14.03 Fall 2004
Problem Set 2 Solutions

October 2, 2004

1 Indirect utility function and expenditure function

Let $U = x^{\frac{1}{3}}y^{\frac{2}{3}}$ be the utility function where x and y are two goods. Denote p_x and p_y as respectively the prices of the two goods x and y, and where M as the income of the consumer.

1. Derive the indirect utility function $V(p_x, p_y, M)$

   The primal problem is:

$$\begin{aligned} \max\ U &= x^{\frac{1}{3}}y^{\frac{2}{3}} \\ s.t.\ p_x x + p_y y &\leq M \end{aligned}$$

   The lagrangian for this problem is:

$$L = x^{\frac{1}{3}}y^{\frac{2}{3}} - \lambda\left(p_x x + p_y y - M\right)$$

   First order conditions:

$$\begin{aligned} \frac{\partial L}{\partial x} &= \frac{1}{3}\left(\frac{y}{x}\right)^{\frac{2}{3}} - \lambda p_x = 0 \\ \frac{\partial L}{\partial y} &= \frac{2}{3}\left(\frac{x}{y}\right)^{\frac{1}{3}} - \lambda p_y = 0 \\ \frac{\partial L}{\partial \lambda} &= M - p_x x - p_y y = 0 \end{aligned}$$

$$
\begin{aligned}
x^*(p_x, p_y, M) &= \frac{1}{3}\frac{M}{p_x} \\
y^*(p_x, p_y, M) &= \frac{2}{3}\frac{M}{p_y} \\
V(p_x, p_y, M) &= \left(\frac{1}{3p_x}\right)^{\frac{1}{3}}\left(\frac{2}{3p_y}\right)^{\frac{2}{3}} M
\end{aligned}
$$

2. Derive the expenditure function $E(p_x, p_y, U)$

   The dual problem is:

$$
\begin{aligned}
\min\ M &= p_x x + p_y y \\
s.t.\ x^{\frac{1}{3}} y^{\frac{2}{3}} &\geq U_0
\end{aligned}
$$

   The lagrangian for this problem is:

$$
L = p_x x + p_y y - \lambda^D\left(x^{\frac{1}{3}} y^{\frac{2}{3}} - U_0\right)
$$

   First order conditions:

$$
\begin{aligned}
\frac{\partial L}{\partial x} &= p_x - \lambda^D \frac{1}{3}\left(\frac{y}{x}\right)^{\frac{2}{3}} = 0 \\
\frac{\partial L}{\partial y} &= p_y - \lambda^D \frac{2}{3}\left(\frac{x}{y}\right)^{\frac{1}{3}} = 0 \\
\frac{\partial L}{\partial \lambda} &= U_0 - x^{\frac{1}{3}} y^{\frac{2}{3}} = 0
\end{aligned}
$$

$$
\begin{aligned}
x^*(p_x, p_y, U_0) &= \left(\frac{p_y}{2p_x}\right)^{\frac{2}{3}} U_0 \\
y^*(p_x, p_y, M) &= \left(\frac{2p_x}{p_y}\right)^{\frac{1}{3}} U_0 \\
E(p_x, p_y, U_0) &= U_0\left(\frac{p_y}{2/3}\right)^{\frac{2}{3}}\left(\frac{p_x}{1/3}\right)^{\frac{1}{3}}
\end{aligned}
$$

3. Let $p_x = 2$, $p_y = 3$, and $M = 200$. Find the utility maximizing bundle at those prices and income.

   We just need to replace these values in the optimal values calculated in part 1.

$$
\begin{aligned}
x^* &= \frac{1}{3}\frac{200}{2} = \frac{100}{3} \\
y^* &= \frac{2}{3}\frac{200}{3} = \frac{400}{9}
\end{aligned}
$$

2 Utility maximization

A consumer has the following utility function:

$$U(x_1, x_2) = (x_1 + 2)^{\frac{1}{2}} x_2^{\frac{1}{2}}$$

find the optimal consumption of x_1 and x_2 given prices $p_1 = 6$, $p_2 = 1$ and income $I = 10$.[Hint: This problem does not have a standard 'interior' solution.]

As you were told in the hint this problem has a corner solution, so we set up the problem using non-negativity constraints along with the usual budget constraint.

$$\begin{aligned}
\max\ U &= (x_1 + 2)^{\frac{1}{2}} x_2^{\frac{1}{2}} \\
s.t.\ 6x_1 + x_2 &\leq 10 \\
x_1 &\geq 0 \\
x_2 &\geq 0
\end{aligned}$$

The lagrangian for this problem is:

$$L = (x_1 + 2)^{\frac{1}{2}} x_2^{\frac{1}{2}} + \lambda (10 - 6x_1 - x_2) + \phi_1 x_1 + \phi_2 x_2$$

Kuhn-Tucker conditions allow us to solve this kind of optimization problems with inequality constraints. We need to find first order conditions and complementary slackness conditions.

