserving the weights, and form the corresponding quotient scheme. Our main theorem constructs a canonical universally injective map from this quotient scheme to the Hilbert scheme of F/Y; further, this map is an embedding in characteristic 0. However, in every positive characteristic, we give an example, in Appendix B, where the map is purely inseparable.

1. Introduction

Recently, there has been much renewed interest in an old and timeless problem: enumerating the r-nodal curves in a linear system on a smooth projective surface. Notably, Tzeng [**30**] and Kool, Shende and Thomas [**19**], in very different ways, proved Göttsche's conjecture [**5**], which was motivated by the Yau–Zaslow formula [**32**] and describes the shape of a corresponding generating series in r. However, Göttsche's expression involves two power series, which remain unknown in general.

On the other hand, Vainsencher [**31**] found enumerating polynomials for $r \leq 7$; they are explicit and general, although he [**31**, Sec. 7] was unsure of the case $r = 7$. In [**16**] and [**17**], the present authors refined and extended Vainsencher's work, by settling the case $r = 7$, handling $r = 8$, producing more compact formulas, and establishing validity under more extensive conditions. Further, this enumeration, unlike Göttsche's, applies to nonconstant families of surfaces; notably, see [**31**, pp. 513–514] and [**17**, pp. 80–83], it proves 17,601,000 is the number of irreducible 6-nodal quintic plane curves on a general quintic 3-fold in 4-space, contrary to the predictions of Clemens' conjecture and of mirror symmetry.

2000 *Mathematics Subject Classification.* Primary 14N10; Secondary 14C20, 14H40, 14K05.

Key words and phrases. Enriques diagrams, arbitrarily near points, Hilbert schemes, property of exchange.

To establish a range of validity for these enumerative formulas, it is necessary to analyze various generalized Severi varieties, namely, the loci of curves of given equisingularity type in the system. Göttsche [**5**, Prp. 5.2] treated nodes in an ad hoc fashion; Tzeng relies on his analysis, whereas Kool, Shende and Thomas [**19**, Prp. 2.1] improved it, thus extending the range of validity.

On the other hand, Vainsencher's approach, as pursued by the present authors, relies on a more extensive and more systematic analysis. It is based on Enriques diagrams. They are directed graphs, similar to resolution graphs, that represent the equisingularity types of the curves. Equivalently, see [**16**, § 3] and the references there, they represent the types of the complete ideals, the ideals formed by the equations of the curves with singularities of the same type or worse at given points.

Specifically, in the authors' paper [**16**], Proposition (3.6) on p. 225 concerns the locus $H(\mathbf{D})$ that sits in the Hilbert scheme of a smooth irreducible complex surface and parametrizes the complete ideals $\mathcal{I}$ with a given minimal Enriques diagram $\mathbf{D}$. The proposition asserts that $H(\mathbf{D})$ is smooth and equidimensional.

The proposition was justified intuitively, then given an ad hoc proof in [**16**]. The intuitive justification was not developed into a formal proof, which is surprisingly long and complicated. However, the proof yields more: it shows $H(\mathbf{D})$ is irreducible; it works for nonminimal $\mathbf{D}$; and it works for *families* of surfaces. Further, it works to a great extent when the characteristic is positive or mixed, but then it only shows $H(\mathbf{D})$ has a finite and universally injective covering by a smooth cover; this covering need not be birational, as examples in Appendix B show.

It is naive to form $H(\mathbf{D})$ as a locus with reduced scheme structure. It is more natural to consider the functor of sequences of arbitrarily near points corresponding to $\mathbf{D}$. This functor is representable by a smooth irreducible scheme, and it admits a natural map into the Hilbert scheme, whose image is $H(\mathbf{D})$. This map is finite and universally injective, so an embedding in characteristic zero, but it may be totally ramified in positive characteristic as the examples show.

Originally, the authors planned to develop this discussion in a paper that also dealt with other loose ends, notably, the details of the enumeration of curves with eight nodes. However, there is so much material involved that it makes more sense to divide it up. Thus the discussion of $H(\mathbf{D})$ alone is developed in the present paper; the result itself is asserted in Corollary 5.8. Here, in more detail, is a description of this paper's contents.

Fix a smooth family of geometrically irreducible surfaces F/Y and an integer $n \geq 0$. Given a Y-scheme T, by a *sequence of arbitrarily near T-points of F/Y*, we mean an $(n+1)$-tuple $(t_0, \dots, t_n)$ where t_0 is a T-point of $F_T^{(0)} := F \times_Y T$ and where t_i, for $i \geq 1$, is a T-point of the blowup $F_T^{(i)}$ of $F_T^{(i-1)}$ at t_{i-1}. (If each t_i is, in fact, a T-point of the exceptional divisor $E_T^{(i)}$ of $F_T^{(i)}$, then $(t_0, \dots, t_n)$ is a sequence of infinitely near points in the traditional sense.) The sequences of arbitrarily near T-points form a functor in T, and it is representable by a smooth Y-scheme $F^{(n)}$, according to Proposition 3.4 below; this result is due, in essence, to Harbourne [**12**, Prp. I.2, p. 104].

We say that the sequence $(t_0, \dots, t_n)$ is *strict* if, for each i, j with $1 \leq j \leq i$, the image $T^{(i)} \subset F_T^{(i)}$ of t_i is either (a) disjoint from, or (b) contained in, the strict transform of the exceptional divisor $E_T^{(j)}$ of $F_T^{(j)}$. If (b) obtains, then we say that t_i is *proximate* to t_j and we write $t_i \succ t_j$.

To each strict sequence, we associate, in Section 3, an unweighted Enriques diagram $\mathbf{U}$ and an ordering $\theta\colon \mathbf{U} \xrightarrow{\sim} \{0,\dots,n\}$. Effectively, $\mathbf{U}$ is just a graph whose vertices are the t_i. There is a directed edge from t_j to t_i provided that $j+1 \le i$ and that the map from $F_T^{(i)}$ to $F_T^{(j+1)}$ is an isomorphism in a neighborhood of $T^{(i)}$ and embeds $T^{(i)}$ in $E_T^{(j+1)}$. In addition, $\mathbf{U}$ inherits the binary relation of proximity. Finally, θ is defined by $\theta(t_i) := i$. This material is discussed in more detail in Section 2. In particular, to aid in passing from $(t_0,\dots,t_n)$ to $(\mathbf{U},\,\theta)$, we develop a new combinatorial notion, which we call a *proximity structure*.

Different strict sequences often give rise to isomorphic pairs $(\mathbf{U},\,\theta)$. If we fix a pair, then the corresponding sequences form a functor, and it is representable by a subscheme $F(\mathbf{U},\,\theta)$ of $F^{(n)}$, which is Y-smooth with irreducible geometric fibers of a certain dimension. This statement is asserted by Theorem 3.10, which was inspired by Roé's Proposition 2.6 in [**27**].

Given another ordering θ', in Section 4 we construct a natural isomorphism

$$\Phi_{\theta,\theta'}\colon F(\mathbf{U},\,\theta) \xrightarrow{\sim} F(\mathbf{U},\,\theta').$$

It is easy to describe $\Phi_{\theta,\theta'}$ on geometric points. A geometric point of $F(\mathbf{U},\,\theta)$ corresponds to a certain sequence of local rings in the function field of the appropriate geometric fiber of F/Y. Then $\theta' \circ \theta^{-1}$ yields a suitable permutation of these local rings, and so a geometric point of $F(\mathbf{U},\,\theta')$. However, it is harder to work with arbitrary T-points. Most of the work is carried out in the proofs of Lemmas 4.1 and 4.2, and the work is completed in the proof of Proposition 4.3.

We easily derive two corollaries. Corollary 4.4 asserts that $\mathrm{Aut}(\mathbf{U})$ acts freely on $F(\mathbf{U},\,\theta)$; namely, $\gamma \in \mathrm{Aut}(\mathbf{U})$ acts as $\Phi_{\theta,\theta'}$ where $\theta' := \theta \circ \gamma$. Corollary 4.5 asserts that $\Psi\colon F(\mathbf{U},\,\theta)\big/\mathrm{Aut}(\mathbf{U})$ is Y-smooth with irreducible geometric fibers.

A different treatment of $F(\mathbf{U},\,\theta)$ is given by A.-K. Liu in [**22**]. In Section 3 on pp. 400–401, he constructs $F^{(n)}$. In Subsection 4.3.1 on pp. 412–414, he discusses his version of an Enriques diagram, which he calls an "admissible graph." In Subsections 4.3.2, 4.4.1, and 4.4.2 on pp. 414–427, he constructs $F(\mathbf{U},\,\theta)$, and proves it is smooth. In Subsection 4.5 on pp. 428–433, he constructs the action of $\mathrm{Aut}(\mathbf{U})$ on $F(\mathbf{U},\,\theta)$. Of course, he uses different notation; also, he doesn't represent functors. But, like the present authors, he was greatly inspired by Vainsencher's approach in [**31**] to enumerating the singular curves in a linear system on a smooth surface.

Our main result is Theorem 5.7. It concerns the Enriques diagram $\mathbf{D}$ obtained by equipping the vertices $V \in \mathbf{U}$ with weights m_V satisfying the *Proximity Inequality*, $m_V \ge \sum_{W \succ V} m_W$. We discuss the theory of such $\mathbf{D}$ in Section 2. Note that $\mathrm{Aut}(\mathbf{D}) \subset \mathrm{Aut}(\mathbf{U})$. Set $d := \sum_V \binom{m_V+1}{2}$. Theorem 5.7 asserts the existence of a universally injective map from the quotient to the Hilbert scheme

$$\Psi\colon F(\mathbf{U},\,\theta)\big/\mathrm{Aut}(\mathbf{D}) \to \mathrm{Hilb}^d_{F/Y}\,.$$

Proposition 5.4 implies that Ψ factors into a finite map followed by an open embedding. So Ψ is an embedding in characteristic 0. However, in any positive characteristic, Ψ can be ramified everywhere; examples are given in Appendix B, whose content is due to Tyomkin. Nevertheless, according to Proposition 5.9, in the important case where every vertex of $\mathbf{D}$ is a root, Ψ is an embedding in any characteristic. Further, adding a nonroot does not necessarily mean there is a characteristic in which Ψ ramifies, as other examples in Appendix B show.

We construct Ψ via a relative version of the standard construction of the complete ideals on a smooth surface over a field, which grew out of Zariski's work in 1938; the standard theory is reviewed in Subsection 5.1. Now, a T-point of $F(\mathbf{U}, \theta)$ represents a sequence of blowing-ups $F_T^{(i)} \to F_T^{(i-1)}$ for $1 \le i \le n+1$. On the final blowup $F_T^{(n+1)}$, for each i, we form the preimage of the ith center $T^{(i)}$. This preimage is a divisor; we multiply it by $m_{\theta^{-1}(i)}$, and we sum over i. We get an effective divisor. We take its ideal, and push down to F_T. The result is an ideal, and it defines the desired T-flat subscheme of F_T. The flatness holds and the formation of the subscheme commutes with base change owing to the generalized property of exchange proved in Appendix A. Appendix A is of independent interest.

It is not hard to see that Ψ is injective on geometric points, and that its image is the subset $H(\mathbf{D}) \subset \mathrm{Hilb}^d_{F/Y}$ parameterizing complete ideals with diagram $\mathbf{D}$ on the fibers of F/Y. To prove that Ψ induces a finite map onto $H(\mathbf{D})$, we use a sort of valuative criterion; the work appears in Lemma 5.2 and Proposition 5.4. An immediate corollary, Corollary 5.5, asserts that $H(\mathbf{D})$ is locally closed. This result was proved for complex analytic varieties by Lossen [**23**, Prp. 2.19, p. 35] and for excellent schemes by Nobile and Villamayor [**25**, Thm. 2.6, p. 250]. Their proofs are rather different from each other and from ours.

In [**28**] and [**29**], Russell studies sets somewhat similar to the $H(\mathbf{D})$. They parameterize isomorphism classes of finite subschemes of F supported at one point.

In short, Section 2 treats weighted and unweighted Enriques diagrams and proximity structures. Section 3 treats sequences of arbitrarily near T-points. To certain ones, the *strict* sequences, we associate an unweighted Enriques diagram $\mathbf{U}$ and an ordering θ. Fixing $\mathbf{U}$ and θ, we obtain a functor, which we represent by a smooth Y-scheme $F(\mathbf{U}, \theta)$. Section 4 treats the variance in θ. We produce a free action on $F(\mathbf{U}, \theta)$ of $\mathrm{Aut}(\mathbf{U})$. Section 5 treats the Enriques diagram $\mathbf{D}$ obtained by equipping $\mathbf{U}$ with suitable weights. We construct a map Ψ from $F(\mathbf{U}, \theta)/\mathrm{Aut}(\mathbf{D})$ to $\mathrm{Hilb}_{F/Y}$, whose image is the locus $H(\mathbf{D})$ of complete ideals. We prove $H(\mathbf{D})$ is locally closed. Our main theorem asserts that Ψ is universally injective, and in fact, in characteristic 0, an embedding. Appendix A treats the generalized property of exchange used in constructing Ψ. Finally, Tyomkin's Appendix B treats a few examples: in some, Ψ is ramified; in others, there's a nonroot, yet Ψ is unramified.

2. Enriques diagrams

In 1915, Enriques [**4**, IV.I, pp. 350–51] explained a way to represent the equisingularity type of a plane curve singularity by means of a directed graph: each vertex represents an arbitrarily near point, and each edge connects a vertex representing a point to a vertex representing a point in its first-order neighborhood; furthermore, the graph is equipped with a binary relation representing the "proximity" of arbitrarily near points. These graphs have, for a long time, been called *Enriques diagrams*, and in 2000, they were given a modern treatment by Casas in [**2**, Sec. 3.9, pp. 98–102].

Based in part on a preliminary edition of Casas' monograph, a more axiomatic treatment was given by the authors in [**17**, § 2], and this treatment is elaborated on here in Subsection 2.1. In this treatment, the vertices are weighted, and the number of vertices is minimized. When the diagram arises from a curve, the vertices correspond to the "essential points" as defined by Greuel et al. [**6**, Sec. 2.2], and the weights are the multiplicities of the points on the strict transforms. Casas'

treatment is similar: the Proximity Inequality is always an equality, and the leaves, or extremal vertices, are of weight 1; so the rest of the weights are determined.

At times, it is convenient to work with unweighted diagrams. For this reason, Roé [**27**, §1], inspired by Casas, defined an "Enriques diagram" to be an unweighted graph, and he imposed five conditions, which are equivalent to our Laws of Proximity and of Succession. Yet another description of unweighted Enriques diagrams is developed below in Subsection 2.3 and Proposition 2.4 under the name of "proximity structure." This description facilitates the formal assignment, in Subsection 2.7, of an Enriques diagram to a plane curve singularity. Similarly, the description facilitates the assignment in Section 3 of the Enriques diagram associated to a strict sequence of arbitrarily near points.

At times, it is convenient to order the elements of the set underlying an Enriques diagram or underlying a proximity structure. This subject is developed in Subsections 2.2 and 2.3 and in Corollary 2.5. It plays a key role in the later sections.

Finally, in Subsection 2.6, we discuss several useful numerical characters. Three were introduced in [**16**, Sct. 2, p. 214], and are recalled here. Proposition 2.8 describes the change in one of the three when a singularity is blown up; this result is needed in [**18**]

2.1 (Enriques diagrams)**.** First, recall some general notions. In a directed graph, a vertex V is considered to be one of its own predecessors and one of its own successors. Its other predecessors and successors W are said to be *proper*. If there are no loops, then W is said to be *remote*, or *distant*, if there is a distinct third vertex lying between V and W; otherwise, then W is said to be *immediate*.

A *tree* is a directed graph with no loops; by definition, it has a single initial vertex, or *root,* and every other vertex has a unique immediate predecessor. A final vertex is called a *leaf*. A disjoint union of trees is called a *forest*.

Next, from [**17**, § 2], recall the definition of a minimal Enriques diagram. It is a finite forest $\mathbf{D}$ with additional structure. Namely, each vertex V is assigned a weight m_V, which is an integer at least 1. Also, the forest is equipped with a binary relation; if one vertex V is related to another U, then we say that V is *proximate* to U, and write $V \succ U$. If U is a remote predecessor of V, then we call V a *satellite of* U; if not, then we say V is *free*. Thus a root is free, and a leaf can be either free or a satellite.

Elaborating on [**17**], call $\mathbf{D}$ an *Enriques diagram* if $\mathbf{D}$ obeys these three laws:

> (Law of Proximity) *A root is proximate to no vertex. If a vertex is not a root, then it is proximate to its immediate predecessor and to at most one other vertex; the latter must be a remote predecessor. If one vertex is proximate to a second, and if a distinct third lies between the two, then it too is proximate to the second.*
>
> (Proximity Inequality) *For each vertex* V*,*
>
> $$m_V \geq \textstyle\sum_{W \succ V} m_W.$$
>
> (Law of Succession) *A vertex may have any number of* free *immediate successors, but at most two immediate successors may be satellites, and they must be satellites of different vertices.*

Notice that, by themselves, the Law of Proximity and the Proximity Inequality imply that a vertex V has at most m_V immediate successors; so, although this

property is included in the statement of the Law of Succession in [**17**, § 2], it is omitted here.

Recovering the notion in [**16**], call an Enriques diagram **D** *minimal* if **D** obeys the following fourth law:

> (Law of Minimality) *There are only finitely many vertices, and every leaf of weight* 1 *is a satellite.*

In [**16**], the Law of Minimality did not include the present finiteness restriction; rather, it was imposed at the outset.

2.2 (Unweighted diagrams)**.** In [**27**, §1], Roé defines an Enriques diagram to be an *unweighted* finite forest that is equipped with a binary relation, called "proximity," that is required to satisfy five conditions. It is not hard to see that his conditions are equivalent to our Laws of Proximity and Succession. Let us call this combinatorial structure an *unweighted Enriques diagram*.

Let **U** be any directed graph on $n+1$ vertices. By an *ordering* of **U**, let us mean a bijective mapping

$$\theta\colon \mathbf{U} \xrightarrow{\sim} \{0,\dots,n\}$$

such that, if one vertex V precedes another W, then $\theta(V) \le \theta(W)$. Let us call the pair $(\mathbf{U}, \theta)$ an *ordered directed graph*.

An ordering θ need not be unique. Furthermore, if one exists, then plainly **U** has no loops. Conversely, if **U** has no loops — if it is a forest — then **U** has at least one ordering. Indeed, then **U** has a leaf L. Let **T** be the complement of L in **U**. Then **T** inherits the structure of a forest. So, by induction on n, we may assume that **T** has an ordering. Extend it to **U** by mapping L to n.

Associated to any ordered unweighted Enriques diagram $(\mathbf{U}, \theta)$ is its *proximity matrix* (p_{ij}), which is the $n+1$ by $n+1$ lower triangular matrix defined by

$$p_{ij} := \begin{cases} 1, & \text{if } i = j; \\ -1, & \text{if } \theta^{-1}i \text{ is proximate to } \theta^{-1}j; \\ 0, & \text{otherwise.} \end{cases}$$

The transpose was introduced by Du Val in 1936, and he named it the "proximity matrix" in 1940; Lipman [**21**, p. 298] and others have followed suit. The definition here is the one used by Roé [**27**] and Casas [**2**, p. 139].

Note that $(\mathbf{U}, \theta)$ is determined up to unique isomorphism by (p_{ij}).

2.3 (Proximity structure)**.** Let **U** be a finite set equipped with a binary relation. Call **U** a *proximity structure,* its elements *vertices,* and the relation *proximity* if the following three laws are obeyed:

- (P1) *No vertex is proximate to itself; no two vertices are each proximate to the other.*
- (P2) *Every vertex is proximate to at most two others; if to two, then one of the two is proximate to the other.*
- (P3) *Given two vertices, at most one other is proximate to them both.*

A proximity structure supports a natural structure of directed graph. Indeed, construct an edge proceeding from one vertex V to another W whenever either W is proximate only to V or W is proximate both to V and U but V is proximate to U (rather than U to V). Of course, this graph may have loops; for example, witness a triangle with each vertex proximate to the one clockwise before it, and

witness a pentagon with each vertex proximate to the two clockwise before it.

Let us say that a proximity structure is *ordered* if its vertices are numbered, say $V_0, \dots, V_n$, such that, if V_i is proximate to V_j, then $i > j$.

Proposition 2.4. *The unweighted Enriques diagrams sit in natural bijective correspondence with the proximity structures whose associated graphs have no loops.*

PROOF. First, take an unweighted Enriques diagram, and let's check that its proximity relation obeys Laws (P1) to (P3).

A vertex is proximate only to a proper successor; so no vertex is proximate to itself. And, if two vertices were proximate to one another, then each would succeed the other; so there would be a loop. Thus (P1) holds.

A root is proximate to no vertex. Every other vertex W is proximate to its immediate predecessor V and to at most one other vertex U, which must be a remote predecessor. Since an immediate predecessor is unique in a forest, V must lie between W and U; whence, V must be proximate to U. Thus (P2) holds.

Suppose two vertices W and X are each proximate to two others U and V. Say V is the immediate predecessor of W. Then U is a remote predecessor of W; so U precedes V. Hence V is also the immediate predecessor of X, and W is also a remote predecessor of X. Thus both W and X are immediate successors of V, and both are satellites of W; so the Law of Succession is violated. Thus (P3) holds.