First order conditions are the same as we have seen in other problems:

$$\begin{aligned}
\frac{\partial L}{\partial x_1} &= \frac{1}{2}\left(\frac{x_2}{x_1 + 2}\right)^{\frac{1}{2}} - 6\lambda + \phi_1 = 0 \\
\frac{\partial L}{\partial x_2} &= \frac{1}{2}\left(\frac{x_1 + 2}{x_2}\right)^{\frac{1}{2}} - \lambda + \phi_2 = 0
\end{aligned}$$

Complementary slackness conditions take the form:

$$\begin{aligned}
\lambda (10 - 6x_1 - x_2) &= 0 \\
\phi_1 x_1 &= 0 \\
\phi_2 x_2 &= 0 \\
\lambda &\geq 0, \phi_1 \geq 0, \phi_2 \geq 0
\end{aligned}$$

which has a simple interpretation. The product $\lambda (10 - 6x_1 - x_2)$ is zero: so either $\lambda = 0$ and $6x_1 + x_2 < 10$ or $\lambda > 0$ and $6x_1 + x_2 = 10$.

Remember the interpretation of the multiplier: if the constraint is binding, then we are consuming the entire income and the shadow value of income is positive. If we are not consuming the entire income ($6x_1 + x_2 < 10$), then relaxing the budget constraint has no value in terms of utility because the consumer is not constrained.

Once you have written down all the Kuhn-Tucker conditions then you have to check case by case. You need to do this cleverly in order to avoid time-consuming calculations. For instance in this case we know that, since preferences exhibit non-satiation, the optimal bundle will be on the budget line, i.e. the budget contraint will be binding, therefore $\lambda > 0$ and $10 = 6x_1 + x_2$.

We also know that the solution is not interior, otherwise we would have both $\phi_1 = 0$ and $\phi_2 = 0$. Then the FOC give us

$$\frac{1}{2}\left(\frac{x_2}{x_1+2}\right)^{\frac{1}{2}} = 6\lambda = 6 \cdot \frac{1}{2}\left(\frac{x_2}{x_1+2}\right)^{-\frac{1}{2}} \tag{1}$$

$$\frac{x_2}{x_1+2} = 6 \tag{2}$$

With the budget constraint, this implies that $x_1 = -\frac{1}{6}$, which obviously violates non-negativity.

So we investigate two cases:

1. $\phi_1 = 0,\ x_1 > 0$ and $\phi_2 > 0,\ x_2 = 0$
2. $\phi_1 > 0,\ x_1 = 0$ and $\phi_2 = 0,\ x_2 > 0$

Let's see them in detail:

1.

$$\begin{aligned}
\frac{1}{2}\left(\frac{x_2}{x_1+2}\right)^{\frac{1}{2}} - 6\lambda &= 0 \\
\frac{1}{2}\left(\frac{x_1+2}{x_2}\right)^{\frac{1}{2}} - \lambda + \phi_2 &= 0 \\
x_2 &= 0 \\
10 &= 6x_1 + x_2
\end{aligned}$$

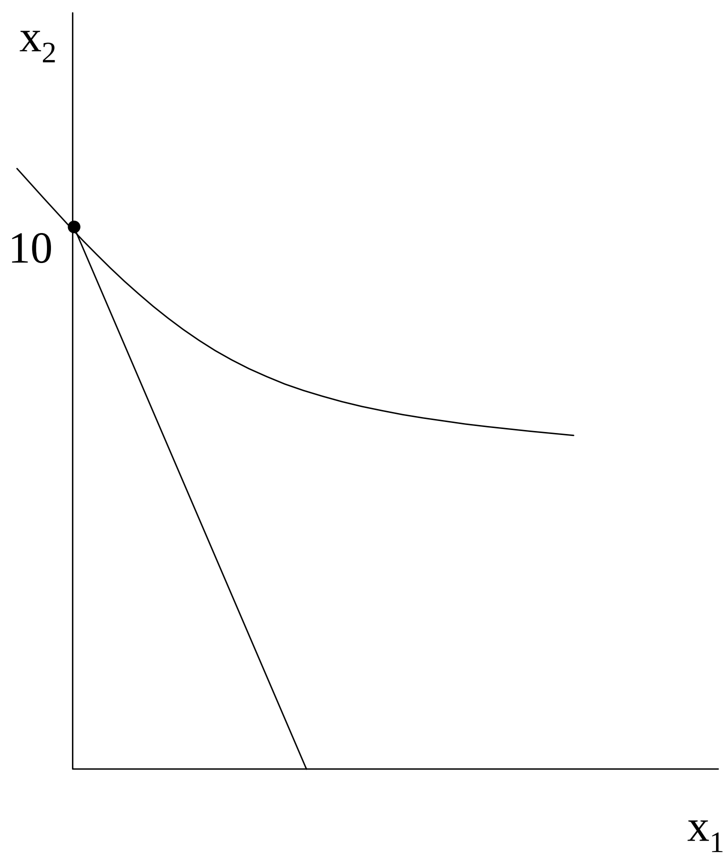
You can verify that replacing $x_2 = 0$ yields $\lambda = 0$ in the first equation and we already know this is not the case as long as the consumer would be strictly better off with more income. Alternatively, we can note that $U(x_1, 0) = 0$ for any x_1,and the consumer can clearly do better than 0 utility in this problem. So we dismiss this case.