Conversely, take a proximity structure whose associated graph has no loops. Plainly, a root is proximate to no vertex. Suppose a vertex W is not a root. Then W has an immediate predecessor V. Plainly, W is proximate to V. Plainly, W is proximate to at most one other vertex U, and if so, then V is proximate to U. Since U cannot also be proximate to V by (P1), it follows that V is the only immediate predecessor to W.

Every vertex is, therefore, preceded by a unique root. Plainly the connected component of each root is a tree. Thus the graph is a finite forest.

Returning to U, V, and W, we must show that U precedes W. Now, V is proximate to U. So V is not a root. Hence V has an immediate predecessor V'. If $V' = U$, then stop. If not, then V' is proximate to U owing to the definition of the associated graph, since V is proximate to U. Hence, similarly, V' has an immediate predecessor V''. If $V'' = U$, then stop. If not, then repeat the process. Eventually, you must stop since the number of vertices is finite. Thus U precedes W. Furthermore, every vertex between U and W is proximate to U. Thus the Law of Proximity holds.

Continuing with U, V, and W, suppose that W' is a second immediate successor of V and that W' is also proximate to a vertex U'. Then $U' \neq U$ since at most one vertex can be proximate to both V and U by (P3).

Finally, suppose that W'' is a third immediate successor of V and that W'' is also proximate to a vertex U''. Then $U'' \neq U$ and $U'' \neq U'$ by what we just proved. But V is proximate to each of U, U', and U''. So (P2) is violated. Thus the Law of Succession holds, and the proof is complete. □

Corollary 2.5. *The ordered unweighted Enriques diagrams sit in natural bijective correspondence with the ordered proximity structures.*

PROOF. Given an unweighted Enriques diagram, its proximity relation obeys Laws (P1) to (P3) by the proof of Proposition 2.4. And, if one vertex V is proximate to another W, then W precedes V. So $\theta(W) < \theta(V)$ for any ordering θ. Hence, if

V is numbered $\theta(V)$ for every V, then the proximity structure is ordered.

Conversely, take an ordered proximity structure. The associated directed graph is, plainly, ordered too, and so has no loops. And, the Laws of Proximity and Succession hold by the proof of Proposition 2.4. Thus the corollary holds. □

2.6 (Numerical characters)**.** In [**16**, Sct. 2, p. 214], a number of numerical characters were introduced, and three of them are useful in the present work.

The first character makes sense for any unweighted Enriques diagram $\mathbf{U}$, although it was not defined in this generality before; namely, the *dimension* $\dim(\mathbf{U})$ is the number of roots plus the number of free vertices in $\mathbf{U}$, including roots. Of course, the definition makes sense for a weighted Enriques diagram $\mathbf{D}$; namely, the *dimension* $\dim(\mathbf{D})$ is simply the dimension of the underlying unweighted diagram.

The second and third characters make sense only for a weighted Enriques diagram $\mathbf{D}$; namely, the *degree* and *codimension* are defined by the formulas

$$\deg(\mathbf{D}) := \textstyle\sum_{V\in\mathbf{D}} \binom{m_V+1}{2};$$
$$\operatorname{cod}(\mathbf{D}) := \deg(\mathbf{D}) - \dim(\mathbf{D}).$$

It is useful to introduce a new character, the *type* of a vertex V of $\mathbf{U}$ or of $\mathbf{V}$. It is defined by the formula

$$\operatorname{type}(V) := \begin{cases} 0, & \text{if } V \text{ is a satellite;} \\ 1, & \text{if } V \text{ is a free vertex, but not a root;} \\ 2, & \text{if } V \text{ is a root.} \end{cases}$$

The type appears in the following two formulas:

$$\dim(\mathbf{A}) = \textstyle\sum_{V\in\mathbf{A}} \operatorname{type}(V); \tag{2.6.1}$$
$$\operatorname{cod}(\mathbf{A}) = \textstyle\sum_{V\in\mathbf{A}} \left[\binom{m_V+1}{2} - \operatorname{type}(V)\right]. \tag{2.6.2}$$

Formula 2.6.2 is useful because every summand is nonnegative in general and positive when $\mathbf{A}$ is a minimal Enriques diagram.

2.7 (The diagram of a curve)**.** Let C be a reduced curve lying on a smooth surface over an algebraically closed ground field; the surface need not be complete. In [**16**, Sec. 2, p. 213] and again in [**17**, Sec. 2, p. 72], we stated that, to C, we can associate a minimal Enriques diagram $\mathbf{D}$. (It represents the equisingularity type of C; this aspect of the theory is treated in [**2**, p. 99] and [**6**, pp. 543–4].) Here is more explanation about the construction of $\mathbf{D}$.

First, form the configuration of all arbitrarily near points of the surface lying on all the branches of the curve through all its singular points. Say that one arbitrarily near point is *proximate* to a second if the first lies above the second and on the strict transform of the exceptional divisor of the blowup centered at the second. Then Laws (P1) to (P3) hold because three strict transforms never meet and, if two meet, then they meet once and transversely. Plainly, there are no loops. Hence, by Proposition 2.4, this configuration is an unweighted Enriques diagram.

Second, weight each arbitrarily near point with its multiplicity as a point on the strict transform of the curve. By the theorem of strong embedded resolution, all but finitely many arbitrarily near points are of multiplicity 1, and are proximate only to their immediate predecessors; prune off all the infinite unbroken successions of such points, leaving finitely many points. Then the Law of Minimality holds.

Finally, the Proximity Inequality holds for this well-known reason: the multiplicity of a point P' on a strict transform C' can be computed as an intersection number m on the blowup at P' of the surface containing C'; namely, m is the intersection number of the exceptional divisor and the strict transform of C'; the desired inequality results now from Noether's formula for m in terms of multiplicities of arbitrarily near points. (In [**2**, p. 83], the inequality is an equality, because no pruning is done.) Therefore, this weighted configuration is a minimal Enriques diagram. It is $\mathbf{D}$.

Notice that, if K is any algebraically closed extension field of the ground field, then the curve C_K also has diagram $\mathbf{D}$.

Proposition 2.8. *Let C be a reduced curve lying on a smooth surface over an algebraically closed field. Let $\mathbf{D}$ be the minimal Enriques diagram of C, and $P \in C$ a singular point of multiplicity m. Form the blowup of the surface at P, the exceptional divisor E, the proper transform C' of C, and the union $C'' := C' \cup E$. Let $\mathbf{D}'$ be the diagram of C', and $\mathbf{D}''$ that of C''. Then*

$$\operatorname{cod}(\mathbf{D}) - \operatorname{cod}(\mathbf{D}') \geq \tbinom{m+1}{2} - 2 \text{ and } \operatorname{cod}(\mathbf{D}) - \operatorname{cod}(\mathbf{D}'') = \tbinom{m}{2} - 2;$$

equality holds in the first relation if and only if P is an ordinary m-fold point.

PROOF. We obtain $\mathbf{D}'$ from $\mathbf{D}$ by deleting the root R corresponding to P and also all the vertices T that are of weight 1, proximate to R, and such that all successors of T are also (of weight 1 and) proximate to R (and so deleted too). Note that an immediate successor of R is free; if it is deleted, then it has weight 1, and if it is not deleted, then it becomes a root of $\mathbf{D}'$. Also, by the Law of Proximity, an undeleted satellite of R becomes a free vertex of $\mathbf{D}'$.

Let σ be the total number of satellites of R, and ρ the number of undeleted immediate successors. Then it follows from the Formula (2.6.2) that

$$\operatorname{cod}(\mathbf{D}) - \operatorname{cod}(\mathbf{D}') = \tbinom{m+1}{2} - 2 + \sigma + \rho.$$

Thus the asserted inequality holds, and it is an equality if and only if $\sigma = 0$ and $\rho = 0$. So it is an equality if P is an ordinary m-fold point.

Conversely, suppose $\sigma = 0$ and $\rho = 0$. Then R has no immediate successor V of weight 1 for the following reason. Otherwise, any immediate successor W of V is proximate to V by the Law of Proximity. So W has weight 1 by the Proximity Inequality. Hence, by recursion, we conclude that V is succeeded by a leaf L of weight 1. So, by the Law of Minimality, L is a satellite. But $\sigma = 0$. Hence V does not exist. But $\rho = 0$. Hence R has no successors whatsoever. So P is an ordinary m-fold point.

Furthermore, we obtain $\mathbf{D}''$ from $\mathbf{D}$ by deleting R and by adding 1 to the weight of each T proximate to R. So a satellite of R becomes a free vertex of $\mathbf{D}''$, and an immediate successor of R becomes a root of $\mathbf{D}''$. In addition, for each smooth branch of C that is transverse at P to all the other branches, we adjoin an isolated vertex (root) of weight 2.

The number of adjoined vertices is $m - \sum_{T \succ R} m_T$. So, by Formula (2.6.2),

$$\begin{aligned}\operatorname{cod}(\mathbf{D}) - \operatorname{cod}(\mathbf{D}'') = \tbinom{m+1}{2} - 2 + \textstyle\sum_{T \succ R} \big[\tbinom{m_T+1}{2} - \operatorname{type}(T)\big] \\ - \textstyle\sum_{T \succ R} \big[\tbinom{m_T+2}{2} - (\operatorname{type}(T)+1)\big] - \big[m - \textstyle\sum_{T \succ R} m_T\big].\end{aligned}$$

The right hand side reduces to $\binom{m}{2} - 2$. So the asserted equality holds. □

3. Infinitely near points

Fix a smooth family of geometrically irreducible surfaces $\pi\colon F \to Y$. In this section, we study sequences of arbitrarily near T-points of F/Y. They are defined in Definition 3.3. Then Proposition 3.4 asserts that they form a representable functor. In essence, this result is due to Harbourne [**12**, Prp. I.2, p. 104], who identified the functor of points of the iterated blow-up that was introduced in [**15**, Sct. 4.1, p. 36] and is recalled in Definition 3.1.

In the second half of this section, we study a special kind of sequence of arbitrarily near T-points, the *strict* sequence, which is defined in Definition 3.5. To each strict sequence is associated a natural ordered unweighted Enriques diagram owing to Propositions 3.8 and 2.4. Finally, Theorem 3.10 asserts that the strict sequences with given diagram $(\mathbf{U}, \theta)$ form a functor, which is representable by a Y-smooth scheme with irreducible geometric fibers of dimension $\dim(\mathbf{U})$. This theorem was inspired by Roé's Proposition 2.6 in [**27**].

Definition 3.1. By induction on $i \geq 0$, let us define more families

$$\pi^{(i)}\colon F^{(i)} \to F^{(i-1)},$$

which are like $\pi\colon F \to Y$. Set $\pi^{(0)} := \pi$. Now, suppose $\pi^{(i)}$ has been defined. Form the fibered product of $F^{(i)}$ with itself over $F^{(i-1)}$, and blow up along the diagonal $\Delta^{(i)}$. Take the composition of the blowup map and the second projection to be $\pi^{(i+1)}$.

In addition, for $i \geq 1$, let $\varphi^{(i)}\colon F^{(i)} \to F^{(i-1)}$ be the composition of the blowup map and the first projection, and let $E^{(i)}$ be the exceptional divisor. Finally, set $\varphi^{(0)} := \pi$; so $\varphi^{(0)} = \pi^{(0)}$.

Lemma 3.2. *Both $\pi^{(i)}$ and $\varphi^{(i)}$ are smooth, and have geometrically irreducible fibers of dimension 2. Moreover, $E^{(i)}$ is equal, as a polarized scheme, to the bundle $\mathbb{P}(\Omega^1_{\pi^{(i-1)}})$ over $F^{(i-1)}$, where $\Omega^1_{\pi^{(i-1)}}$ is the sheaf of relative differentials.*

PROOF. The first assertion holds for $i = 0$ by hypothesis. Suppose it holds for i. Consider the fibered product formed in Definition 3.1. Then both projections are smooth, and have geometrically irreducible fibers of dimension 2; also, the diagonal $\Delta^{(i)}$ is smooth over both factors. It follows that the first assertion holds for $i + 1$.

The second assertion holds because $\Omega^1_{\pi^{(i-1)}}$ is the conormal sheaf of $\Delta^{(i)}$. □

Definition 3.3. Let T be a Y-scheme. Given a sequence of blowups

$$F_T^{(n+1)} \xrightarrow{\varphi_T^{(n+1)}} F_T^{(n)} \to \cdots \to F_T' \xrightarrow{\varphi_T^{(1)}} F_T := F \times_Y T$$

whose ith center $T^{(i)} \subset F_T^{(i)}$ is the image of a section t_i of $F_T^{(i)}/T$ for $0 \leq i \leq n$, call $(t_0, \dots, t_n)$ a *sequence of arbitrarily near T-points* of F/Y.

For $1 \leq i \leq n+1$, denote the exceptional divisor in $F_T^{(i)}$ by $E_T^{(i)}$.

The following result is a version of Harbourne's Proposition I.2 in [**12**, p. 104].

Proposition 3.4 (Harbourne)**.** *As T varies, the sequences $(t_0, \dots, t_n)$ of arbitrarily near T-points of F/Y form a functor, which is represented by $F^{(n)}/Y$.*

Given $(t_0, \dots, t_n)$ and i, say $(t_0, \dots, t_i)$ is represented by $\tau_i\colon T \to F^{(i)}$. Then $\pi^{(i)}\tau_i = \tau_{i-1}$ where τ_{-1} is the structure map. Also, $F_T^{(i+1)} = F^{(i+1)} \times_{F^{(i)}} T$ where $F^{(i+1)} \to F^{(i)}$ is $\pi^{(i+1)}$; correspondingly, $t_i = (\tau_i, 1)$ and $E_T^{(i+1)} = E^{(i+1)} \times_{F^{(i)}} T$;

moreover, $T^{(i)}$ is the scheme-theoretic image of $E_T^{(i+1)}$ under $\varphi_T^{(i+1)}\colon F_T^{(i+1)} \to F_T^{(i)}$. Finally, $\varphi_T^{(i+1)}$ is induced by $\varphi^{(i+1)}$, and $F_T^{(i+1)} \to T$ is induced by $\pi^{(i+1)}$.

PROOF. First, observe that, given a section of any smooth map $a\colon A \to B$, blowing up A along the section's image, C say, commutes with changing the base B. Indeed, let $\mathcal{I}$ be the ideal of C, and for each $m \geq 0$, consider the exact sequence

$$0 \to \mathcal{I}^{m+1} \to \mathcal{I}^m \to \mathcal{I}^m\big/\mathcal{I}^{m+1} \to 0.$$

Since a is smooth, $\mathcal{I}^m/\mathcal{I}^{m+1}$ is a locally free $\mathcal{O}_C$-module, so B-flat. Hence forming the sequence commutes with changing B. However, the blowup of A is just $\operatorname{Proj}\bigoplus_m \mathcal{I}^m$. Hence forming it commutes too.

Second, observe in addition that C is the scheme-theoretic image of the exceptional divisor, E say, of this blowup. Indeed, this image is the closed subscheme of C whose ideal is the kernel of the comorphism of the map $E \to C$. However, this comorphism is an isomorphism, because $E = \mathbb{P}(I/I^2)$ since a is smooth.

The first observation implies that the sequences $(t_0, \dots, t_n)$ form a functor, because, given any Y-map $T' \to T$, each induced map

$$F_T^{(i+1)} \times_T T' \to F_T^{(i)} \times_T T'$$

is therefore the blowing-up along the image of the induced section of $F_T^{(i)} \times_T T'/T'$.

To prove this functor is representable by $F^{(n)}/Y$, we must set up a functorial bijection between the sequences $(t_0, \dots, t_n)$ and the Y-maps $\tau_n\colon T \to F^{(n)}$. Of course, n is arbitrary. So $(t_0, \dots, t_i)$ then determines a Y-map $\tau_i\colon T \to F^{(i)}$, and correspondingly we want the remaining assertions of the proposition to hold as well.

So given $(t_0, \dots, t_n)$, let us construct appropriate Y-maps $\tau_i\colon T \to F^{(i)}$ for $-1 \leq i \leq n$. We proceed by induction on i. Necessarily, $\tau_{-1}\colon T \to Y$ is the structure map, and correspondingly, $F_T^{(0)} = F^{(0)} \times_{F^{(-1)}} T$ owing to the definitions.

Suppose we've constructed τ_{i-1}. Then $F_T^{(i)} = F^{(i)} \times_{F^{(i-1)}} T$. Set $\tau_i := p_1 t_i$ where $p_1\colon F_T^{(i)} \to F^{(i)}$ is the projection. Then $\tau_{i-1} = \pi^{(i)}\tau_i$. Also, $t_i = (\tau_i, 1)$; so t_i is the pullback, under the map $(1, \tau_i)$, of the diagonal map of $F^{(i)}/F^{(i-1)}$. Therefore, owing to the first observation, $F_T^{(i+1)} = F^{(i+1)} \times_{(F^{(i)} \times F^{(i)})} F_T^{(i)}$ where $F_T^{(i)} \to F^{(i)} \times_{F^{(i-1)}} F^{(i)}$ is equal to $1 \times \tau_i$. Hence $F_T^{(i+1)} = F^{(i+1)} \times_{F^{(i)}} T$ where $F^{(i+1)} \to F^{(i)}$ is $\pi^{(i+1)}$. It follows formally that $E_T^{(i+1)} = E^{(i+1)} \times_{F^{(i)}} T$, that $F_T^{(i+1)} \to F_T^{(i)}$ is induced by $\varphi^{(i+1)}$, and that $F_T^{(i+1)} \to T$ is induced by $\pi^{(i+1)}$.

By the second observation above, $T^{(i)}$ is the scheme-theoretic image of $E_T^{(i+1)}$.

Conversely, given a map $\tau_n\colon T \to F^{(n)}$, set $\tau_{i-1} := \pi^{(i)} \cdots \pi^{(n)}\tau_n$ for $0 \leq i \leq n$; so $\tau_{i-1}\colon T \to F^{(i-1)}$. Set $F_T^{(i)} := F^{(i)} \times_{F^{(i-1)}} T$ where the map $F^{(i)} \to F^{(i-1)}$ is $\pi^{(i)}$ for $0 \leq i \leq n+1$. Then τ_i defines a section t_i of $F_T^{(i)}/T$. Furthermore, blowing up its image yields the map $F_T^{(i+1)} \to F_T^{(i)}$ induced by $\varphi^{(i+1)}$, because, as noted above, forming the blowup along $\Delta^{(i)}$ commutes with changing the base via $1 \times \tau_i$. Thus $(t_0, \dots, t_n)$ is a sequence of arbitrarily near T-points of F/Y.

Plainly, for each T, we have set up the bijection we sought, and it is functorial in T. Since we have checked all the remaining assertions of the proposition, the proof is now complete. □

Definition 3.5. Given a sequence $(t_0, \dots, t_n)$ of arbitrarily near T-points of F/Y, let us call it *strict* if, for $0 \leq i \leq n$, the image $T^{(i)}$ of t_i satisfies the following i

conditions, defined by induction on i. There are, of course, no conditions on $T^{(0)}$. Fix i, and suppose, for $0 \le j < i$, the conditions on $T^{(j)}$ are defined and satisfied.

The i conditions on $T^{(i)}$ involve the natural embeddings

$$e_T^{(j,i)}\colon E_T^{(j)} \hookrightarrow F_T^{(i)} \quad \text{for } 1 \le j \le i,$$

which we assume defined by induction; see the next paragraph. (The image $e_T^{(j,i)}E_T^{(j)}$ can be regarded as the "strict transform" of $E_T^{(j)}$ on $F_T^{(i)}$.) The jth condition requires $e_T^{(j,i)}E_T^{(j)}$ either (a) to be disjoint from $T^{(i)}$ or (b) to contain $T^{(i)}$ as a subscheme.

Define $e_T^{(i+1,\,i+1)}$ to be the inclusion. Now, for $1 \le j \le i$, we have assumed that $e_T^{(j,i)}$ is defined, and required that its image satisfy either (a) or (b). If (a) is satisfied, then the blowing-up $F_T^{(i+1)} \to F_T^{(i)}$ is an isomorphism on a neighborhood of $e_T^{(j,\,i)}E_T^{(j)}$, namely, the complement of $T^{(i)}$; so then $e_T^{(j,\,i)}$ lifts naturally to an embedding $e_T^{(j,\,i+1)}$. If (b) is satisfied, then $T^{(i)}$ is a relative effective divisor on the T-scheme $e^{(j,i)}E_T^{(j)}$, because $E_T^{(j)}$ and $T^{(i)}$ are flat over T, and the latter's fibers are effective divisors on the former's fibers, which are $\mathbb{P}^1$s; hence, then blowing up $e_T^{(j,i)}E_T^{(j)}$ along $T^{(i)}$ yields an isomorphism. But the blowup of $e_T^{(j,i)}E_T^{(j)}$ embeds naturally in $F_T^{(i)}$. Thus, again, $e_T^{(j,\,i)}$ lifts naturally.

Definition 3.6. Given a strict sequence $(t_0, \dots, t_n)$ of arbitrarily near T-points of F/Y, say that t_i is *proximate* to t_j if $j < i$ and $e^{(j+1,\,i)}E_T^{(j+1)}$ contains $T^{(i)}$.

Lemma 3.7. *Let $(t_0, \dots, t_n)$ be a strict sequence of arbitrarily near T-points of F/Y. Fix $n+1 \ge i \ge j \ge k \ge 1$. Then $\varphi_T^{(j+1)} \cdots \varphi_T^{(i)} e_T^{(k,i)} = e_T^{(k,j)}$, and $T^{(j-1)}$ is the scheme-theoretic image of $e_T^{(j,i)}E_T^{(j)}$ under $\varphi_T^{(j)} \cdots \varphi_T^{(i)}$. Set*

$$Z_T^{(i)} := e_T^{(k,i)}E_T^{(k)} \bigcap e_T^{(j,i)}E_T^{(j)}.$$

If $j > k$ and $Z_T^{(i)} \ne \emptyset$, then $\varphi_T^{(j)} \cdots \varphi_T^{(i)}$ induces an isomorphism $Z_T^{(i)} \xrightarrow{\sim} T^{(j-1)}$, and t_{j-1} is proximate to t_{k-1}; moreover, then $Z_T^{(i)}$ meets no $e_T^{(l,i)}E_T^{(l)}$ for $l \ne j, k$.

PROOF. The formula $\varphi_T^{(j+1)} \cdots \varphi_T^{(i)} e_T^{(k,i)} = e_T^{(k,j)}$ is trivial if $i = j$. It holds by construction if $i = j+1$. Finally, it follows by induction if $i > j+1$. With $k := j$, this formula implies that $E_T^{(j)}$ is the scheme-theoretic image of $e_T^{(j,i)}E_T^{(j)}$ under $\varphi_T^{(j+1)} \cdots \varphi_T^{(i)}$; whence, Proposition 3.4 implies that $T^{(j-1)}$ is the scheme-theoretic image of $e_T^{(j,i)}E_T^{(j)}$ under $\varphi_T^{(j)} \cdots \varphi_T^{(i)}$.

Suppose $j > k$ and $Z_T^{(i)} \ne \emptyset$. Now, for any l such that $i \ge l \ge j$, both $e_T^{(k,l)}E_T^{(k)}$ and $e_T^{(j,l)}E_T^{(j)}$ are relative effective divisors on $F_T^{(l)}/T$, because they're flat and divisors on the fibers. Hence, on either of $e_T^{(k,l)}E_T^{(k)}$ and $e_T^{(j,l)}E_T^{(j)}$, their intersection $Z_T^{(l)}$ is a relative effective divisor, since each fiber of $Z_T^{(l)}$ is correspondingly a divisor. In fact, each nonempty fiber of $Z_T^{(l)}$ is a reduced point on a $\mathbb{P}^1$.

Since $\varphi_T^{(j+1)} \cdots \varphi_T^{(i)} e_T^{(j,i)} = e_T^{(j,j)}$ and since $e_T^{(j,j)}$ is the inclusion of $E_T^{(j)}$, which is the exceptional divisor of the blowing-up $\varphi^{(j)}\colon F^{(j)} \to F^{(j-1)}$ along $T^{(j-1)}$, the map $\varphi_T^{(j)} \cdots \varphi_T^{(i)}$ induces a proper map $e\colon Z_T^{(i)} \to T^{(j-1)}$. Since the fibers of e are isomorphisms, e is a closed embedding. So since $Z_T^{(i)}$ and $T^{(j-1)}$ are T-flat, e is an isomorphism onto an open and closed subscheme.

Since $\varphi_T^{(j)} \cdots \varphi_T^{(i)} e_T^{(k,i)} = e_T^{(k,\,j-1)}$, it follows that $e_T^{(k,\,j-1)} E_T^{(k)}$ contains a nonempty subscheme of $T^{(j-1)}$. So since $(t_0, \ldots, t_n)$ is strict, $e_T^{(k,\,j-1)} E_T^{(k)}$ contains all of $T^{(j-1)}$ as a subscheme. Thus t_{j-1} is proximate to t_{k-1}.

It follows that $\varphi_T^{(j)}$ induces a surjection $Z_T^{(j)} \twoheadrightarrow T^{(j-1)}$. If $i = j$, then this surjection is just e, and so e is an isomorphism, as desired.

Suppose $i > j$. Then $Z_T^{(j)} \bigcap T^{(j)} = \emptyset$. Indeed, suppose not. Then both $e_T^{(k,j)} E_T^{(k)}$ and $E_T^{(j)}$ meet $T^{(j)}$. So since $(t_0, \ldots, t_n)$ is strict, $Z_T^{(j)}$ contains $T^{(j)}$ as a closed subscheme. Both these schemes are T-flat, and their fibers are reduced points; hence, they coincide. It follows that $e_T^{(k,\,j+1)} E_T^{(k)}$ and $e_T^{(j,\,j+1)} E_T^{(j)}$ are disjoint on $F^{(j+1)}$. But these subschemes intersect in $Z_T^{(j+1)}$. And $Z_T^{(j+1)} \neq \emptyset$ since $Z_T^{(i)} \neq \emptyset$ and $Z_T^{(i)}$ maps into $Z_T^{(j+1)}$. We have a contradiction, so $Z_T^{(j)} \bigcap T^{(j)} = \emptyset$.

Therefore, $\varphi_T^{(j+1)}$ induces an isomorphism $Z_T^{(j+1)} \xrightarrow{\sim} Z_T^{(j)}$. Similarly, $\varphi_T^{(l+1)}$ induces an isomorphism $Z_T^{(l+1)} \xrightarrow{\sim} Z_T^{(l)}$ for $l = j, \ldots, i-1$. Hence $\varphi_T^{(j)} \cdots \varphi_T^{(i)}$ induces an isomorphism $Z_T^{(i)} \xrightarrow{\sim} T^{(j-1)}$.

Finally, suppose $Z_T^{(i)}$ meets $e_T^{(l,i)} E_T^{(l)}$ for $l \neq j, k$, and let's find a contradiction. If $l < j$, then interchange l and j. Then, by the above, $T^{(j-1)}$ lies in both $e_T^{(k,\,j-1)} E_T^{(k)}$ and $e_T^{(l,\,j-1)} E_T^{(l)}$. Therefore, $T^{(j-1)}$ is equal to their intersection, because $T^{(j-1)}$ is flat and its fibers are equal to those of the intersection. It follows that $e_T^{(k,\,j)} E_T^{(k)}$ and $e_T^{(l,\,j)} E_T^{(l)}$ are disjoint on $F^{(j)}$. But both these subschemes contain the image of $Z_T^{(i)}$, which is nonempty. We have a contradiction, as desired. The proof is now complete. □

Proposition 3.8. *Let $(t_0, \ldots, t_n)$ be a strict sequence of arbitrarily near T-points of F/Y. Equip the abstract ordered set of t_i with the relation of proximity of* Definition 3.6. *Then this set becomes an ordered proximity structure.*

PROOF. Law (P1) holds trivially.

As to (P2), suppose t_i is proximate to t_j and to t_k with $j > k$. Then $T^{(i)}$ lies in $e_T^{(k+1,i)} E_T^{(k+1)} \bigcap e_T^{(j+1,i)} E_T^{(j+1)}$. So Lemma 3.7 implies t_j is proximate to t_k. Furthermore, the lemma implies the intersection meets no $e_T^{(l+1,i)} E_T^{(l+1)}$ for $l \neq j, k$. So t_i is proximate to no third vertex t_l. Thus (P2) holds.

As to (P3), suppose t_i and t_j are each proximate to both t_k and t_l where $i > j > k > l$. Given $p > k$, set $Z^{(p)} := e_T^{(l+1,p)} E_T^{(l+1)} \bigcap e_T^{(k+1,p)} E_T^{(k+1)}$. Then $T^{(i)} \subseteq Z^{(i)}$. Now, $Z^{(i)}$ is T-flat with reduced points as fibers by Lemma 3.7. But $T^{(i)}$ is a similar T-scheme. Hence $T^{(i)} = Z^{(i)}$. Similarly, $T^{(j)} := Z^{(j)}$.

Lemma 3.7 yields $\varphi_T^{(j+1)} \cdots \varphi_T^{(i)} e_T^{(m,i)} = e_T^{(m,j)}$ for $m = k, l$. So $\varphi_T^{(j+1)} \cdots \varphi_T^{(i)}$ carries $T^{(i)}$ into $T^{(j)}$. Now, this map is proper, and both $T^{(i)}$ and $T^{(j)}$ are T-flat with reduced points as fibers; hence, $T^{(i)} \xrightarrow{\sim} T^{(j)}$. It follows that

$$\varphi_T^{(j+2)} \cdots \varphi_T^{(i)} T^{(i)} \subseteq Z^{(j+1)} \subset \left(\varphi_T^{(j+1)}\right)^{-1} T^{(j)} = E_T^{(j+1)}.$$

Hence $Z^{(j+1)}$ meets $E_T^{(j+1)}$, contrary to Lemma 3.7. Thus (P3) holds. □

Definition 3.9. Let's say that a strict sequence of arbitrarily near T-points of F/Y has *diagram* $(\mathbf{U}, \theta)$ if $(\mathbf{U}, \theta)$ is isomorphic to the ordered unweighted Enriques diagram coming from Propositions 3.8 and 2.4.

The following result was inspired by Roé's Proposition 2.6 in [**27**].

Theorem 3.10. *Fix an ordered unweighted Enriques diagram* $(\mathbf{U}, \theta)$ *on* $n+1$ *vertices. Then the strict sequences of arbitrarily near* T*-points of* F/Y *with diagram* $(\mathbf{U}, \theta)$ *form a functor; it is representable by a subscheme* $F(\mathbf{U}, \theta)$ *of* $F^{(n)}$*, which is* Y*-smooth with irreducible geometric fibers of dimension* $\dim(\mathbf{U})$.

PROOF. If a strict sequence of arbitrarily near T-points has diagram $(\mathbf{U}, \theta)$, then, for any map $T' \to T$, the induced sequence of arbitrarily near T'-points plainly also has diagram $(\mathbf{U}, \theta)$. So the sequences with diagram $(\mathbf{U}, \theta)$ form a subfunctor of the functor of all sequences, which is representable by $F^{(n)}/Y$ by Proposition 3.4.

Suppose $n = 0$. Then $\mathbf{U}$ has just one vertex. So the two functors coincide, and both are representable by F, which is Y-smooth with irreducible geometric fibers of dimension 2. However, $2 = \dim(\mathbf{U})$. Thus the theorem holds when $n = 0$.

Suppose $n \geq 1$. Set $L := \theta^{-1}n$. Then L is a leaf. Set $\mathbf{T} := \mathbf{U} - L$. Then $\mathbf{T}$ inherits the structure of an unweighted Enriques diagram, and it is ordered by the restriction $\theta|\mathbf{T}$. By induction on n, assume the theorem holds for $(\mathbf{T}, \theta|\mathbf{T})$.

Set $G := F(\mathbf{T}, \theta|\mathbf{T}) \subset F^{(n-1)}$ and $H := \pi_n^{-1}G \subset F^{(n)}$. Then H represents the functor of sequences $(t_0, \dots, t_n)$ of arbitrarily near T-points such that $(t_0, \dots, t_{n-1})$ has diagram $(\mathbf{T}, \theta|\mathbf{T})$ since $\pi^{(i)}\tau_i = \tau_{i-1}$ by Proposition 3.4. Moreover, H is G-smooth with irreducible geometric fibers of dimension 2 by Lemma 3.2. And G is Y-smooth with irreducible geometric fibers of dimension $\dim(\mathbf{T})$ as the theorem holds for $(\mathbf{T}, \theta|\mathbf{T})$. Thus H is Y-smooth with irreducible geometric fibers of dimension $\dim(\mathbf{T}) + 2$.

Let $(h_0, \dots, h_n)$ be the universal sequence of arbitrarily near H-points, and $H^{(i)} \subset F_H^{(i)}$ the image of h_i. We must prove that H has a largest subscheme S over which $(h_0, \dots, h_n)$ restricts to a sequence with diagram $(\mathbf{U}, \theta)$; we must also prove that S is Y-smooth with irreducible geometric fibers of dimension $\dim(\mathbf{U})$.

But, $(h_0, \dots, h_{n-1})$ has diagram $(\mathbf{T}, \theta|\mathbf{T})$. So $H^{(i)}$ satisfies the i conditions of Definition 3.5 for $i = 0, \dots, n-1$. Hence S is defined simply by the n conditions on $H^{(n)}$: the jth requires $e_H^{(j,n)}E_H^{(j)}$ either (a) to be disjoint from $H^{(n)}$ or (b) to contain it as a subscheme; (b) applies if L is proximate to $\theta^{-1}(j-1)$, and (a) if not, according to Definition 3.6. Let P be the set of j for which (b) applies. Set

$$S := h_n^{-1}\left(\bigcap_{j\in P} e_H^{(j,n)}E_H^{(j)} - \bigcup_{j\notin P} e_H^{(j,n)}E_H^{(j)}\right)$$

Plainly, S is the desired largest subscheme of H.

It remains to analyze the geometry of S. First of all, $F_G^{(n)} = F^{(n)} \times_{F^{(n-1)}} G$ by Proposition 3.4; so $F_G^{(n)} = H$ since $H := \pi_n^{-1}G$. Also, $F_H^{(n)} = F^{(n)} \times_{F^{(n-1)}} H$ and $h_n = (\zeta_n, 1)$ where $\zeta_n \colon H \hookrightarrow F$, again by Proposition 3.4. Hence

$$F_H^{(n)} = F_G^{(n)} \times_G H = H \times_G H \text{ and } h_n = (1, 1).$$

Plainly, forming $e_T^{(j,n)}$ is functorial in T; whence, $e_H^{(j,n)}E_H^{(j)} = (e_G^{(j,n)}E_G^{(j)}) \times_G H$. Hence, $h_n^{-1}e_H^{(j,n)}E_H^{(j)} = e_G^{(j,n)}E_G^{(j)}$. Therefore,

$$S := \bigcap_{j\in P} e_G^{(j,n)}E_G^{(j)} - \bigcup_{j\notin P} e_G^{(j,n)}E_G^{(j)}.$$

There are three cases to analyze, depending on $\text{type}(L)$. In any case,

$$\dim(\mathbf{T}) + \text{type}(L) = \dim(\mathbf{U})$$

owing to Formula 2.6.1. Furthermore, each $e_G^{(j,n)}$ is an embedding. So $e_G^{(j,n)}E_G^{(j)}$ has the form $\mathbb{P}(\Omega)$ for some locally free sheaf Ω of rank 2 on G by Lemma 3.2 and Proposition 3.4. Hence $e_G^{(j,n)}E_G^{(j)}$ is Y-smooth with irreducible geometric fibers of dimension $\dim(\mathbf{T})+1$.

Suppose $\operatorname{type}(L)=2$. Then L is a root. So P is empty, and by convention, the intersection $\bigcap_{j\in P} e_H^{(j,n)}E_H^{(j)}$ is all of H. So S is open in H, and maps onto Y. Hence S is Y-smooth with irreducible geometric fibers of dimension $\dim(H/Y)$, and

$$\dim(H/Y)=\dim(\mathbf{T})+2=\dim(\mathbf{U}).$$

Thus the theorem holds in this case.

Suppose $\operatorname{type}(L)=1$. Then L is a free vertex, but not a root. So L has an immediate predecessor, M say. Set $m:=\theta(M)$. Then $P=\{m\}$. So S is open in $e_G^{(m,n)}E_G^{(m)}$, and maps onto Y. Hence S is Y-smooth with irreducible geometric fibers of dimension $\dim(e_G^{(m,n)}E_G^{(m)}/Y)$, and

$$\dim(e_G^{(m,n)}E_G^{(m)}/Y)=\dim(\mathbf{T})+1=\dim(\mathbf{U}).$$

Thus the theorem holds in this case too.

Finally, suppose $\operatorname{type}(L)=0$. Then L is a satellite. So L is proximate to two vertices: an immediate predecessor, M say, and a remote predecessor, R say. Set $m:=\theta(M)$ and $r:=\theta(R)$. Then $P=\{r,\,m\}$. Set $Z:=e_G^{(r,n)}E_G^{(r)}\bigcap e_G^{(m,n)}E_G^{(m)}$. Then $Z\xrightarrow{\sim} G$ and Z meets no $e_G^{(j,n)}E_G^{(m)}$ with $j\notin P$ owing to Lemma 3.7, because $(h_0,\,\dots,\,h_{n-1})$ is strict with diagram $(\mathbf{T},\,\theta|\mathbf{T})$. Hence $S=Z$. Therefore, S is Y-smooth with irreducible geometric fibers of dimension $\dim(G/Y)$, and

$$\dim(G/Y)=\dim(\mathbf{T})+0=\dim(\mathbf{U}).$$

Thus the theorem holds in this case too, and the proof is complete. □

4. Isomorphism and enlargement

Fix a smooth family of geometrically irreducible surfaces $\pi\colon F\to Y$. In this section, we study the scheme $F(\mathbf{U},\,\theta)$ introduced in Theorem 3.10. First, we work out the effect of replacing the ordering θ by another one θ'. Then we develop, in our context, much of Roé's Subsections 2.1–2.3 in [**27**]; specifically, we study a certain closed subset $E(\mathbf{U},\,\theta)\subset F^{(n)}$ containing $F(\mathbf{U},\,\theta)$ set-theoretically. Notably, we prove that, if the sets $F(\mathbf{U}',\,\theta')$ and $E(\mathbf{U},\,\theta)$ meet, then $E(\mathbf{U}',\,\theta')$ lies in $E(\mathbf{U},\,\theta)$; furthermore, $E(\mathbf{U}',\,\theta')=E(\mathbf{U},\,\theta)$ if and only if $(\mathbf{U},\,\theta)\cong(\mathbf{U}',\,\theta')$.

Proposition 4.3 below asserts that there is a natural isomorphism $\Phi_{\theta,\theta'}$ from $F(\mathbf{U},\,\theta)$ to $F(\mathbf{U},\,\theta')$. On geometric points, $\Phi_{\theta,\theta'}$ is given as follows. A geometric point with field K represents a sequence of arbitrarily near K-points $(t_0,\dots,t_n)$ of F/Y. To give t_i is the same as giving the local ring A_i of the surface $F_K^{(i)}$ at the K-point $T^{(i)}$, the image of t_i. Set $\alpha:=\theta'\circ\theta^{-1}$. Then $\alpha i>\alpha j$ if t_i is proximate to t_j. So there is a unique sequence $(\tilde t_0,\dots,\tilde t_n)$ whose local rings $\widetilde A_j$ satisfy $A_i=\widetilde A_{\alpha i}$ in the function field of F_K. The sequences $(t_0,\dots,t_n)$ and $(\tilde t_0,\dots,\tilde t_n)$ correspond under $\Phi_{\theta,\theta'}$.

To construct $\Phi_{\theta,\theta'}$, we must work with a sequence $(t_0,\dots,t_n)$ of T-points for an arbitrary T. To do so, instead of the A_i, we use the transforms $e_T^{(i+1,n+1)}E_T^{(i+1)}$. The notation becomes more involved, and it is harder to construct $(\tilde t_0,\dots,\tilde t_n)$. We proceed by induction on n: we omit t_n, apply induction, and "reinsert" t_n as $\tilde t_{\alpha n}$.

Most of the work is done in Lemma 4.2; the reinsertion is justified by Lemma 4.1.

Lemma 4.1. *Let $(\tilde{t}_0,\dots,\tilde{t}_{n-1})$ be a strict sequence of arbitrarily near T-points of F/Y, say with blowups $\widetilde{F}_T^{(i)}$ and so on. Fix l, and let $T^{(l)}\subset\widetilde{F}_T^{(l)}$ be the image of a section t_l of $\widetilde{F}_T^{(l)}/T$. Set $t_i:=\tilde{t}_i$ for $0\le i<l$, and assume the sequence $(t_0,\dots,t_l)$ is strict. Set $T_l:=\widetilde{T}^{(l)}$ and $T_i:=\widetilde{\varphi}_T^{(l+1)}\cdots\widetilde{\varphi}_T^{(i)}\widetilde{T}^{(i)}$ for $l<i<n$, and assume $T^{(l)}$ and the T_i are disjoint. Then $(t_0,\dots,t_l)$ extends uniquely to a strict sequence $(t_0,\dots,t_n)$, say with blowups $F_T^{(i)}$ and so on, such that t_l is a leaf and $F_T^{(l+1)}\times_{F_T^{(l)}}\widetilde{F}_T^{(i-1)}=F_T^{(i)}$ for $l<i\le n$. Furthermore, the diagram of $(t_0,\dots,t_n)$ induces that of $(\tilde{t}_0,\dots,\tilde{t}_{n-1})$.*

PROOF. Set $F_T^{(l)}:=\widetilde{F}_T^{(l)}$; let $F_T^{(l+1)}$ be the blowup of $F_T^{(l)}$ with center $T^{(l)}$, and $E_T^{(l+1)}$ be its exceptional divisor. For $l<i\le n$, set $F_T^{(i+1)}:=F_T^{(l+1)}\times_{F_T^{(l)}}\widetilde{F}_T^{(i)}$ and $T^{(i)}:=F_T^{(l+1)}\times_{F_T^{(l)}}\widetilde{T}^{(i-1)}$. Now, $T^{(l)}$ and T_i are disjoint for $l\le i<n$. So $F_T^{(i+1)}$ is the blowup of $F_T^{(i)}$ with center $T^{(i)}$. Also, $T^{(i)}$ is the image of a section t_i of $F_T^{(i)}/T$. Moreover, since $(t_0,\dots,t_l)$ and $(\tilde{t}_0,\dots,\tilde{t}_{n-1})$ are strict sequences, it follows that $(t_0,\dots,t_n)$ is a strict sequence too. Furthermore, t_l is a leaf, and the diagram of $(t_0,\dots,t_n)$ induces that of $(\tilde{t}_0,\dots,\tilde{t}_{n-1})$. Plainly, $(t_0,\dots,t_n)$ is unique. □

Lemma 4.2. *Let α be a permutation of $\{0,\dots,n\}$. Let $(t_0,\dots,t_n)$ be a strict sequence of arbitrarily near T-points of F/Y. Assume that, if t_i is proximate to t_j, then $\alpha i>\alpha j$. Then there is a unique strict sequence $(\tilde{t}_0,\dots,\tilde{t}_n)$, say with blowups $\widetilde{F}_T^{(i)}$, exceptional divisors $\widetilde{E}_T^{(i)}$, and so on, such that $F_T^{(n+1)}=\widetilde{F}_T^{(n+1)}$ and $e_T^{(i,n+1)}E_T^{(i)}=\tilde{e}_T^{(\alpha' i,n+1)}\widetilde{E}_T^{(\alpha' i)}$ with $\alpha' i:=\alpha(i-1)+1$ for $1\le i\le n+1$; furthermore, t_i is proximate to t_j if and only if $\tilde{t}_{\alpha i}$ is proximate to $\tilde{t}_{\alpha j}$.*

PROOF. Assume $(\tilde{t}_0,\dots,\tilde{t}_n)$ exists. Let's prove, by induction on j, that both the sequence $(\tilde{t}_0,\dots,\tilde{t}_j)$ and the map $F_T^{(n+1)}\to\widetilde{F}_T^{(j+1)}$ are determined by the equality $F_T^{(n+1)}=\widetilde{F}_T^{(n+1)}$ and the $n+1$ equalities $e_T^{(i,n+1)}E_T^{(i)}=\tilde{e}_T^{(\alpha' i,n+1)}\widetilde{E}_T^{(\alpha' i)}$ where $1\le i\le n+1$. If $j=-1$, there's nothing to prove. So suppose $j\ge 0$. Then $\widetilde{T}^{(j+1)}$ is determined as the scheme-theoretic image of $\tilde{e}_T^{(j+2,n+1)}\widetilde{E}_T^{(j+2)}$ by Lemma 3.7. So $\tilde{t}_{j+1}$ is determined. But then $\widetilde{F}_T^{(j+2)}$ is determined as the blowup of $\widetilde{T}^{(j+1)}$. And $F_T^{(n+1)}\to\widetilde{F}_T^{(j+2)}$ is determined, because the preimage of $\widetilde{T}^{(j+1)}$ in $F_T^{(n+1)}$ is a divisor. Thus $(\tilde{t}_0,\dots,\tilde{t}_n)$ is unique.

To prove $(\tilde{t}_0,\dots,\tilde{t}_n)$ exists, let's proceed by induction on n. Assume $n=0$. Then $\alpha=1$. So plainly $\tilde{t}_0$ exists; just take $\tilde{t}_0:=t_0$.

So assume $n\ge 1$. Set $l:=\alpha n$. Define a permutation β of $\{0,\dots,n-1\}$ by $\beta i:=\alpha i$ if $\alpha i<l$ and $\beta i:=\alpha i-1$ if $\alpha i>l$.

Suppose t_i is proximate to t_j with $i<n$, and let us check that $\beta i>\beta j$. The hypothesis yields $\alpha i>\alpha j$. So if either $\alpha i<l$ or $\alpha j>l$, then $\beta i>\beta j$. Now, $\alpha i\ne l$ since $i<n$ and $l:=\alpha n$. Similarly, $\alpha j\ne l$ since $j<i$ as t_i is proximate to t_j. But if $\alpha i>l$, then $\beta i:=\alpha i-1\ge l$, and if $\alpha j<l$, then $\beta j:=\alpha j<l$. Thus $\beta i>\beta j$.

Since $(t_0,\dots,t_{n-1})$ is strict, induction applies: there exists a strict sequence $(\hat{t}_0,\dots,\hat{t}_{n-1})$, say with blowups $\widehat{F}_T^{(i)}$ and so forth, such that $F_T^{(n)}=\widehat{F}_T^{(n)}$ and $e_T^{(i,n)}E_T^{(i)}=\hat{e}_T^{(\beta' i,n)}\widehat{E}_T^{(\beta' i)}$ with $\beta' i:=\beta(i-1)+1$ for $1\le i\le n$; furthermore, t_i is proximate to t_j if and only if $\tilde{t}_{\beta i}$ is proximate to $\tilde{t}_{\beta j}$. Set $\tilde{t}_i:=\hat{t}_i$ for $0\le i<l$.

Set $\tilde{t}_l := \widehat{\varphi}_T^{(l+1)} \cdots \widehat{\varphi}_T^{(n)} t_n$ and $\widetilde{T}^{(l)} := \widehat{\varphi}_T^{(l+1)} \cdots \widehat{\varphi}_T^{(n)} T^{(n)}$. Then $\tilde{t}_l$ is a section of $\widehat{F}_T^{(l)}/T$, and $\widetilde{T}^{(l)}$ is its image. Note that, if $T^{(n)}$ meets $\hat{e}_T^{(j,n)} \widehat{E}_T^{(j)}$ with $1 \le j \le n$, then $T^{(n)}$ is contained in $\hat{e}_T^{(j,n)} \widehat{E}_T^{(j)}$, because $\hat{e}_T^{(j,n)} \widehat{E}_T^{(j)} = e_T^{(i,n)} E_T^{(i)}$ for $i := \beta'^{-1} j$ and because $(t_0, \dots, t_n)$ is strict. Furthermore, if so, then $l > j$, because t_n is proximate to t_i, and so $\alpha n > \alpha i$, or $l > \beta i = j$; moreover, then $\widetilde{T}^{(l)}$ is contained in $\hat{e}_T^{(j,l)} \widehat{E}_T^{(j)}$, because the latter is equal to $\widehat{\varphi}_T^{(l+1)} \cdots \widehat{\varphi}_T^{(n)} \hat{e}_T^{(j,n)} \widehat{E}_T^{(j)}$ since $l > j$.

Suppose $\widetilde{T}^{(l)}$ meets $\hat{e}_T^{(k,l)} \widehat{E}_T^{(k)}$. Then $T^{(n)}$ meets $(\widehat{\varphi}_T^{(l+1)} \cdots \widehat{\varphi}_T^{(n)})^{-1} \hat{e}_T^{(k,l)} \widehat{E}_T^{(k)}$. So $T^{(n)}$ meets one of the latter's components, which is a $\hat{e}_T^{(j,n)} \widehat{E}_T^{(j)}$ for some j. Hence $T^{(l)} \subset \hat{e}_T^{(j,l)} \widehat{E}_T^{(j)}$, as was noted above. Now, the map $\hat{e}_T^{(j,n)} \widehat{E}_T^{(j)} \to \widehat{F}_T^{(l)}$ factors through $\widehat{E}_T^{(k)}$, and its image is $\hat{e}_T^{(j,l)} \widehat{E}_T^{(j)}$, as was noted above. So $\hat{e}_T^{(j,l)} \widehat{E}_T^{(j)}$ is contained in $\hat{e}_T^{(k,l)} \widehat{E}_T^{(k)}$; whence, the two coincide, since they are flat and coincide on the fibers over T. Thus $\widetilde{T}^{(l)}$ is contained in $\hat{e}_T^{(k,l)} \widehat{E}_T^{(k)}$. Hence, since $(\tilde{t}_0, \dots, \tilde{t}_{l-1})$ is strict, so is $(\tilde{t}_0, \dots, \tilde{t}_l)$. Furthermore, $T^{(n)}$ is contained in $\hat{e}_T^{(k,n)} \widehat{E}_T^{(k)}$. Thus if $\tilde{t}_l$ is proximate to $\tilde{t}_k$, then t_n is proximate to t_i for $i := \beta'^{-1} k$. Moreover, the converse follows from what was noted above.

Set $T_l := \widetilde{T}^{(l)}$ and $T_i := \widetilde{\varphi}_T^{(l+1)} \cdots \widetilde{\varphi}_T^{(i)} \widetilde{T}^{(i)}$ for $l < i < n$. Then $\widetilde{T}^{(l)}$ meets no $\widetilde{T}_i$, because, otherwise, $T^{(n)}$ would meet $(\widehat{\varphi}_T^{(l+1)} \cdots \widehat{\varphi}_T^{(n)})^{-1} \widetilde{T}_{i+1}$, and so $T^{(n)}$ would meet some $\hat{e}_T^{(j,n)} \widehat{E}_T^{(j)}$ with $l < j$, contrary to the note above. So Lemma 4.1 implies $(\tilde{t}_0, \dots, \tilde{t}_l)$ extends to a strict sequence $(\tilde{t}_0, \dots, \tilde{t}_n)$ such that $\tilde{t}_l$ is a leaf and $\widetilde{F}_T^{(l+1)} \times_{\widetilde{F}_T^{(l)}} \widehat{F}_T^{(i)} = \widetilde{F}_T^{(i+1)}$ for $l < i \le n$; furthermore, the diagram of $(\tilde{t}_0, \dots, \tilde{t}_n)$ induces that of $(\hat{t}_0, \dots, \hat{t}_{n-1})$.

Therefore, t_i is proximate to t_j if and only if $\tilde{t}_{\alpha i}$ is proximate to $\tilde{t}_{\alpha j}$ for $0 \le i \le n$, because t_i is proximate to t_j if and only if $t_{\beta i}$ is proximate to $t_{\beta j}$ for $0 \le i < n$ and because t_n is proximate to t_j if and only if $\tilde{t}_l$ is proximate to $\tilde{t}_k$ for $k := \beta' j$.

Recall from above that $F_T^{(n)} = \widehat{F}_T^{(n)}$ and $\widetilde{F}_T^{(l+1)} \times_{\widetilde{F}_T^{(l)}} \widehat{F}_T^{(n)} = \widetilde{F}_T^{(n+1)}$. But this product is equal to the blowup of $\widehat{F}_T^{(n)}$ along $T^{(n)}$ since $\widetilde{T}^{(l)}$ meets no $\widehat{T}_i$. And the blowup of $F_T^{(n)}$ along $T^{(n)}$ is $F_T^{(n+1)}$. Thus $F_T^{(n+1)} = \widetilde{F}_T^{(n+1)}$.

Recall $e_T^{(i,n)} E_T^{(i)} = \hat{e}_T^{(\beta' i,n)} \widehat{E}_T^{(\beta' i)}$ for $1 \le i \le n$. Hence, $e_T^{(i,n+1)} E_T^{(i)}$ is equal to the image of a natural embedding of $\hat{e}_T^{(\beta' i,n)} \widehat{E}_T^{(\beta' i)}$ in $\widetilde{F}_T^{(n+1)}$. In turn, this image is equal to $\tilde{e}_T^{(\alpha' i,n)} \widetilde{E}_T^{(\alpha' i)}$ since $\widetilde{T}^{(l)}$ meets no $\widehat{T}_i$. Similarly, $E_T^{(n+1)} = \tilde{e}_T^{(l+1,n+1)} \widetilde{E}_T^{(l+1)}$. Thus $e_T^{(i,n+1)} E_T^{(i)} = \tilde{e}_T^{(\alpha' i,n+1)} \widetilde{E}_T^{(\alpha' i)}$ for $1 \le i \le n+1$. □

Proposition 4.3. *Fix an unweighted Enriques diagram* $\mathbf{U}$*. Then, given two orderings* θ *and* θ'*, there exists a natural isomorphism*

$$\Phi_{\theta,\theta'} \colon F(\mathbf{U}, \theta) \xrightarrow{\sim} F(\mathbf{U}, \theta').$$

Furthermore, $\Phi_{\theta,\theta} = 1$*, and* $\Phi_{\theta',\theta''} \circ \Phi_{\theta,\theta'} = \Phi_{\theta,\theta''}$ *for any third ordering* θ''*.*

PROOF. Say $\mathbf{U}$ has $n+1$ vertices. Set $\alpha := \theta' \circ \theta^{-1}$. Then α is a permutation of $\{0, \dots, n\}$.

Each T-point of $F(\mathbf{U}, \theta)$ corresponds to a strict sequence $(t_0, \dots, t_n)$ owing to Theorem 3.10. For each i, say t_i corresponds to the vertex V_i of $\mathbf{U}$. Then $\theta(V_i) = i$, and if t_i is proximate to t_j, then V_i is proximate to V_j. So $\theta'(V_i) > \theta'(V_j)$ since θ' is an ordering. Hence $\alpha i > \alpha j$.

Therefore, by Lemma 4.2, there is a unique strict sequence $(\tilde{t}_0, \dots, \tilde{t}_n)$ such

that t_i is proximate to t_j if and only if $\tilde{t}_{\alpha i}$ is proximate to $\tilde{t}_{\alpha j}$. Plainly $(\tilde{t}_0, \dots, \tilde{t}_n)$ has $(\mathbf{U}, \theta')$ as its diagram. Hence $(\tilde{t}_0, \dots, \tilde{t}_n)$ corresponds to a T-point of $F(\mathbf{U}, \theta')$ owing to Theorem 3.10.

Due to uniqueness, sending $(t_0, \dots, t_n)$ to $(\tilde{t}_0, \dots, \tilde{t}_n)$ gives a well-defined map of functors. It is represented by a map $\Phi_{\theta,\theta'}\colon F(\mathbf{U}, \theta) \to F(\mathbf{U}, \theta')$. Again due to uniqueness, $\Phi_{\theta,\theta} = 1$ and $\Phi_{\theta',\theta''} \circ \Phi_{\theta,\theta'} = \Phi_{\theta,\theta''}$ for any θ''. So $\Phi_{\theta',\theta} \circ \Phi_{\theta,\theta'} = 1$ and $\Phi_{\theta,\theta'} \circ \Phi_{\theta',\theta} = 1$. Thus $\Phi_{\theta,\theta'}$ is an isomorphism, and the proposition is proved. □

Corollary 4.4. *Fix an ordered unweighted Enriques diagram* $(\mathbf{U}, \theta)$*. Then there is a natural free right action of* $\mathrm{Aut}(\mathbf{U})$ *on* $F(\mathbf{U}, \theta)$*; namely,* $\gamma \in \mathrm{Aut}(\mathbf{U})$ *acts as* $\Phi_{\theta,\theta'}$ *where* $\theta' := \theta \circ \gamma$*.*

PROOF. Let $V \in \mathbf{U}$ be a vertex that precedes another W. Then $\gamma(V)$ precedes $\gamma(W)$ because $\gamma \in \mathrm{Aut}(\mathbf{U})$. Since θ is an ordering, $\theta(\gamma(V)) \le \theta(\gamma(W))$. Hence $\theta'(V) \le \theta'(W)$. Thus θ' is an ordering.

So there is a natural isomorphism $\Phi_{\theta,\theta'}\colon F(\mathbf{U}, \theta) \xrightarrow{\sim} F(\mathbf{U}, \theta')$ by Proposition 4.3. Now, γ induces an isomorphism of ordered unweighted Enriques diagrams from $(\mathbf{U}, \theta')$ to $(\mathbf{U}, \theta)$; hence, $F(\mathbf{U}, \theta')$ and $F(\mathbf{U}, \theta)$ are the same subscheme of $F^{(n)}$, and $\Phi_{\theta,\theta'}$ is an automorphism of $F(\mathbf{U}, \theta)$.

Note that, if $\gamma = 1$, then $\theta' = \theta$; moreover, $\Phi_{\theta,\theta} = 1$.

Given $\delta \in \mathrm{Aut}(\mathbf{U})$, set $\theta'' := \theta' \circ \delta$ and $\theta^* := \theta \circ \delta$. Then γ also induces an isomorphism from $(\mathbf{U}, \theta'')$ to $(\mathbf{U}, \theta^*)$, and so $\Phi_{\theta',\theta''}$ and Φ_{θ,θ^*} coincide. Now, $\Phi_{\theta',\theta''} \circ \Phi_{\theta,\theta'} = \Phi_{\theta,\theta''}$. Thus $\mathrm{Aut}(\mathbf{U})$ acts on $F(\mathbf{U}, \theta)$, but it acts on the right because θ'' is equal to $\theta \circ (\gamma\delta)$, not to $\theta \circ (\delta\gamma)$.

Suppose γ has a fixed T-point. Then the T-point is fixed under $\Phi_{\theta,\theta'}$. Now, we defined $\Phi_{\theta,\theta'}$ by applying Lemma 4.2 with $\alpha := \theta' \circ \theta^{-1}$. And the lemma asserts that α is determined by its action on the $e_T^{(i,n+1)} E_T^{(i)}$. But this action is trivial because the T-point is fixed. Hence $\alpha = 1$. But $\alpha = \theta \circ \gamma \circ \theta^{-1}$. Therefore, $\gamma = 1$. Thus the action of $\mathrm{Aut}(\mathbf{U})$ is free, and the corollary is proved. □

Corollary 4.5. *Fix an ordered unweighted Enriques diagram* $(\mathbf{U}, \theta)$*, and let* $G \subset \mathrm{Aut}(\mathbf{U})$ *be a subgroup. Then the quotient* $F(\mathbf{U}, \theta)/G$ *is* Y*-smooth with irreducible geometric fibers of dimension* $\dim(\mathbf{U})$*.*

PROOF. The action of G on $F(\mathbf{U}, \theta)$ is free by Corollary 4.4. So G defines a finite flat equivalence relation on $F(\mathbf{U}, \theta)$. Therefore, the quotient exists, and the map $F(\mathbf{U}, \theta) \to F(\mathbf{U}, \theta)/G$ is faithfully flat. Now, $F(\mathbf{U}, \theta)$ is Y-smooth with irreducible geometric fibers of dimension $\dim(\mathbf{U})$ by Theorem 3.10; so $F(\mathbf{U}, \theta)/G$ is too. □

Definition 4.6. For $1 \le i \le j$, set $E^{(i,i)} := E^{(i)}$ and

$$E^{(i,j)} := (\varphi^{(i+1)} \cdots \varphi^{(j)})^{-1} E^{(i)} \text{ if } i < j.$$

Given an ordered unweighted Enriques diagram $(\mathbf{U}, \theta)$ on $n+1$ vertices, say with proximity matrix (p_{ij}), let $E(\mathbf{U}, \theta) \subset F^{(n)}$ be the set of scheme points $\mathbf{t}$ such that, on the fiber $F_{\mathbf{t}}^{(n+1)}$, for $1 \le k \le n$, the divisors $\sum_{i=k}^{n+1} p_{ik} E_{\mathbf{t}}^{(i,n+1)}$ are effective.

Proposition 4.7. *Let* $(\mathbf{U}, \theta)$ *be an ordered unweighted Enriques diagram. Then* $E(\mathbf{U}, \theta)$ *is closed and contains* $F(\mathbf{U}, \theta)$ *set-theoretically.*

PROOF. Say $\mathbf{U}$ has $n+1$ vertices. Fix $\mathbf{t} \in F^{(n)}$ and $1 \le k \le n$. If $\mathbf{t} \in F(\mathbf{U}, \theta)$, then, as is easy to see by induction on j for $k \le j \le n$, the divisor $\sum_{i=k}^{j+1} p_{ik} E_{\mathbf{t}}^{(i,j+1)}$ is equal to the strict transform on $F_{\mathbf{t}}^{(j+1)}$ of $E_{\mathbf{t}}^{(k)}$, in other words, to $e_T^{(k,j+1)} E_T^{(k)}$ where $T := \operatorname{Spec} \kappa(\mathbf{t})$. Hence $E(\mathbf{U}, \theta)$ contains $F(\mathbf{U}, \theta)$.

Set $\widetilde{E}^{(k)} := \sum_{i=k}^{n+1} p_{ik} E^{(i,n+1)}$. Then $\mathrm{h}^0\big(F_{\mathbf{t}}^{(n+1)}, \mathcal{O}(\widetilde{E}_{\mathbf{t}}^{(k)})\big) \le 1$ for any $\mathbf{t}$, and equality holds if and only if $\mathbf{t} \in F(\mathbf{U}, \theta)$, as the following essentially standard argument shows. Plainly, it suffices to show that, if $\widetilde{E}_{\mathbf{t}}^{(k)}$ is linearly equivalent to an effective divisor D, then $\widetilde{E}_{\mathbf{t}}^{(k)} = D$.

Let H be the preimage on $F_{\mathbf{t}}^{(n+1)}$ of an ample divisor on $F_{\mathbf{t}}$. Then the intersection number $\widetilde{E}_{\mathbf{t}}^{(k)} \cdot H$ vanishes by the projection formula because each component of $\widetilde{E}_{\mathbf{t}}^{(k)}$ maps to a point in $F_{\mathbf{t}}$. So $D \cdot H$ vanishes too. Hence each component of D must also map to a point in $F_{\mathbf{t}}$ because D is effective and H is ample. Hence D is some linear combination of the $E_{\mathbf{t}}^{(i,n+1)}$ because they form a basis of the group of divisors whose components each map to a point. Furthermore, the combining coefficients must be the p_{ik} because these coefficients are given by the intersection numbers with the $E_{\mathbf{t}}^{(i,n+1)}$. Thus $\widetilde{E}_{\mathbf{t}}^{(k)} = D$.

Thus $E(\mathbf{U}, \theta)$ is the set of $\mathbf{t} \in F^{(n)}$ such that $\mathrm{h}^0\big(F_{\mathbf{t}}^{(n+1)}, \mathcal{O}(\widetilde{E}_{\mathbf{t}}^{(k)})\big) \ge 1$ for all k. Hence $E(\mathbf{U}, \theta)$ is closed by semi-continuity [**8**, Thm. (7.7.5), p. 67]. □

Proposition 4.8. *Let $(\mathbf{U}, \theta)$ and $(\mathbf{U}', \theta')$ be two ordered unweighted Enriques diagrams on $n+1$ vertices, and let $\mathbf{P}$ and $\mathbf{P}'$ be their proximity matrices. Then the following conditions are equivalent:*

(1) *The sets $F(\mathbf{U}', \theta')$ and $E(\mathbf{U}, \theta)$ meet.*
(2) *The set $E(\mathbf{U}', \theta')$ is contained in the set $E(\mathbf{U}, \theta)$.*
(3) *The matrix $\mathbf{P}'^{-1}\mathbf{P}$ only has nonnegative entries.*

Furthermore, $E(\mathbf{U}', \theta') = E(\mathbf{U}, \theta)$ if and only if $(\mathbf{U}, \theta) \cong (\mathbf{U}', \theta')$.

PROOF. Fix $\mathbf{t} \in F^{(n)}$, and define two sequences of divisors on $F_{\mathbf{t}}^{(n+1)}$ by these matrix equations:

$$(\widetilde{E}_{\mathbf{t}}^{(1)}, \dots, \widetilde{E}_{\mathbf{t}}^{(n+1)}) = (E_{\mathbf{t}}^{(1,n+1)}, \dots, E_{\mathbf{t}}^{(n+1,n+1)})\mathbf{P};$$
$$(\widetilde{E}_{\mathbf{t}}^{(1)\prime}, \dots, \widetilde{E}_{\mathbf{t}}^{(n+1)\prime}) = (E_{\mathbf{t}}^{(1,n+1)}, \dots, E_{\mathbf{t}}^{(n+1,n+1)})\mathbf{P}'.$$

These two equations imply the following one:

$$(\widetilde{E}_{\mathbf{t}}^{(1)}, \dots, \widetilde{E}_{\mathbf{t}}^{(n+1)}) = (\widetilde{E}_{\mathbf{t}}^{(1)\prime}, \dots, \widetilde{E}_{\mathbf{t}}^{(n+1)\prime})\mathbf{P}'^{-1}\mathbf{P}; \tag{4.8.1}$$

in other words, $\widetilde{E}_{\mathbf{t}}^{(j)} = \sum_{k=1}^{n+1} q_{kj} \widetilde{E}_{\mathbf{t}}^{(k)\prime}$ where say $(q_{kj}) := \mathbf{P}'^{-1}\mathbf{P}$.

Suppose $\mathbf{t} \in F(\mathbf{U}', \theta')$. Then $\widetilde{E}_{\mathbf{t}}^{(k)\prime}$ is the proper transform on $F_{\mathbf{t}}^{(n+1)}$ of $E_{\mathbf{t}}^{(k)}$, as we noted at the beginning of the proof of Proposition 4.7. So the $\widetilde{E}_{\mathbf{t}}^{(k)\prime}$ form a basis of the group of divisors whose components each map to a point in $F_{\mathbf{t}}$. Hence, by (4.8.1), if $\widetilde{E}_{\mathbf{t}}^{(j)}$ is effective, then $q_{kj} \ge 0$ for all k. Thus (1) implies (3).

Suppose $\mathbf{t} \in E(\mathbf{U}', \theta')$. Then $\widetilde{E}_{\mathbf{t}}^{(k)\prime}$ is effective. Suppose too $q_{kj} \ge 0$ for all k, j. Then $\widetilde{E}_{\mathbf{t}}^{(j)}$ is effective for all j by (4.8.1). So $\mathbf{t} \in E(\mathbf{U}, \theta)$. Thus (3) implies (2).

By Proposition 4.7, $E(\mathbf{U}, \theta)$ contains $F(\mathbf{U}, \theta)$. By Theorem 3.10, $F(\mathbf{U}, \theta)$ is nonempty. Thus (2) implies (1). So (1), (2), and (3) are equivalent.

Furthermore, suppose $E(\mathbf{U}', \theta') = E(\mathbf{U}, \theta)$. Then both $\mathbf{P}'^{-1}\mathbf{P}$ and $\mathbf{P}^{-1}\mathbf{P}'$ have nonnegative entries since (2) implies (3). But each matrix is the inverse of the other,

and both are lower triangular. Hence both are the identity. So $\mathbf{P}' = \mathbf{P}$; whence, $(\mathbf{U}, \theta) \cong (\mathbf{U}', \theta')$. The converse is obvious. Thus the proposition is proved. □

5. The Hilbert scheme

Fix a smooth family of geometrically irreducible surfaces $\pi\colon F \to Y$. In this section, we prove our main result, Theorem 5.7. It asserts that, given an Enriques diagram $\mathbf{D}$ and an ordering θ, there exists a natural map Ψ from the quotient $F(\mathbf{D}, \theta)/\mathrm{Aut}(\mathbf{D})$ into the Hilbert scheme $\mathrm{Hilb}^d_{F/Y}$ with $d := \deg \mathbf{D}$ and with $F(\mathbf{D}, \theta) := F(\mathbf{U}, \theta)$ where $\mathbf{U}$ is the unweighted diagram underlying $\mathbf{D}$.

The quotient $F(\mathbf{D}, \theta)/\mathrm{Aut}(\mathbf{D})$ parameterizes the strict sequences of arbitrarily near points of F/Y with diagram $(\mathbf{U}, \theta)$, up to automorphism of $\mathbf{D}$. The image of Ψ parameterizes the (geometrically) complete ideals of F/Y with diagram $\mathbf{D}$. The map Ψ is universally injective. In fact, Ψ is an embedding in characteristic 0. However, in positive characteristic, Ψ can be purely inseparable; Appendix B discusses examples found by Tyomkin.

We close this section with Proposition 5.9, which addresses the important special case where every vertex of $\mathbf{D}$ is a root; here, Ψ is an embedding in any characteristic. Further, other examples in Appendix B show that Ψ can remain an embedding even after a nonroot is added.

5.1 (Geometrically complete ideals)**.** Let K be a field, $(t_0, \dots, t_n)$ a sequence of arbitrarily near K-points of F/Y. Since $\mathrm{Spec}(K)$ consists of a single reduced point, the sequence is strict. Let $(\mathbf{U}, \theta)$ be its diagram in the sense of Definition 3.9.

Suppose $\mathbf{U}$ underlies an Enriques diagram $\mathbf{D}$, say with weights m_V for $V \in \mathbf{U}$. Using the divisors $E_K^{(i,n+1)}$ on $F_K^{(n+1)}$ of Definition 4.6, set

$$E_K := \textstyle\sum_V m_V E_K^{(\theta(V)+1,\, n+1)} \text{ and } \mathcal{L}_K := \mathcal{O}_{F_K^{(n+1)}}(-E_K).$$

Given $V \in \mathbf{U}$, set $j := \theta(V)$ and $D_V := e_K^{(j+1,n+1)} E_K^{(j+1)}$. Inspired by Lipman's remark [**21**, p. 306], let's compute the intersection number $-(E_K \cdot D_V)$, that is, $\deg(\mathcal{L}|D_V)$. Plainly, $(E_K^{(j+1,\, n+1)} \cdot D_V) = -1$. And, for $W \neq V$, plainly $(E_K^{(\theta(W)+1,\, n+1)} \cdot D_V)$ is equal to 1 if $W \succ V$, and to 0 if not. Hence $-(E_K \cdot D_V)$ is equal to $m_V - \sum_{W \succ V} m_W$, which is at least 0 by the Proximity Inequality.

Set $\varphi_K := \varphi_K^{(1)} \cdots \varphi_K^{(n+1)}$, and form $\mathcal{I} := \varphi_{K*}\mathcal{L}_K$ on F_K. Then $\mathcal{I}$ is a complete ideal, one that is integrally closed; also, $\mathcal{I}\mathcal{O}_{F_K^{(n+1)}} = \mathcal{L}_K$ and $R^q\varphi_{K*}\mathcal{L}_K = 0$ for $q \geq 1$. These three statements hold since $(E_K \cdot D_V) \leq 0$ for all V and, as is well known, $R^q\varphi_{K*}\mathcal{O}_{F_K^{(n+1)}} = 0$ for $q \geq 1$; see Lipman's discussion [**20**, §18, p. 238] and his Part (ii) of [**20**, Thm. (12.1), p. 220]; also see Deligne's Théorème 2.13 [**3**, p. 22]. Furthermore,

$$\dim_K \mathrm{H}^0(\mathcal{O}_{F_K}/\mathcal{I}) = d \quad \text{where } d := \deg \mathbf{D}.$$

This formula is a modern version of Enriques' formula [**4**, Vol. II, p. 426]; it was proved in different ways independently by Hoskin [**13**, 5.2, p. 85], Deligne [**3**, 2.13, p. 22], and Casas [**1**, 6.1, p. 438]; Hoskin and Deligne worked in greater generality, Casas worked over $\mathbb{C}$.

The m_V are determined by $\mathcal{I}$ because the divisors $E_K^{(i,n+1)}$ are numerically independent; their intersection numbers with divisors are defined because they are complete. The m_V may be found as follows. Let $\mathcal{P}$ be the ideal of the image $T^{(0)}$ of t_0, which is a K-point of F_K. Let m be the largest integer such that $\mathcal{P}^m \supset \mathcal{I}$.

Then $m = m_V$ where $V := \theta^{-1}(0)$, since $\mathcal{P}\mathcal{O}_{F_K^{(n+1)}} = \mathcal{O}_{F_K^{(n+1)}}(-E_K^{(1,n+1)})$. Note in passing that $\mathcal{P}$ is a minimal prime of $\mathcal{I}$ since $m_V \geq 1$.

The remaining m_W can be found by recursion. Indeed, on $F_K^{(1)}$, form the ideal $\mathcal{I}' := \mathcal{I}\mathcal{O}(m_V E^{(1)})$. Then $\mathcal{I}'$ is the direct image from $F_K^{(n+1)}$ of $\mathcal{O}(-E'_K)$ where $E'_K := \sum_{W \neq V} m_W E_K^{(\theta(W)+1,\, n+1)}$. Hence $\mathcal{I}'$ is the complete ideal associated to the sequence $(t_1, \ldots, t_n)$ of arbitrarily near K-points of $F^{(1)}/Y$ and to the ordered Enriques diagram $(\mathbf{D}', \theta')$ where $\mathbf{D}' := \mathbf{D} - V$ and $\theta'(W) := \theta(W) - 1$.

The ideal $\mathcal{I}$ determines the diagram $\mathbf{D}$. Indeed, for $0 \leq i \leq n$, let A_i, $\mathbf{m}_i$ be the local ring of the surface $F_K^{(i)}$ at the K-point that is the image of t_i. Then according to Lipman's preliminary discussion in [**21**, p. 294–295], the set $\{A_i\}$ consists precisely of 2-dimensional regular local K-domains whose fraction field is that of F_K and whose maximal ideal contains the stalk of $\mathcal{I}$ at some point of F_K. Furthermore, t_i is proximate to t_j if and only if A_i is contained in the ring of the valuation v_j defined by the formula: $v_j(f) := \max\{m \mid f \in \mathbf{m}_j^m\}$. Finally, if $W := \theta^{-1}(j)$, then the weight m_W is the largest integer m such that $\mathbf{m}_j^m$ contains the appropriate stalk of $\mathcal{I}$.

Let $\mathcal{J}$ be an arbitrary ideal on F_K of finite colength. Let L/K be an arbitrary field extension. If the extended ideal $\mathcal{J}_L$ on F_L is complete, then $\mathcal{J}$ is complete, and the converse holds if L/K is separable; see Nobile and Villamayor's proof of [**25**, Prp. (3.2), p. 251]. Let us say that $\mathcal{J}$ is *geometrically complete* if $\mathcal{J}_L$ on F_L is complete for every L, or equivalently, for some algebraically closed L. In characteristic 0, if $\mathcal{J}$ is complete, then it is geometrically complete.

The extended ideal $\mathcal{I}_L$ on F_L is, plainly, the complete ideal associated to the extension of the sequence $(t_0, \ldots, t_n)$ and to the same ordered Enriques diagram $(\mathbf{D}, \theta)$. Hence $\mathcal{I}$ is geometrically complete.

Suppose that K is algebraically closed. Suppose that $\mathcal{J}$ is complete and that $\dim_K \mathrm{H}^0(\mathcal{O}_{F_K}/\mathcal{J})$ is finite and nonzero. Then $\mathcal{J}$ arises from some sequence $(s_0, \ldots, s_n)$ and some ordered Enriques diagram. Indeed, choose a minimal prime $\mathcal{P}$ of $\mathcal{J}$. Then $K \xrightarrow{\sim} \mathcal{O}_{F_K}/\mathcal{P}$ since K is algebraically closed. Hence $\mathcal{P}$ defines a K-point $S^{(0)}$ of F_K, so a section s_0 of F_K/K. Set $m_0 := \max\{\, m \mid \mathcal{P}^m \supset \mathcal{J} \,\}$.

Let F'_K be the blowup of F_K at $S^{(0)}$, and E'_K the exceptional divisor. Set $\mathcal{J}' := \mathcal{J}\mathcal{O}_{F'_K}(m_0 E'_K)$. Then $\mathcal{J}'$ is complete by Zariski and Samuel's [**33**, Prp. 5, p. 381]. If $\mathcal{J}' = \mathcal{O}_{F'_K}$, then stop. If not, then repeat the process again and again, obtaining a sequence $(s_0, s_1, \ldots)$. Only finitely many repetitions are necessary because, as Lipman [**21**, p. 295] points out, the local ring of $F_K^{(i)}$ at $S^{(i)}$ is dominated by a Rees valuation of $\mathcal{J}$, that is, the valuation associated to an exceptional divisor of the normalized blowup of $\mathcal{J}$. Then $\mathcal{J}'$ arises from the sequence of s_i weighted by the $m_{\theta^{-1}(i)}$ owing to Lipman's [**20**, prp. (6.2), p. 208] and discussion before it.

Lemma 5.2. *Let A be a discrete valuation ring, set $T := \operatorname{Spec} A$, and denote by $\eta \in T$ the generic point and by $y \in T$ the closed point. Fix a map $T \to Y$. Let $\mathbf{D}$ be an Enriques diagram, say with $n+1$ vertices, and $\mathcal{I}$ a coherent ideal on F_T that generates geometrically complete ideals on F_η and F_y, each with diagram $\mathbf{D}$. Let θ be an ordering of $\mathbf{D}$, and $\tilde{\mathbf{t}}$ a $k(\eta)$-point of $F(\mathbf{D}, \theta)$ such that $\mathcal{I}_\eta$ generates an invertible sheaf on $F_\eta^{(n+1)}$. Then $\tilde{\mathbf{t}}$ extends to a T-point $\mathbf{t}$ of $F(\mathbf{D}, \theta)$.*

PROOF. Let θ' be a second ordering. By the construction of the isomorphism

$\Phi_{\theta,\theta'}$ in the proof of Proposition 4.3, a T-point of $F(\mathbf{D}, \theta)$ corresponds to the T-point of $F(\mathbf{D}, \theta')$ given by Lemma 4.2 with $\alpha := \theta' \circ \theta^{-1}$. Moreover, the lemma says that $F_T^{(n+1)}$ is unchanged. It follows that, to construct $\mathbf{t}$, we may replace θ by θ'. Thus we may assume that $E(\mathbf{D}, \theta)$ is a minimal element among the various closed subsets $E(\mathbf{D}, \theta')$ of $F^{(n)}$.

Let $R \in \mathbf{D}$ be a root, and temporarily set $i := \theta(R)$. Say $\tilde{\mathbf{t}}$ corresponds to the sequence of blowups $F_\eta^{(j+1)} \to F_\eta^{(j)}$ with centers η_j. The image of η_i in F_T is a $k(\eta)$-point; denote its closure by T_R. Since A is a discrete valuation ring, the structure map is an isomorphism $T_R \xrightarrow{\sim} T$.

Let $Z \subset F_T$ be the subscheme with ideal $\mathcal{I}$. Its fibers Z_η and Z_y are finite, and both have degree $\deg(\mathbf{D})$ since the two ideals are geometrically complete with diagram $\mathbf{D}$ by hypothesis. Since T is reduced, Z is T-flat.

As R varies, the points $(T_R)_\eta$ are exactly the components of Z_η again because its ideal $\mathcal{I}_\eta$ is geometrically complete with diagram $\mathbf{D}$. Hence the several T_R are just the components of Z that meet Z_η. But every component of Z meets Z_η since Z is T-flat. Thus the T_R are the the components of Z.

Since $T_R \xrightarrow{\sim} T$ for each R, the fiber $(T_R)_y$ is a single point, so a component of the discrete set Z_y. The number of T_R is the number of roots of $\mathbf{D}$, which is also the number of points of Z_y. Hence the several T_R are disjoint.

Given R, let m_R be its weight, $\mathcal{P}_R$ the ideal of T_R in F_T. Then $(\mathcal{P}_R^{m_R})_\eta \supset \mathcal{I}_\eta$. Let's see that $\mathcal{P}_R^{m_R} \supset \mathcal{I}$. Indeed, form the image, $\mathcal{M}$ say, of $\mathcal{I}$ in $\mathcal{O}_{F_T}/\mathcal{P}_R^{m_R}$. Then $\mathcal{M}_\eta = 0$. Let $u \in A$ be a uniformizing parameter. Then $\mathcal{M}$ is annihilated by a power of u. Now, $\mathcal{P}_R$ is quasi-regular by [**10**, (17.12.3), p. 83] since $T_R \xrightarrow{\sim} T$ and F_T is T-smooth. Hence $\mathcal{P}_R^j/P_R^{j+1}$ is T-flat for all j by [**10**, (16.9.4), p. 47]. Hence $\mathcal{O}_{F_T}/\mathcal{P}_R^{m_R}$ is T-flat. So u is a nonzerodivisor on $\mathcal{O}_{F_T}/\mathcal{P}_R^{m_R}$. Hence $\mathcal{M} = 0$. Thus $\mathcal{P}_R^{m_R} \supset \mathcal{I}$.

Let n_R be the largest integer such that $(\mathcal{P}_R^{n_R})_y \supset \mathcal{I}_y$. Then $n_R \geq m_R$. Now, $\mathcal{I}_y$ is geometrically complete with diagram $\mathbf{D}$. Hence n_R is the weight of the root corresponding to $(\mathcal{P}_R)_y$. Hence $\sum_R n_R = \sum_R m_R$. But $n_R \geq m_R$. Therefore, $n_R = m_R$ for every root R.

Let $\mathbf{D}'$ be the diagram obtained from $\mathbf{D}$ by omitting the roots. Let θ' be the ordering of $\mathbf{D}'$ induced by θ; namely, $\theta'(V) := \theta(V) - r_V$ where r_V denotes the number of roots R of $\mathbf{D}$ such that $\theta(R) < \theta(V)$. Let F_T' be obtained from F_T by blowing up $\bigcup T_R$, and for each R, let E_R' be the preimage of T_R. Set

$$\mathcal{I}' := \mathcal{I}\mathcal{O}_{F_T'}\big(\textstyle\sum_R m_R E_R'\big).$$

Finally, let n' be the number of vertices of $\mathbf{D}'$.

Then $\mathcal{I}'$ generates geometrically complete ideals on F_η' and F_y', each with diagram $\mathbf{D}'$ owing to the theory of geometrically complete ideals over a field; see Subsection 5.1. (To ensure that the ideals on F_η' and F_y' have the same diagram, it is necessary to omit all the roots of $\mathbf{D}$. Indeed, $\mathbf{D}$ might have two roots with the same multiplicity, but the diagram obtained by omitting one root might differ from that obtained by eliminating the other. Conceivably, the two roots get interchanged under the specialization.)

Plainly, $\tilde{\mathbf{t}}$ induces a $k(\eta)$-point $\tilde{\mathbf{t}}'$ of $F(\mathbf{D}', \theta')$ such that $\mathcal{I}_\eta'$ generates an invertible sheaf on the corresponding $F_\eta'^{(n'+1)}$, which is equal to $F_\eta^{(n+1)}$. Hence, by induction on n, we may assume that $\tilde{\mathbf{t}}'$ extends to T-point $\mathbf{t}'$ of $F(\mathbf{D}', \theta')$ such that,

on the corresponding scheme $F_T'^{(n'+1)}$, the ideal $\mathcal{I}'$ generates an invertible ideal. It remains to show that $\mathbf{t}'$ and the several isomorphisms $T_R \xrightarrow{\sim} T$ yield an extension $\mathbf{t}$ of $\tilde{\mathbf{t}}$.

Proceed by induction on i where $0 \le i \le n$. Suppose we have constructed a sequence $(t_0, \dots, t_{i-1})$ extending the sequence $(\tilde{t}_0, \dots, \tilde{t}_{i-1})$ coming from $\tilde{\mathbf{t}}$; suppose also that, if we blow up $F_T^{(i)}$ along the preimage of $\bigcup_{k\ge i} T_k$, then we get $F_T'^{(i')}$ where, for $0 \le j \le n$, we let j' denote j diminished by the number of roots R of $\mathbf{D}$ such that $\theta(R) < j$. Note that the base case $i := 0$ obtains: the sequence $(t_0, \dots, t_{i-1})$ is empty; furthermore, $F_T^{(i)} = F_T$ and $F_T'^{(i')} = F_T'$, which is the blowup of F_T along $\bigcup_{k\ge i} T_k$.

Note that $F_T^{(i)} \to F_T$ is an isomorphism off $\bigcup_{k<i} T_k$. Indeed, given $j < i$, let $R' \in \mathbf{D}$ be the root preceding $\theta^{-1}(j)$, and set $k := \theta(R')$. Since θ is an ordering, $k \le j$. Since $(t_0, \dots, t_{i-1})$ extends $(\tilde{t}_0, \dots, \tilde{t}_{i-1})$, the image of $T_\eta^{(j)}$ in F_T is just $(T_k)_\eta$. So $T^{(j)}$ maps into T_k, and $k < i$.

Set $V := \theta^{-1}(i) \in \mathbf{D}$. First suppose V is a root of $\mathbf{D}$. Then $(i+1)' = i'$. Also, T_i is defined, and the isomorphism $T_i \xrightarrow{\sim} T$ provides a section t_i of $F_T^{(i)}$ owing to the preceding note. By the same token, the blowup of $F_T^{(i+1)}$ along the preimage of $\bigcup_{k\ge i+1} T_k$ is equal to the blowup of $F_T^{(i)}$ along the preimage of $\bigcup_{k\ge i} T_k$. But the latter blowup is equal to $F_T'^{(i')}$. It follows that t_i does the trick.

Next suppose V is not a root, so $V \in \mathbf{D}'$. Also $\bigcup_{k\ge i} T_k = \bigcup_{k\ge i+1} T_k$. Now, by the induction assumption, $F_T'^{(i')}$ is equal to $F_T^{(i)}$ off the preimage of $\bigcup_{k\ge i} T_k$. Take $t_i := t_i'$ where $(t_0', \dots, t_i')$ comes from $\mathbf{t}'$. It is not hard to see that t_i does the trick.

It is not immediately obvious that $(t_0, \dots, t_n)$ is strict, even though $(t_0', \dots, t_{n'}')$ is strict. However, $\mathbf{t}$ is a T-point of $F^{(n)}(T)$ and $\mathbf{t}_\eta$ is a $k(\eta)$-point of $F(\mathbf{D}, \theta)$; furthermore, $\mathbf{t}_y$ is a $k(y)$-point of $F(\mathbf{D}, \phi)$ for some ordering ϕ of $\mathbf{D}$. Since T is irreducible, $\mathbf{t}_y$ is a point of the closure of $F(\mathbf{D}, \theta)$ in $F^{(n)}$, so is a point of $E(\mathbf{D}, \theta)$. Hence $E(\mathbf{D}, \theta)$ contains $E(\mathbf{D}, \phi)$ by Proposition 4.8. But, by the initial reduction, $E(\mathbf{U}, \theta)$ is minimal, so equal to $E(\mathbf{D}, \phi)$. Hence $(\mathbf{D}, \theta) \cong (\mathbf{D}, \phi)$ again by Proposition 4.8. So $\mathbf{t}_y$ is a point of $F(\mathbf{D}, \theta)$. Since T is reduced, $\mathbf{t}$ is therefore a T-point of $F(\mathbf{D}, \theta)$, as desired. □

Definition 5.3. Given an Enriques diagram $\mathbf{D}$, say with $d := \deg \mathbf{D}$, let $H(\mathbf{D}) \subset \mathrm{Hilb}^d_{F/Y}$ denote the subset parameterizing the geometrically complete ideals with diagram $\mathbf{D}$ on the geometric fibers of F/Y; see Subsection 5.1.

Proposition 5.4. *Let $\mathbf{D}$ be an Enriques diagram, set $d := \deg \mathbf{D}$, and choose an ordering θ. Then there exists a natural map $\Upsilon_\theta \colon F(\mathbf{D}, \theta) \to \mathrm{Hilb}^d_{F/Y}$, whose formation commutes with base extension of Y. Its image is $H(\mathbf{D})$, and it factors into a finite map $F(\mathbf{D}, \theta) \to U$ and an open embedding $U \hookrightarrow \mathrm{Hilb}^d_{F/Y}$. Moreover, $\Upsilon_\theta = \Upsilon_{\theta'} \circ \Phi_{\theta,\theta'}$ for any second ordering θ'.*

PROOF. Say $\mathbf{D}$ has $n+1$ vertices V with weights m_V. On $F^{(n+1)}$, set

$$E := \textstyle\sum_V m_V E^{(\theta(V)+1,\, n+1)} \text{ and } \mathcal{L} := \mathcal{O}(-E).$$

Consider the standard short exact sequence:

$$0 \to \mathcal{L} \to \mathcal{O}_{F^{(n+1)}} \to \mathcal{O}_E \to 0.$$

It remains exact on the fibers of $\pi^{(n+1)} \colon F^{(n+1)} \to F^{(n)}$. And $\pi^{(n+1)}$ is flat by

Lemma 3.2. Hence $\mathcal{L}$ and $\mathcal{O}_E$ are flat over $F^{(n)}$ owing to the local criterion.

Fix a T-point of $F(\mathbf{D}, \theta) \subset F^{(n)}$. It corresponds to a strict sequence of arbitrarily near T-points of F/Y by Theorem 3.10. Set $\varphi := \varphi_T^{(1)} \cdots \varphi_T^{(n+1)}$. Let $t \in T$. Then $R^i\varphi_{t*}(\mathcal{L}_t) = 0$ and $R^i\varphi_{t*}(\mathcal{O}_{F_t^{(n+1)}}) = 0$ for $i \geq 1$ by [**3**, Thm. 2.13, p. 22]. Therefore, by Lemma A.2, the induced sequence on F_T,

$$0 \to \varphi_*\mathcal{L}_T \to \varphi_*\mathcal{O}_{F_T^{(n+1)}} \to \varphi_*\mathcal{O}_{E_T} \to 0, \tag{5.4.1}$$

is an exact sequence of T-flat sheaves, and forming it commutes with extending T.

The middle term in (5.4.1) is equal to $\mathcal{O}_{F_T}$: the comorphism $\mathcal{O}_{F_T} \to \varphi_*\mathcal{O}_{F_T^{(n+1)}}$ is an isomorphism, since forming it commutes with passing to the fibers of F_T/T, and on the fibers, it is an isomorphism as it is the comorphism of a birational map between smooth varieties. The third term in (5.4.1) is a locally free $\mathcal{O}_T$-module of rank d because its fibers are vector spaces of dimension d owing again to [**3**, Thm. 2.13, p. 22]. Therefore, (5.4.1) defines a T-point of $\mathrm{Hilb}^d_{F/Y}$.

The construction of this T-point is, plainly, functorial in T, and commutes with base extension of Y. Hence it yields a map $\Upsilon_\theta \colon F(\mathbf{D}, \theta) \to \mathrm{Hilb}^d_{F/Y}$, whose formation commutes with extension of Y.

To see that $H(\mathbf{D})$ is the image of Υ_θ, just observe that, in view of Subsection 5.1, if T is the spectrum of an algebraically closed field, then $\varphi_*\mathcal{L}_T$ is a geometrically complete ideal on F_T with diagram $\mathbf{D}$, and every such ideal on F_T is of this form for some choice of T-point of $F(\mathbf{D}, \theta)$.

Let θ' be a second ordering. Then by the construction of $\Phi_{\theta,\theta'}$ in the proof of Proposition 4.3, our T-point of $F(\mathbf{D}, \theta)$ is carried to that of $F(\mathbf{D}, \theta')$ given by Lemma 4.2 with $\alpha := \theta' \circ \theta^{-1}$. Moreover, the lemma says that $F_T^{(n+1)}$ is unchanged and implies that $E^{(\theta(V)+1,\, n+1)} = E^{(\theta'(V)+1,\, n+1)}$ for all V. Hence $\Upsilon_\theta = \Upsilon_{\theta'} \circ \Phi_{\theta,\theta'}$.

By Zariski's Main Theorem in the form of [**9**, Thm. (8.12.6), p. 45], there exists a factorization

$$\Upsilon_\theta \colon F(\mathbf{D}, \theta) \xrightarrow{\Omega} H \xrightarrow{\Theta} \mathrm{Hilb}^d_{F/Y},$$

where Ω is an open embedding and Θ is a finite map. Let W be the image of Ω, so $\Theta(W) = H(\mathbf{D})$. Replace H by the closure of W, and let us prove $W = \Theta^{-1}H(\mathbf{D})$.

Let $v \in \Theta^{-1}H(\mathbf{D})$. Then v is the specialization of a point $w \in W$ since H is the closure of W. And w is the image of a point $\mathbf{w} \in F(\mathbf{D}, \theta)$. Hence, by [**7**, Thm. (7.1.9), p. 141], there is a map $\tau \colon T \to H$ where T is the spectrum of a discrete valuation ring, such that the closed point $y \in T$ maps to v and the generic point $\eta \in T$ maps to w; also there is a $k(\eta)$-point $\tilde{\mathbf{t}}$ of $F(\mathbf{D}, \theta)$ supported at $\mathbf{w}$.

The map $\Theta \circ \tau$ corresponds to a coherent ideal $\mathcal{I}$ on F_T. Now, both $\Theta(w)$ and $\Theta(v)$ lie in $H(\mathbf{D})$; so $\mathcal{I}$ generates geometrically complete ideals on F_η and F_y, each with diagram $\mathbf{D}$. And $\Upsilon_\theta(\tilde{\mathbf{t}})$ corresponds to $\mathcal{I}_\eta$ on F_η; so $\mathcal{I}_\eta$ generates an invertible sheaf on $F_\eta^{(n+1)}$. Hence, by Lemma 5.2, the $k(\eta)$-point $\tilde{\mathbf{t}}$ extends to T-point $\mathbf{t}$ of $F(\mathbf{D}, \theta)$.

Then $\Upsilon_\theta(\mathbf{t}) \colon T \to W$ carries η to w. But H/Y is separated. Hence $\Upsilon_\theta(\mathbf{t}) = \tau$ by the valuative criterion [**7**, Prp. (7.2.3), p. 142]. But $\tau(y) = v$. Hence $v \in W$. Thus $W \supset \Theta^{-1}H(\mathbf{D})$. But $\Theta(W) = H(\mathbf{D})$. Therefore, $W = \Theta^{-1}H(\mathbf{D})$.

But W is open in H, and Θ is finite. So $\Theta(H)$ and $\Theta(H - W)$ are closed in $\mathrm{Hilb}^d_{F/Y}$. Hence $H(\mathbf{D})$ is open in $\Theta(H)$. So there is an open subscheme U of $\mathrm{Hilb}^d_{F/Y}$ such that $U \bigcap \Theta(H) = H(\mathbf{D})$. Furthermore, $W \to U$ is finite, as it is the restriction of Θ. So $F(\mathbf{D}, \theta) \to U$ is finite. The proof is now complete. □

Corollary 5.5. *Let* $\mathbf{D}$ *be an Enriques diagram, and set* $d := \deg \mathbf{D}$. *Then* $H(\mathbf{D})$ *is a locally closed subset of* $\mathrm{Hilb}^d_{F/Y}$.

PROOF. By Proposition 5.4, $H(\mathbf{D})$ is the image of a finite map into an open subscheme U of $\mathrm{Hilb}^d_{F/Y}$. So $H(\mathbf{D})$ is closed in U, so locally closed in $\mathrm{Hilb}^d_{F/Y}$. □

Remark 5.6. Lossen [**23**, Prp. 2.19, p. 35] proved a complex analytic version of Corollary 5.5. Independently, Nobile and Villamayor [**25**, Thm. 2.6, p. 250] proved the corollary assuming $\mathrm{Hilb}^d_{F/Y}$ is reduced and excellent; in fact, they worked with an arbitrary flat family of ideals on a reduced excellent scheme, but of course, any flat family is induced by a map to the Hilbert scheme. All three approaches are rather different.

Theorem 5.7. *Let* $\mathbf{D}$ *be an Enriques diagram, and set* $d := \deg \mathbf{D}$. *Choose an ordering* θ, *and form the map* Υ_θ *of Proposition* 5.4. *Then* Υ_θ *induces a map*

$$\Psi\colon F(\mathbf{D},\,\theta)\big/\mathrm{Aut}(\mathbf{D}) \to \mathrm{Hilb}^d_{F/Y}\,.$$

It is universally injective; in fact, it is an embedding in characteristic 0. *Furthermore,* Ψ *is independent of the choice of* θ, *up to a canonical isomorphism.*

PROOF. By Corollary 4.4, $\mathrm{Aut}(\mathbf{D})$ acts freely. Hence, the quotient map

$$\Pi\colon F(\mathbf{D},\,\theta) \to F(\mathbf{D},\,\theta)\big/\mathrm{Aut}(\mathbf{D})$$

is faithfully flat. By Proposition 5.4, the action of $\mathrm{Aut}(\mathbf{D})$ is compatible with Υ_θ, and is compatible with a second choice of ordering θ', up to the isomorphism $\Phi_{\theta,\theta'}$. Hence, by descent theory, Υ_θ induces the desired map Ψ. Plainly, its formation commutes with base change.

Plainly, a map is universally injective if it is injective on geometric points. Furthermore, since Π is surjective, Proposition 5.4 also implies that Ψ too factors into a finite map followed by an open embedding. Now, a finite map is a closed embedding if its comorphism is surjective. Hence, to prove that Ψ is an embedding, it suffices to prove that its fibers over Y are embeddings. Now, forming Ψ commutes with extending Y. Therefore, we may assume Y is the spectrum of an algebraically closed field K.

To prove Ψ is universally injective, plainly we need only prove Ψ is injective on K-points. Since Π is surjective, every K-point of $F(\mathbf{D},\,\theta)\big/\mathrm{Aut}(\mathbf{D})$ is the image of a K-point of $F(\mathbf{D},\,\theta)$. Hence we need only observe that, if two K-points $\mathbf{t}'$ and $\mathbf{t}''$ of $F(\mathbf{D},\,\theta)$ have the same image in $\mathrm{Hilb}^d_{F/Y}(K)$ under Υ_θ, then the two differ by an automorphism γ of $\mathbf{D}$. But that image corresponds to a geometrically complete ideal $\mathcal{I}$ on F_K with diagram $\mathbf{D}$. In turn, as explained in Subsection 5.1, $\mathcal{I}$ determines a set $\mathbf{A}$ of 2-dimensional regular local K-domains whose fraction field is that of F_K, and $\mathbf{A}$ has a proximity structure, under which it is isomorphic to $\mathbf{D}$. Say $\mathbf{t}' \in F(\mathbf{A},\,\theta')$ and $\mathbf{t}'' \in F(\mathbf{A},\,\theta'')$. Then $\theta'^{-1} \circ \theta''$ induces the desired automorphism $\gamma \in \mathrm{Aut}(\mathbf{D})$.

By Corollary 4.5, $F(\mathbf{D},\,\theta)\big/\mathrm{Aut}(\mathbf{D})$ is smooth and irreducible. By Corollary 5.5, $H(\mathbf{D})$ is a locally closed subset of $\mathrm{Hilb}^d_{F/Y}$, so carries an induced reduced structure. And Ψ induces a bijective finite map $\beta\colon F(\mathbf{D},\,\theta)\big/\mathrm{Aut}(\mathbf{D}) \to H(\mathbf{D})$.

Suppose K is of characteristic 0. Then β is birational. If, perchance, $\mathbf{D}$ is minimal in the sense of [**16**, Section 2, p. 213], then $H(\mathbf{D})$ is smooth by the direct, alternative proof of [**16**, Prp. (3.6), p. 225]; hence, β is an isomorphism. In any

case, it follows from Proposition 3.3.14 on p. 70 of [**11**] that β is unramified; hence, β is an isomorphism. The proof is now complete. □

Corollary 5.8. *Fix an Enriques diagram* $\mathbf{D}$*, and set* $d := \deg \mathbf{D}$*. Assume the characteristic is* 0*. Then* $H(\mathbf{D}) \subset \mathrm{Hilb}^d_{F/Y}$ *supports a natural structure of* Y*-smooth subscheme with irreducible geometric fibers of dimension* $\dim(\mathbf{D})$*.*

PROOF. By Theorem 5.7, Υ_θ induces an embedding of $F(\mathbf{D}, \theta)/\mathrm{Aut}(\mathbf{D})$ into $\mathrm{Hilb}^d_{F/Y}$. By Proposition 5.4, the image is $H(\mathbf{D})$. And by Corollary 4.5, the source is Y-smooth, and has irreducible geometric fibers of dimension $\dim(\mathbf{D})$. □

Proposition 5.9. *Given positive integers* $r_1, \dots, r_k$*, let* $G(r_i) \subset \mathrm{Hilb}^{r_i}_{F/Y}$ *be the open subscheme over which the universal family is smooth, and let*

$$G(r_1, \dots, r_k) \subset G(r_1) \times_Y \cdots \times_Y G(r_k)$$

be the open subscheme over which, for $i \neq j$*, the fibers of the universal families over* $G(r_i)$ *and* $G(r_j)$ *have empty intersection. Set* $r := \sum r_i$*.*

Given distinct integers $m_1, \dots, m_k \geq 2$*, let* $\mathbf{D}$ *be the the weighted Enriques diagram with* r *vertices, each a root, and an ordering* θ *such that the first* r_1 *vertices are roots of weight* m_1*, the next* r_2 *are of weight* m_2*, and so on. Set* $d := \sum \binom{m_i+1}{2} r_i$*.*

Then $F(\mathbf{D}, \theta)$ *is equal to the complement in the relative direct product* $F^{\times_Y r}$ *of the* $\binom{r}{2}$ *large diagonals, and* $F(\mathbf{D}, \theta)/\mathrm{Aut}(\mathbf{D})$ *is equal to* $G(r_1, \dots, r_k)$*. Further,* Υ_θ *always induces an embedding*

$$\Psi \colon G(r_1, \dots, r_k) \hookrightarrow \mathrm{Hilb}^d_{F/Y};$$

on T*-points,* Ψ *acts by taking a* k*-tuple* $(W_1, \dots, W_k)$ *where* W_i *is a smooth length-*r_i *subscheme of* F_T*, say with ideal* $\mathcal{I}_i$*, to the length-*d *subscheme* W *with ideal* $\prod \mathcal{I}_i^{m_i}$*.*

PROOF. Let $(t_0, \dots, t_{r-1})$ be a strict sequence of arbitrarily near T-points of F/Y with diagram $(\mathbf{D}, \theta)$. Plainly, the t_i are just sections of F_T, and their images are disjoint. So $F(\mathbf{D}, \theta)$ is equal to the asserted complement.

Plainly, $\mathrm{Aut}(\mathbf{D})$ is the product of k groups, the ith being the full symmetric group on the r_i roots in the ith set. So the quotient $F(\mathbf{D}, \theta)/\mathrm{Aut}(\mathbf{D})$ is equal to the open subscheme of $\mathrm{Hilb}^{r_1}_{F/Y} \times_Y \cdots \times_Y \mathrm{Hilb}^{r_k}_{F/Y}$ whose geometric points parameterize the k-tuples whose ith component is an unordered set of r_i geometric points of F such that all r points are distinct; in other words, the quotient is equal to the asserted open subscheme.

Since each vertex is a root of some weight m_i, plainly Ψ acts on T-points in the asserted way, owing to the following standard general result, which is easily proved by descending induction: let A be a locally Noetherian scheme, $\mathcal{I}$ a regular ideal, $b \colon B \to A$ the blow-up of $\mathcal{I}$, and E the exceptional divisor; let $m \geq 0$ and set $\mathcal{L} := \mathcal{O}_B(-mE)$; then $R^q b_* \mathcal{L} = 0$ for $q \geq 1$ and $b_* \mathcal{L} = \mathcal{I}^m$.

Finally, to prove that Ψ is always an embedding, we may assume that Y is the spectrum of an algebraically closed field K, owing to the proof of Theorem 5.7. By the same token, Ψ is universally injective, and factors into a finite map followed by an open embedding. Hence, we need only show that Ψ is unramified.

Let v be a K-point of $\mathrm{Hilb}^r_{F/Y}$; let $V \subset F$ be the corresponding subscheme, and $\mathcal{I}$ its ideal. Recall the definition of the isomorphism from the tangent space at v to the normal space $\mathrm{Hom}(\mathcal{I}, \mathcal{O}_V)$; the definition runs as follows. Let $K[\epsilon]$ be the ring of dual numbers, and set $T := \mathrm{Spec}(K[\epsilon])$. An element of the tangent

space corresponds to a T-point of $\mathrm{Hilb}^r_{F/Y}$ supported at v; so it represents a T-flat subscheme $V_\epsilon \subset F_T$ that deforms V. The natural splitting $K[\epsilon] = K \oplus K\epsilon$ induces a splitting $\mathcal{O}_{V_\epsilon} = \mathcal{O}_V \oplus \mathcal{O}_V\epsilon$. Similarly, the ideal $\mathcal{I}_\epsilon$ of V_ϵ splits: $\mathcal{I}_\epsilon = \mathcal{I} \oplus \mathcal{I}\epsilon$. Then the natural map $\mathcal{O}_{F_T} \to \mathcal{O}_{V_\epsilon}$ restricts to a map $\mathcal{I} \to \mathcal{O}_V\epsilon$, which is equal to the desired map $\zeta\colon \mathcal{I} \to \mathcal{O}_V$.

Assume $v \in G(r_1, \dots, r_k)$. Then V is the union of k sets of reduced K-points of F. The ith set has r_i points; let $\mathcal{I}_i$ be the ideal of its union. Further, Ψ carries V and V_ϵ to the subschemes W and W_ϵ defined by $\mathcal{I}_1^{m_1} \cdots \mathcal{I}_k^{m_k}$ and $\mathcal{I}_{1,\epsilon}^{m_1} \cdots \mathcal{I}_{k,\epsilon}^{m_k}$. So Ψ is unramified at v if the induced map on tangent spaces is injective:

$$\psi\colon T_{G(r_1,\dots,r_k),v} \hookrightarrow \mathrm{Hom}(\mathcal{I}_1^{m_1} \cdots \mathcal{I}_k^{m_k}, \mathcal{O}_W).$$

Say $v = (v_1, \dots, v_k)$ with $v_i \in G(r_i)$, and say v_i represents $V_i \subset F$. Then

$$T_{G(r_1,\dots,r_k),v} = \bigoplus T_{\mathrm{Hilb}^{r_i}_{F/Y},v_i} = \bigoplus \mathrm{Hom}(\mathcal{I}_i, \mathcal{O}_{V_i}).$$

Given any $\zeta \in T_{G(r_1,\dots,r_k),v}$, its image $\psi(\zeta)$ is equal to the restriction of the canonical map $\mathcal{O}_{F_T} \to \mathcal{O}_{W_\epsilon}$. So ψ splits into a direct sum of local components

$$\psi_x\colon \mathrm{Hom}(\mathcal{I}_{i,x}, \mathcal{O}_{V,x}) \to \mathrm{Hom}(\mathcal{I}_{i,x}^{m_i}, \mathcal{O}_{W,x}) \quad \text{for } x \in V_i \text{ and } i = 1, \dots, k.$$

It remains to prove that each ψ_x is injective. Fix an x.

Set $\mathcal{I} := \mathcal{I}_i$ and $m := m_i$. Fix generators $\mu, \nu \in \mathcal{I}_x$. Set $a := \zeta_x\mu$ and $b := \zeta_x\nu$ in $\mathcal{O}_{V_i,x} = K$. Then $\mathcal{I}_{\epsilon,x}$ is generated by $\mu - a\epsilon$ and $\nu - b\epsilon$; so $\mathcal{I}^m_{\epsilon,x}$ is generated by

$$\mu^m - m\mu^{m-1}a\epsilon,\ \mu^{m-1}\nu - (m-1)\mu^{m-2}\nu a\epsilon - \mu^{m-1}b\epsilon,\ \dots,$$
$$\mu\nu^{m-1} - a\nu^{m-1}\epsilon - (m-1)b\mu\nu^{m-2}\epsilon,\ \nu^m - m\nu^{m-1}b\epsilon.$$

Hence, modulo $\mathcal{I}^m_{\epsilon,x}$, the generators $\mu^{m-1}\nu$ and $\mu\nu^{m-1}$ of $\mathcal{I}^m_x$ are congruent to $(m-1)\mu^{m-2}\nu a\epsilon + \mu^{m-1}b\epsilon$ and $a\nu^{m-1}\epsilon + (m-1)b\mu\nu^{m-2}\epsilon$. (They're equal if $m = 2$.)

Form the latter's classes in $\mathcal{O}_{W,x}$. Then, therefore, these classes are the images of those generators under the map $\psi_x\zeta_x$. Hence, in any characteristic, we can recover a and b from the images of $\mu^{m-1}\nu$ and $\mu\nu^{m-1}$. But a and b determine ζ_x. Thus ψ_x is injective, and the proof is complete. □

Appendix A. Generalized property of exchange

This appendix proves two lemmas of general interest, which we need. The first lemma generalizes the property of exchange to a triple (T, f, F) where T is a (locally Noetherian) scheme, $f\colon P \to Q$ is a proper map of T-schemes of finite type, and $\mathcal{F}$ is a T-flat coherent sheaf on P. The original treatment was made by Grothendieck and Dieudonné in [**8**, Sec. 7.7, pp. 65–72], and somewhat surprisingly, deals only with the case of $Q = T$. (Although they replace F by a complex of flat and coherent sheaves bounded below, this extension is minor and we do not need it.)

The first lemma is proved by generalizing the treatment in Section II, 5 of [**24**, pp. 46–55]. Alternatively, as Illusie pointed out in a private conversation, the lemma can be proved using the methods that he developed in [**14**].

The first lemma is used to prove the second. The second is used in the proof of Proposition 5.4, which constructs the map from the scheme of T-points with given Enriques diagram to the Hilbert scheme.

Lemma A.1 (Generalized property of exchange)**.** *Let T be a scheme, $f\colon P \to Q$ a proper map of T-schemes of finite type, and $\mathcal{F}$ a T-flat coherent sheaf on P. Let $q \in Q$ be a point, $t \in T$ its image, and $i \geq 0$ an integer. Assume that, on the fiber Q_t, the base-change map of sheaves*

$$\rho_t^i\colon (R^i f_*\mathcal{F})_t \to R^i f_{t*}\mathcal{F}_t$$

is surjective at q. Then there exists a neighborhood U of q in Q such that, for any T-scheme T', the base-change map of sheaves

$$\rho_{T'}^i\colon (R^i f_*\mathcal{F})_{T'} \to R^i f_{T'*}\mathcal{F}_{T'}$$

is bijective on the open subset $U_{T'}$ of $Q_{T'}$. Furthermore, the map ρ_t^{i-1} is also surjective at q if and only if sheaf $R^i f_*\mathcal{F}$ is T-flat at q.

PROOF. The question is local on Q; so we may assume that $T = \operatorname{Spec} A$ and $Q = \operatorname{Spec} B$ where A is a Noetherian ring and B is a finitely generated A-algebra. Also, we may assume that B is A-flat by expressing B as a quotient of a polynomial ring over A and then replacing B with that ring. For convenience, when given a B-module or a map of B-modules, let us say that it has a certain property at q to mean that it acquires this property on localizing at the prime corresponding to q.

There is a finite complex $K^\bullet$ of A-flat finitely generated B-modules, and on the category of A-algebras C, there is, for every $j \geq 0$, an isomorphism of functors

$$\mathrm{H}^j(K^\bullet \otimes_A C) \xrightarrow{\sim} \mathrm{H}^j(P \otimes_A C,\ \mathcal{F} \otimes_A C).$$

Indeed, this statement results, mutatis mutandis, from the proof of the theorem on page 46 of [**24**].

Let k be the residue field of t. Then there is a natural map of exact sequences

$$\begin{array}{ccccccc} K^{i-1}\otimes k & \to & \mathrm{Z}^i(K^\bullet)\otimes k & \to & \mathrm{H}^i(K^\bullet)\otimes k & \to & 0 \\ \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle z_k^i} & & \downarrow{\scriptstyle h_k^i} & & \\ K^{i-1}\otimes k & \to & \mathrm{Z}^i(K^\bullet\otimes k) & \to & \mathrm{H}^i(K^\bullet\otimes k) & \to & 0. \end{array} \tag{A.1.1}$$

Since ρ_k^i is surjective at q, so is h_k^i. Hence z_k^i is surjective at q.

Consider the following map of exact sequences:

$$\begin{array}{ccccccc} \mathrm{Z}^i(K^\bullet)\otimes k & \to & K^i\otimes k & \to & \mathrm{B}^{i+1}(K^\bullet)\otimes k & \to & 0 \\ \downarrow{\scriptstyle z_k^i} & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle b_k^{i+1}} & & \\ \mathrm{Z}^i(K^\bullet\otimes k) & \to & K^i\otimes k & \to & \mathrm{B}^{i+1}(K^\bullet\otimes k) & \to & 0. \end{array}$$

Now, z_k^i is surjective at q. Hence b_k^{i+1} is bijective at q.

Hence $\mathrm{B}^{i+1}(K^\bullet)\otimes k \to K^{i+1}\otimes k$ is injective at q. Set $L := K^{i+1}/\mathrm{B}^{i+1}(K^\bullet)$. Since K^{i+1} is A-flat, the local criterion of flatness implies that L is A-flat at q. Hence, by the openness of flatness, there is a $g \in B$ outside the prime corresponding to q such that the localization L_g is A-flat. We can replace B by B_g, and so assume L is A-flat.

Let C be any A-algebra. Then the following sequence is exact:

$$0 \to \mathrm{Z}^i(K^\bullet)\otimes C \to K^i\otimes C \to K^{i+1}\otimes C \to L\otimes C \to 0. \tag{A.1.2}$$

It follows that, in the map of exact sequences

$$\begin{array}{ccccccc} K^{i-1}\otimes C & \to & \mathrm{Z}^i(K^\bullet)\otimes C & \to & \mathrm{H}^i(K^\bullet)\otimes C & \to & 0 \\ \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle z^i_C} & & \downarrow{\scriptstyle h^i_C} & & \\ K^{i-1}\otimes C & \to & \mathrm{Z}^i(K^\bullet\otimes C) & \to & \mathrm{H}^i(K^\bullet\otimes C) & \to & 0, \end{array}$$

z^i_C is bijective. Hence h^i_C is bijective. Thus the first assertion holds: ρ^i_C is bijective.

If $\mathrm{H}^i(K^\bullet)$ is A-flat at q, then plainly the sequence

$$0 \to \mathrm{B}^i(K^\bullet)\otimes k \to \mathrm{Z}^i(K^\bullet)\otimes k \to \mathrm{H}^i(K^\bullet)\otimes k \to 0 \tag{A.1.3}$$

is exact. The converse holds too by the local criterion for flatness, because $\mathrm{Z}^i(K^\bullet)$ is A-flat owing to the exactness of (A.1.2) with $C := A$ and to the flatness of L.

Since z^i_k is bijective, (A.1.3) is exact if and only if b^i_k is injective. The latter holds if and only if z^{i-1}_k is surjective, owing to the map of exact sequences

$$\begin{array}{ccccccc} Z^{i-1}(K^\bullet)\otimes k & \to & K^{i-1}\otimes k & \to & \mathrm{B}^i(K^\bullet)\otimes k & \to & 0 \\ \downarrow{\scriptstyle z^{i-1}_k} & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle b^i_k} & & \\ 0 \to Z^{i-1}(K^\bullet\otimes k) & \to & K^{i-1}\otimes k & \to & \mathrm{B}^i(K^\bullet\otimes k) & \to & 0. \end{array}$$

Finally, z^{i-1}_k is surjective if and only if h^{i-1}_k is so, owing to (A.1.1) with $i-1$ in place of i. Putting it all together, we've proved that h^{i-1}_k is surjective if and only if $\mathrm{H}^i(K^\bullet)$ is A-flat at q. In other words, the second assertion holds too. □

Lemma A.2. *Let T be a scheme, $f\colon P \to Q$ a proper map of T-schemes of finite type, and*

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0 \tag{A.2.1}$$

a short exact sequence of T-flat coherent sheaves on P. For each point $t \in T$, let f_t and $\mathcal{F}_t$ and $\mathcal{G}_t$ denote the restrictions to the fiber P_t, and assume that

$$R^i f_{t*}(\mathcal{F}_t) = 0 \text{ and } R^i f_{t*}(\mathcal{G}_t) = 0 \text{ for } i \geq 1. \tag{A.2.2}$$

Then the induced sequence on Q,

$$0 \to f_*\mathcal{F} \to f_*\mathcal{G} \to f_*\mathcal{H} \to 0, \tag{A.2.3}$$

is a short exact sequence of T-flat coherent sheaves, and forming it commutes with base extension.

PROOF. Since $\mathcal{H}$ is T-flat, the sequence (A.2.1) remains exact after restriction to the fiber P_t for each $t \in T$, and so the restricted sequence induces a long exact sequence of cohomology. Hence, (A.2.2) yields

$$R^i f_{t*}(\mathcal{H}_t) = 0 \text{ for all } i \geq 1.$$

By hypothesis, $\mathcal{F}$, $\mathcal{G}$, $\mathcal{H}$ are T-flat. Hence, by the generalized property of exchange, Lemma A.1, the sheaves $f_*\mathcal{F}$, $f_*\mathcal{G}$, $f_*\mathcal{H}$ are T-flat, and forming them commutes with extending T. By the same token, $R^1 f_*(\mathcal{F}) = 0$; whence, Sequence (A.2.3) is exact. The assertion follows. □

Appendix B. A few examples by Ilya TYOMKIN

Let F be the affine plane over the spectrum $Y := \mathrm{Spec}(K)$ of an algebraically closed field K of positive characteristic p. In this appendix, we analyze a few simple examples of minimal Enriques diagrams $\mathbf{D}$. Some depend on p, and have an ordering θ for which the universally injective map of Theorem 5.7,

$$\Psi\colon F(\mathbf{D},\,\theta)/\mathrm{Aut}(\mathbf{D}) \to \mathrm{Hilb}^d_{F/Y},$$

is purely inseparable. Others are independent of p; they have several vertices, but only one root, yet they have an ordering θ for which Ψ is an embedding. In fact, in every case, θ is unique, and $\mathrm{Aut}(\mathbf{D})$ is trivial.

We take F to be the affine plane just to simplify the presentation. With little modification, everything works for any smooth irreducible surface F.

It is unknown what conditions on an arbitrary Enriques diagram $\mathbf{D}$ serve to guarantee here that Ψ is unramified, so an embedding. Nevertheless, in view of the analysis in this appendix, it is reasonable to make the following guess.

Guess B.1. *If $p > \frac{1}{2}\sum_{V\in\mathbf{D}} m_V$, then Ψ is unramified.*

This guess is sharp in the sense that, if $p \le \frac{1}{2}\sum_{V\in\mathbf{D}} m_V$, then Ψ may be ramified. For example, consider the plane curve $C : x_2^p = x_1^{p+1}$. In the notation of Definition B.2, the minimal diagram of C is $\mathbf{M}_{p,p}$. It has $p+1$ vertices with $m_V = p, 1, 1, \ldots, 1$. So $p = \frac{1}{2}\sum_{V\in\mathbf{D}} m_V$. And Ψ is ramified by Proposition B.4.

Similarly, consider $C : y(y - x^p) = 0$. Its minimal diagram has p vertices V with $m_V = 2$. So $p = \frac{1}{2}\sum_{V\in\mathbf{D}} m_V$. And Ψ is ramified by an argument similar to the proof of Proposition B.4.

On the other hand, if $\mathbf{D}$ has a single vertex of weight $2p$, then Ψ is unramified by Proposition 5.9, and of course, $p = \frac{1}{2}\sum_{V\in\mathbf{D}} m_V$.

In general, if a branch has tangency of order divisible by p to an exceptional divisor E, then the multiplicity of the root must be at least p and there must be at least p other vertices. So $p \le \frac{1}{2}\sum_{V\in\mathbf{D}} m_V$. Instead, if, at a point $P \in F$, all the branches have a tangency of order divisible by p to the same smooth curve D, then there must be at least p vertices V with $m_V \ge 2$. So again, $p \le \frac{1}{2}\sum_{V\in\mathbf{D}} m_V$. Thus, if we guess that Ψ can be ramified in only these two ways, then we arrive at Guess B.1.

Further, although Ψ does not sense first-order deformations either along E or along D, nevertheless after we add a transverse branch at P, then Ψ does sense first-order deformations of the new branch; thus Ψ becomes unramified. This intuition is developed into a rigorous proof for the ordinary tacnode in Proposition B.7, and a similar procedure works if the tacnode is replaced by an ordinary cusp.

Definition B.2. Fix $m \ge p$. Let $\mathbf{M}_{p,m}$ denote the minimal Enriques diagram of the plane curve singularity with $1 + m - p$ branches whose tangent lines are distinct, whose first branch is $\{\, x_2^p = x_1^{p+1} \,\}$, and whose remaining $m - p$ branches are smooth.

Example B.3. For motivation, consider the following special case. Take $p := 2$ and $m := 2$. Then $\mathbf{M}_{p,m}$ is the minimal Enriques diagram $\mathbf{A}_2$ of the cuspidal curve $C : x_2^2 = x_1^3$. This diagram has three vertices and a unique ordering θ.

Take $F := \mathbb{A}^2_K$ and $T := \mathrm{Spec}(K)$. In $F(\mathbf{A}_2, \theta) \subset F^{(2)}$, form the locus L of sequences (t_0, t_1, t_2) of arbitrarily near T-points of F/K such that t_0 is the constant

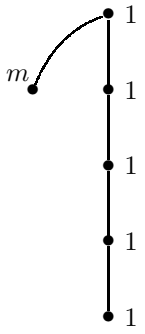

FIGURE 1. The Enriques diagram $\mathbf{M}_{m,p}$, with $m \geq p = 5$, of Definition B.2.

map from T to the origin. Plainly, the second projection induces an isomorphism $L \xrightarrow{\sim} E'_K$ where E'_K is the exceptional divisor of the blow up F'_K of F at the origin.

The strict transform C' of C is tangent to E'_K with order 2, and C' is given by the equation $s^2 = x_1$ where $s := x_2/x_1$. Notice that this equation is preserved by any first order deformation along E'_K of the point of contact; indeed,

$$(s + b\epsilon)^2 = s^2$$

as $p = 2$ and $\epsilon^2 = 0$. This observation suggests that the restriction of Ψ,

$$(\Psi|L)\colon L \to \mathrm{Hilb}^5_{F/K},$$

is purely inseparable; and indeed, $\Psi|L$ is so, as we check next.

Let $\mathbf{D}'$ be the diagram obtained from $\mathbf{A}_2$ by omitting the root, let θ' be the unique ordering of $\mathbf{D}'$, and consider the corresponding map

$$\Psi'\colon F'(\mathbf{D}',\, \theta') \to \mathrm{Hilb}^2_{F'_K/K}\,.$$

Plainly, the projection $(t_0, t_1, t_2) \mapsto (t_1, t_2)$ embeds L into $F'(\mathbf{D}',\, \theta')$.

So Ψ' induces a map $\Psi'_L\colon L \to \mathrm{Hilb}^2_{F'_K/K}$. It carries (t_0, t_1, t_2) to the subscheme of F'_T with ideal $\mathcal{I}'$ defined by the formula

$$\mathcal{I}' := \bigl(\varphi^{(2)}_T \varphi^{(3)}_T\bigr)_* \mathcal{O}_{F^{(3)}_T}\bigl(-E^{(2,3)}_T - E^{(3,3)}_T\bigr).$$

But $E^{(2,3)}_T + E^{(3,3)}_T \leq E^{(1,3)}_T$. So

$$\mathcal{O}_{F^{(3)}_T}\bigl(-E^{(2,3)}_T - E^{(3,3)}_T\bigr) \supseteq \mathcal{O}_{F^{(3)}_T}\bigl(-E^{(1,3)}_T\bigr).$$

Hence $\mathcal{I}'$ contains the ideal of E'_T. Therefore, Ψ'_L factors through $\mathrm{Hilb}^2_{E'_K/K}$, which is isomorphic to $\mathrm{Sym}^2(L)$. The corresponding map $L \to \mathrm{Sym}^2(L)$ is the diagonal map since $\Psi'_L(t_0, t_1, t_2)$ has the same support as t_1. This diagonal map is purely inseparable as $p = 2$.

Finally, $\Psi'_L\colon L \to \mathrm{Hilb}^2_{E'_K/K}$ is a factor of $\Psi|L$ because $\Psi(t_0, t_1, t_2)$ is the subscheme of F_T with ideal $\bigl(\varphi^{(1)}_T\bigr)_* \mathcal{I}'(2E'_T)$. Thus $\Psi|L$ is, indeed, purely inseparable. In fact, Ψ is purely inseparable by Proposition B.4 below.

Proposition B.4. *Fix $m \geq p$. Set $\mathbf{D} := \mathbf{M}_{p,m}$ and $d := \binom{m+1}{2} + p$. Then $\mathbf{D}$ has a unique ordering θ; also $\mathrm{Aut}(\mathbf{D}) = 1$ and $\deg \mathbf{D} = d$. Take $F = \mathbb{A}^2_K$. Then $\dim F(\mathbf{D}, \theta) = 3$, and $\Upsilon_\theta \colon F(\mathbf{D}, \theta) \to \mathrm{Hilb}^d_{F/Y}$ is purely inseparable; also, $\Psi = \Upsilon_\theta$.*

PROOF. Plainly, $\mathbf{D}$ has $p+1$ vertices, say $V_0, \dots, V_p$ ordered by succession. Then proximity is given by $V_k \succ V_{k-1}$ and $V_k \succ V_0$ for $k > 0$. Further, the weights are given by $m_{V_0} = m$ and $m_{V_k} = 1$ for $k > 0$. Set $\theta(V_k) := k$; plainly, θ is an ordering of $\mathbf{D}$, and is the only one. Also, plainly, $\mathrm{Aut}(\mathbf{D}) = 1$ and $\deg \mathbf{D} = d$.

Theorem 3.10 says that $\dim F(\mathbf{D}, \theta) = \dim \mathbf{D}$, but plainly $\dim \mathbf{D} = 3$. Now, $\Psi = \Upsilon_\theta$ because $\mathrm{Aut}(\mathbf{D}) = 1$. Further, Theorem 5.7 says that Ψ is universally injective. Hence Ψ is purely inseparable, because it is everywhere ramified owing to the following lemma. □

Lemma B.5. *Under the conditions of Proposition* B.4*, let $t \in F(\mathbf{D}, \theta)$ be a K-point. Then $\mathrm{Ker}(d_t \Upsilon_\theta)$ is of dimension* 1*.*

PROOF. Say t represents the sequence $(t_0, \dots, t_p)$ of arbitrarily near K-points of F/Y. Choose coordinates x_1, x_2 on F such that $t_0 : x_1 = x_2 = 0$ and such that t_1 is the point of intersection of the exceptional divisor E_0 with the proper transform of the x_1-axis. Set $s_0 := x_2/x_1$, set $s_1 := x_1/s_0$, and set $s_k := s_{k-1}/s_0$ for $2 \leq k \leq p-1$. Then $t_1 : s_0 = x_1 = 0$, and $t_k : s_0 = s_{k-1} = 0$ for $2 \leq k \leq p$.

Set $z := \Upsilon_\theta(t) \in \mathrm{Hilb}^d_{F/Y}(K)$. Let Z denote the corresponding subscheme, and $\mathcal{I}$ its ideal. Recall from the proof of Proposition 5.4 that $\mathcal{I} = \varphi_{K*}\mathcal{O}(-E_K)$ where $E_K = \sum_{i=0}^p m_{V_i} E^{(i+1,\,p+1)}$. Recall from the proof of Proposition B.4 that $m_{V_0} = m$ and $m_{V_k} = 1$ for $k > 0$ and that $V_k \succ V_0$ for $k > 0$. It follows that

$$E_K = m e_K^{(1,\,p+1)} E^{(1)} + \sum_{k=1}^{p} k(m+1) e_K^{(k+1,\,p+1)} E_K^{(k+1)}.$$

Set $\delta(r) := 0$ if $0 \leq r < p$ and $\delta(r) := 1$ if $p \leq r \leq m$. Set

$$f_r := x_1^{m+1-r-\delta(r)} x_2^r \quad \text{for } 0 \leq r \leq m.$$

Let's now show that the f_r generate $\mathcal{I}$.

First, note that, for each r and for $1 \leq k \leq p-1$,

$$f_r = x_1^{m+1-\delta(r)} s_0^r = s_k^{m+1-\delta(r)} s_0^{k(m+1-\delta(r))+r}.$$

Hence, the pullback of f_r vanishes along $e_K^{(1,\,p+1)} E_K^{(1)}$ to order at least m, and along $e_K^{(k+1,\,p+1)} E_K^{(k+1)}$ to order at least $k(m+1)$ for $k \geq 1$, since $r - k\delta(r) \geq 0$. Thus $f_r \in \mathcal{I}$ for each r.

Let $\mathcal{J}$ be the ideal generated by the f_r. Then $\mathcal{J} \subset \mathcal{I}$. Now, $K[x_1, x_2]/\mathcal{J}$ is spanned as a K-vector space by the monomials $x_1^{m+1-r-\delta(r)} x_2^l$ for $0 \leq l < r \leq m$ and by $x_1^{m+1-p} x_2^l$ for $0 \leq l < p$. Hence $\mathcal{J} = \mathcal{I}$ because

$$\dim\bigl(K[x_1, x_2]/\mathcal{J}\bigr) \leq \textstyle\sum_{r=0}^m r + p = d = \dim K[x_1, x_2]/\mathcal{I}.$$

Let $K[\epsilon]$ be the ring of dual numbers, and set $T := \mathrm{Spec}(K[\epsilon])$. Let $(t'_0, \dots, t'_p)$ be a strict sequence of arbitrarily near T-points of F/Y lifting $(t_0, \dots, t_p)$. Then there are $a_1, a_2, b \in K$ so that, after setting $x'_1 := x_1 + a_1\epsilon$ and $x'_2 := x_2 + a_2\epsilon$ and setting $s'_0 := x'_2/x'_1 + b\epsilon$ and $s'_1 := x'_1/s'_0$ and $s'_k := s'_{k-1}/s'_0$ for $2 \leq k \leq p-1$, we have $t'_0 : x'_1 = x'_2 = 0$ and $t'_1 : s'_0 = x'_1 = 0$ and $t'_k : s'_0 = s'_{k-1} = 0$ for $2 \leq k \leq p$.

Let $t' \in F(\mathbf{D}, \theta)(T)$ represent $(t'_0, \dots, t'_p)$. Set $z' := \Upsilon_\theta(t') \in \mathrm{Hilb}^d_{F/Y}(K)(T)$.

Let Z' denote the corresponding subscheme, and $\mathcal{I}'$ its ideal. Let's show that $\mathcal{I}'$ is generated by the following elements:

$$f'_r := (x'_1)^{m+1-r-\delta(r)}(x'_2)^r \quad \text{for } 0 \le r \le m.$$

The f'_r reduce to the f_r, which generate $\mathcal{I}$. Further, $\mathcal{I}'$ reduces to $\mathcal{I}$ as Z' is flat over $K[\epsilon]$. Hence it suffices to prove that $\mathcal{I}'$ contains the f'_r.

Note that $(s'_0 - b\epsilon)^p = (s'_0)^p$ as the characteristic is p. Hence, for each r,

$$f'_r = (x'_1)^{m+1-\delta(r)}(s'_0 - b\epsilon)^r = (s'_k)^{m+1-\delta(r)}(s'_0)^{k(m+1)+(p-k)\delta(r)}(s'_0 - b\epsilon)^{r-p\delta(r)}$$

for $1 \le k \le p-1$. Therefore, the pullback of f'_r vanishes along $e_T^{(1,\,p+1)}E_T^{(1)}$ to order at least m, and along $e_T^{(k+1,\,p+1)}E_T^{(k+1)}$ to order at least $k(m+1)$ for $k \ge 1$ since $(p-k)\delta(r) \ge 0$ and $r - p\delta(r) \ge 0$. Thus $\mathcal{I}'$ contains the f'_r.

Recall that $T_z\mathrm{Hilb}^d_{F/Y}(K) = \mathrm{Hom}(\mathcal{I}, \mathcal{O}_Z)$. Furthermore, it follows from the computations above that

$$d_t\Upsilon_\theta(t')(f'_r) = (m+1-r-\delta(r))x_1^{m-r-\delta(r)}x_2^r a_1 + r x_1^{m+1-r-\delta(r)}x_2^{r-1}a_2$$

for $0 \le r \le m$. Therefore,

$$\ker(d_t\Upsilon_\theta) = \big\{(a_1, a_2, b) \;\big|\; a_1 = a_2 = 0\big\},$$

and we are done. □

Definition B.6. Fix $m \ge 3$. Let $\mathbf{N}_m$ denote the minimal Enriques diagram of the following plane curve singularity: an ordinary tacnode $\{\, x_2(x_2 - x_1^2) = 0 \,\}$ union with $m-2$ smooth branches whose tangent lines are distinct and different from the common tangent line of the two branches of the tacnode.

Proposition B.7. *Fix $m \ge 3$. Set $\mathbf{D} := \mathbf{N}_m$ and $d := \binom{m+1}{2} + 3$. Then $\mathbf{D}$ has a unique ordering θ; also $\mathrm{Aut}(\mathbf{D}) = 1$ and $\deg \mathbf{D} = d$. Take $F = \mathbb{A}^2_K$. Then $\dim F(\mathbf{D}, \theta) = 3$, and $\Upsilon_\theta \colon F(\mathbf{D}, \theta) \to \mathrm{Hilb}^d_{F/K}$ is an embedding; also, $\Psi = \Upsilon_\theta$.*

PROOF. Plainly, $\mathbf{D}$ has 2 vertices, say V_0 and V_1 ordered by succession. Then proximity is given by $V_1 \succ V_0$. Further, the weights are given by $m_{V_0} = m$ and $m_{V_1} = 2$. Set $\theta(V_k) := k$; plainly, θ is an ordering of $\mathbf{D}$, and is the only one. Also, plainly, $\mathrm{Aut}(\mathbf{D}) = 1$ and $\deg \mathbf{D} = d$. Theorem 3.10 says that $\dim F(\mathbf{D}, \theta) = \dim \mathbf{D}$, but plainly $\dim \mathbf{D} = 3$. Now, $\Psi = \Upsilon_\theta$ because $\mathrm{Aut}(\mathbf{D}) = 1$. Further, Theorem 5.7 says that Ψ is universally injective. Hence Ψ is an embedding because it is nowhere ramified owing to the following lemma. □

Lemma B.8. *Under the conditions of Proposition* B.7*, let $t \in F(\mathbf{D}, \theta)$ be a K-point. Then* $\mathrm{Ker}(d_t\Upsilon_\theta) = 0$.

PROOF. Say t represents the sequence (t_0, t_1) of arbitrarily near K-points of F/Y. Choose coordinates $x_1,\ x_2$ on F such that $t_0 : x_1 = x_2 = 0$ and such that t_1 is the point of intersection of the exceptional divisor E_0 with the proper transform of the x_1-axis. Set $s := x_2/x_1$. Then $t_1 : s = x_1 = 0$.

Set $z := \Upsilon_\theta(t) \in \mathrm{Hilb}^d_{F/Y}(K)$. Let Z denote the corresponding subscheme, and $\mathcal{I}$ its ideal. Recall from the proof of Proposition 5.4 that $\mathcal{I} = \varphi_{K*}\mathcal{O}(-E_K)$ where $E_K = \sum_{i=0}^{1} m_{V_i}E^{(i+1,\,2)}$. Recall from the proof of Proposition B.7 that $m_{V_0} = m$ and $m_{V_1} = 2$ and that $V_1 \succ V_0$. It follows that

$$E_K = m e_K^{(1,\,2)}E^{(1)} + (m+2)E_K^{(2)}.$$

Set $\delta(0) := 2$, set $\delta(1) := 1$, and set $\delta(r) := 0$ if $r \geq 2$. Set

$$f_r := x_1^{m-r+\delta(r)} x_2^r \quad \text{for } 0 \leq r \leq m.$$

Let's now show that the f_r generate $\mathcal{I}$.

First, note that, for each r,

$$f_r = x_1^{m+\delta(r)} s^r.$$

Hence, the pullback of f_r vanishes along $e_K^{(1,2)} E_K^{(1)}$ to order at least m, and along $E_K^{(2)}$ to order at least $m+2$, since $m + r + \delta(r) \geq m + 2$. Thus $f_r \in \mathcal{I}$ for each r.

Let $\mathcal{J}$ be the ideal generated by the f_r. Then $\mathcal{J} \subset \mathcal{I}$. Now, $K[x_1, x_2]/\mathcal{J}$ is spanned as a K-vector space by the monomials $x_1^{m-r+\delta(r)} x_2^l$ for $0 \leq l < r \leq m$ and by x_1^{m-1}, $x_1^{m-1}x_2$, and x_1^{m+1}. Hence $\mathcal{J} = \mathcal{I}$ because

$$\dim\bigl(K[x_1, x_2]/\mathcal{J}\bigr) \leq \textstyle\sum_{r=0}^{m} r + 3 = d = \dim K[x_1, x_2]/\mathcal{I}.$$

Furthermore, the monomials x_1^{m-1} and $x_1^{m-1}x_2$ and x_1^{m+1}, and $x_1^{m-r+\delta(r)}x_2^l$ for $0 \leq l < r \leq m$ form a basis of the K-vector space $K[x_1, x_2]/\mathcal{I}$.

Let $K[\epsilon]$ be the ring of dual numbers, and set $T := \operatorname{Spec}(K[\epsilon])$. Let (t_0', t_1') be a strict sequence of arbitrarily near T-points of F/Y lifting (t_0, t_1). Then there are $a_1, a_2, b \in K$ so that, after setting $x_1' := x_1 + a_1\epsilon$ and $x_2' := x_2 + a_2\epsilon$ and $s' := x_2'/x_1' + b\epsilon$, we have $t_0' : x_1' = x_2' = 0$ and $t_1' : s' = x_1' = 0$.

Let $t' \in F(\mathbf{D}, \theta)(T)$ represent (t_0', t_1'). Set $z' := \Upsilon_\theta(t') \in \operatorname{Hilb}^d_{F/Y}(T)$. Let Z' denote the corresponding subscheme, and $\mathcal{I}'$ its ideal. Let's show that $\mathcal{I}'$ is generated by the following elements:

$$f_r' := (x_1')^{m-r+\delta(r)}(x_2')^r + rb\epsilon(x_1')^{m-r+1+\delta(r)}(x_2')^{r-1} \quad \text{for } 0 \leq r \leq m.$$

The f_r' reduce to the f_r, which generate $\mathcal{I}$. Further, $\mathcal{I}'$ reduces to $\mathcal{I}$ as Z' is flat over $K[\epsilon]$. Hence it suffices to prove that $\mathcal{I}'$ contains the f_r'.

The equation $x_2'/x_1' = s' - b\epsilon$ yields

$$f_r' = (x_1')^{m+\delta(r)}(s')^r \quad \text{for } 0 \leq r \leq m.$$

Hence, the pullback of f_r' vanishes along $e_T^{(1,2)} E_T^{(1)}$ to order at least m, and along $E_T^{(2)}$ to order at least $m+2$ since $m + r + \delta(r) \geq m + 2$. Thus $\mathcal{I}'$ contains the f_r'.

Recall that $T_z\operatorname{Hilb}^d_{F/Y}(K) = \operatorname{Hom}(\mathcal{I}, \mathcal{O}_Z)$. Furthermore, it follows from the computations above that

$$d_t\Upsilon_\theta(t')(f_r') = x_1^{m-r+\delta(r)-1} x_2^{r-1} \bigl((m - r + \delta(r))x_2a_1 + rx_1a_2 + rx_1^2b\bigr)$$

for $0 \leq r \leq m$. In particular, $rx_1^{m-2}x_2^2b \in \mathcal{I}$ yields

$$\begin{aligned} d_t\Upsilon_\theta(t')(f_1') &= mx_1^{m-1}x_2a_1 + x_1^m a_2 + x_1^{m+1}b \\ d_t\Upsilon_\theta(t')(f_0') &= (m+2)x_1^{m+1}a_1, \quad \text{and} \\ d_t\Upsilon_\theta(t')(f_3') &= (m-3)x_1^{m-4}x_2^3a_1 + 3x_1^{m-3}x_2^2a_2. \end{aligned}$$

Recall that, in $K[x_1, x_2]/\mathcal{I}$, the monomials

$$x_1^{m-1},\ x_1^{m-1}x_2,\ x_1^{m+1},\ \text{and } x_1^{m-r+\delta(r)}x_2^l \text{ for } 0 \leq l < r \leq m$$

are linearly independent. But $m \geq 3$, so at least one of the coefficients m, $m+2$, and $m-3$ is prime to the characteristic. Thus, $\ker(d_t\Upsilon_\theta) = 0$, and we are done. □

References

[1] Casas, E., *Infinitely near imposed singularities and singularities of polar curves,* Math. Ann. **287** (1990), 429–454. 5.1

[2] Casas, E., "Singularities of plane curves," London Math Society Lecture Note Series **276**, Cambridge Univ. Press, 2000. 2, 2.2, 2.7

[3] Deligne, P. *Exp. X, Intersections sur les surfaces régulières,* in "SGA 7 II, Groupes de Monodromie en géométrie algébrique," Lecture Notes in Math. **340**, Springer-Verlag, 1973, pp. 1–38. 5.1, 5, 5

[4] Enriques, F., and Chisini, O., "Lezioni sulla teoria geometrica delle equazioni e delle funzioni algebriche," Zanichelli, Bologna, 1915. 2, 5.1

[5] Göttsche, L. *A conjectural generating function for numbers of curves on surfaces,* Comm. Math. Phys. **196** (1998), 523–533. 1

[6] Greuel, G.-M., Lossen, C., and Shustin, E., *Plane curves of minimal degree with prescribed singularities*, Invent. Math. **133** (1998), 539–580. 2, 2.7

[7] Grothendieck, A., and Dieudonné, J. "Eléments de Géométrie Algébrique II," Publ. Math. IHES, Vol. 8, 1961. 5

[8] Grothendieck, A., and Dieudonné, J. "Eléments de Géométrie Algébrique III-2," Publ. Math. IHES, Vol. 17, 1963. 4, A

[9] Grothendieck, A., and Dieudonné, J. "Eléments de Géométrie Algébrique IV-3," Publ. Math. IHES, Vol. 28, 1966. 5

[10] Grothendieck, A., and Dieudonné, J. "Eléments de Géométrie Algébrique IV-4," Publ. Math. IHES, Vol. 32, 1967. 5

[11] Gustavsen, T. S., "Topics in deformation and moduli theory for singularities on curves and surfaces," Thesis, U. Oslo, 1998. 5

[12] Harbourne, B., *Iterated blow-ups and moduli for rational surfaces*, in "Algebraic Geometry, Sundance 1986," A. Holme, R. Speiser (Eds.), Lecture Notes in Math **1311** (1988), 101–117. 1, 3, 3

[13] Hoskin, M. A., *Zero-dimensional valuation ideals associated with plane curve branches,* Proc. London Math. Soc. **6** (1956), 70–99. 5.1

[14] Illusie, L., *Conditions de finitude relative,* Exposé III in SGA 6, "Théorie des intersections et théorème de Riemann–Roch," Lecture Notes in Math **225** (1971), 222–273. A

[15] Kleiman, S. L., *Multiple-point formulas I: Iteration,* Acta Math **147** (1981), 13–49. 3

[16] Kleiman, S., and Piene, R., *Enumerating singular curves on surfaces*, in "Algebraic geometry — Hirzebruch 70," Cont. Math. **241** (1999), 209–38 (corrections and revision in math.AG/9903192). 1, 2, 2.1, 2.6, 2.7, 5

[17] Kleiman, S., and Piene, R., *Node polynomials for families: methods and applications,* Math. Nachr. **271** (2004), 1–22. 1, 2, 2.1, 2.7

[18] Kleiman, S., and Piene, R., *Node polynomials for curves on surfaces*, in preparation. 2

[19] Kool, M., Shende, V., and Thomas, R. P. *A short proof of the Gttsche conjecture,* arXiv:1010.3211. 1

[20] Lipman, J., *Rational singularities, with applications to algebraic surfaces and unique factorization*, Publ. Math. IHES, **36** (1969), 195–279. 5.1

[21] Lipman, J., *Proximity inequalities for complete ideals in two-dimensional regular local rings*, Cont. Math, **159** (1994), 293–306. 2.2, 5.1

[22] Liu, A.-K., *Family blowup formula, admissible graphs and the enumeration of singular curves. I*, J. Differential Geom. **56** (2000), no. 3, 381–579. 1

[23] Lossen, C.: "The geometry of equisingular and equianalytical families of curves on a surface," Dr. dissertation, Universität Kaiserslautern, 1998. 1, 5.6

[24] Mumford, D., "Abelian varieties," Tata Institute Studies in Math., Oxford U. Press, 1966. A, A

[25] Nobile, A., and Villamayor, O. E., *Equisingular stratifications associated to families of planar ideals*, J. Alg. **193** (1997), 239–59. 1, 5.1, 5.6

[26] Roé, J. *On the existence of plane curves with imposed multiple points*, J. Pure Appl. Algebra **156** (2001), no. 1, 115–126.

[27] Roé, J. *Varieties of clusters and Enriques diagrams,* Math. Proc. Cambridge Philos. Soc. **137** (2004), no. 1, 69–94. 1, 2, 2.2, 3, 3, 4

[28] Russell, H. *Counting singular plane curves via Hilbert schemes,* Adv. Math. 179 (2003), no. 1, 38–58. 1
[29] Russell, H. *Alignment correspondences*, J. Algebra **278** (2004), no. 2, 553–570. 1
[30] Tzeng, Y. *A proof of the Göttsche–Yau–Zaslow formula*, arXiv:1009.5371. 1
[31] Vainsencher, I. *Enumeration of n-fold tangent hyperplanes to a surface*, J. Alg. Geom. **4** (1995), 503–26. 1
[32] Yau, S.-T., and Zaslow, E. *BPS states, string duality, and nodal curves on K3*, Nucl. Phys. B**471**, (1996), 503–512. 1
[33] Zariski, O., and Samuel, P. "Commutative algebra. Vol. II," D. Van Nostrand Co., Inc., 1960.

5.1

DEPARTMENT OF MATHEMATICS, MIT, 77 MASSACHUSETTS AVENUE, CAMBRIDGE, MA 02139, USA
E-mail address: kleiman@math.mit.edu

CMA AND DEPARTMENT OF MATHEMATICS, UNIVERSITY OF OSLO, PO BOX 1053, BLINDERN, NO-0316 OSLO, NORWAY
E-mail address: ragnip@math.uio.no

DEPARTMENT OF MATHEMATICS, BEN GURION UNIVERSITY OF THE NEGEV, P.O.B. 653, BE'ER SHEVA 84105, ISRAEL
E-mail address: tyomkin@cs.bgu.ac.il