2.

$$\begin{aligned}
\frac{1}{2}\left(\frac{x_2}{x_1+2}\right)^{\frac{1}{2}} - 6\lambda + \phi_1 &= 0 \\
\frac{1}{2}\left(\frac{x_1+2}{x_2}\right)^{\frac{1}{2}} - \lambda &= 0 \\
x_1 &= 0 \\
10 &= 6x_1 + x_2
\end{aligned}$$

You can verify that in this case $x_2 = 10$, $\lambda = \frac{1}{2}\left(\frac{1}{5}\right)^{\frac{1}{2}}$ and $\phi_1 = \frac{6}{2}\left(\frac{1}{5}\right)^{\frac{1}{2}} - \frac{1}{2}5^{\frac{1}{2}} = \frac{1}{10}5^{\frac{1}{2}}$ So the solution to this problem is the following:

$$\begin{aligned}
x_1^* &= 0 \\
x_2^* &= 10
\end{aligned}$$



3 In-kind and cash transfers

A consumer has the following utility over childcare c and food f

$$U(x,y) = c^{\frac{1}{5}} f^{\frac{4}{5}}$$

The price of childcare is $p_c = 2$, the price of food is $p_f = 4$ and income is $I = 20$

1. What is the consumer's demand for childcare and food?

   The Cobb-Douglas utility function allows a shortcut in the calculation of optimal consumption basket. In particular given the function above you can directly find c^* and f^* as follows:

   $$\begin{aligned} c^* &= \frac{1}{5}\frac{I}{p_c} \\ f^* &= \frac{4}{5}\frac{I}{p_f} \end{aligned}$$

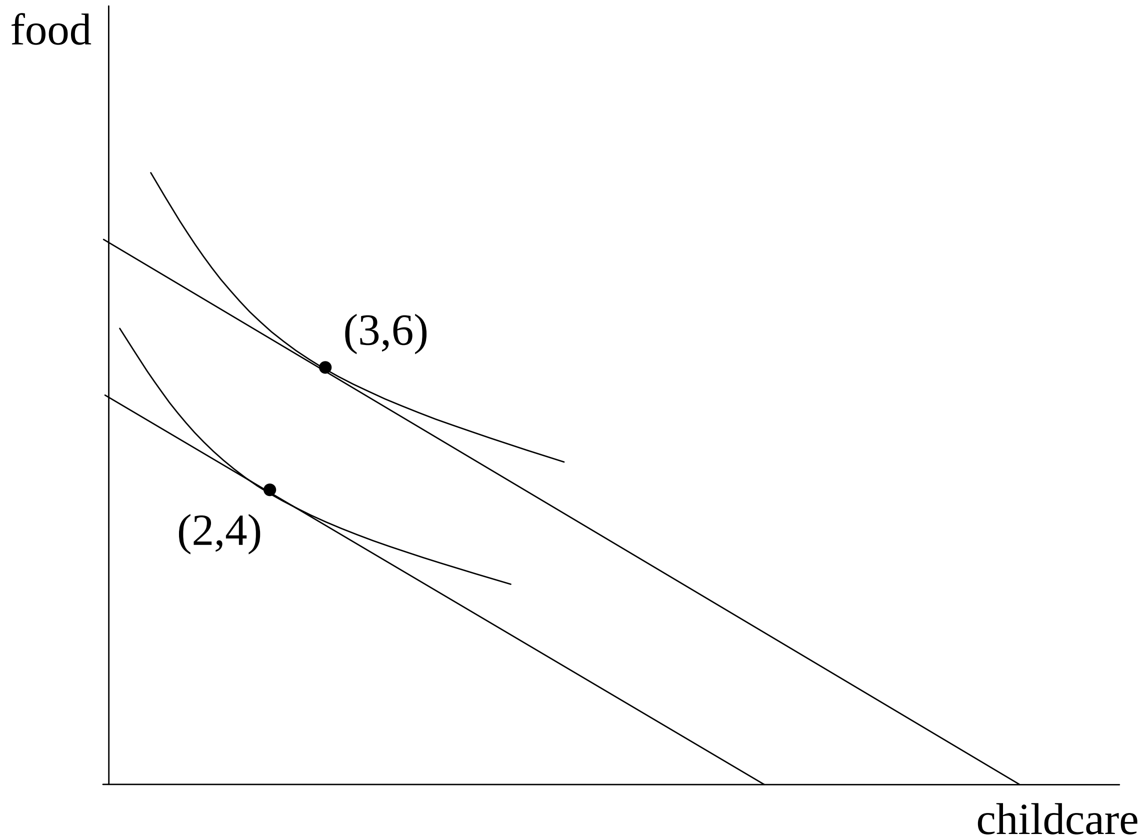
   That is $\frac{1}{5}$ and $\frac{4}{5}$ are expenditure shares: one fifth of monetary income is spend on childcare and fourth fifths are spent on food. Therefore:

   $$\begin{aligned} c^* &= 2 \\ f^* &= 4 \end{aligned}$$

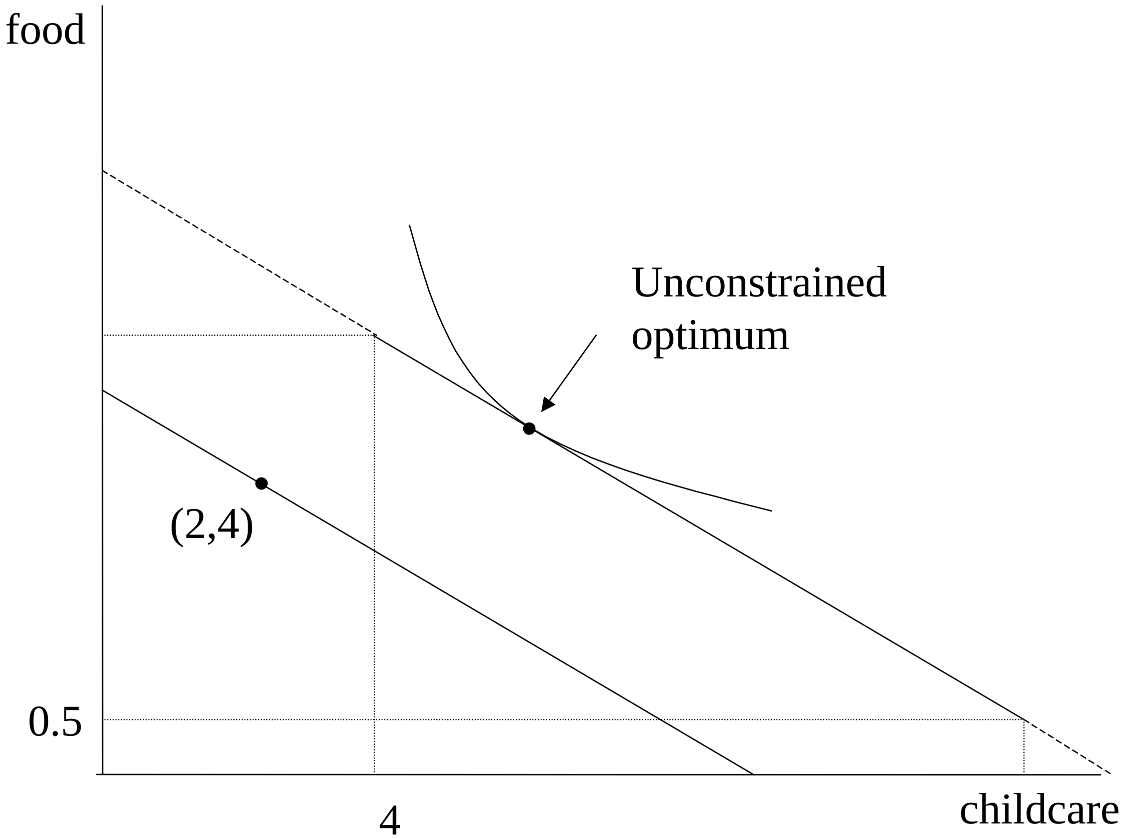
2. Suppose the government gives the consumer an income subsidy of $S = 10$. How will the consumer allocate the subsidy in the consumption of goods c and f? [That is, what is c^* and f^* given the subsidy.]

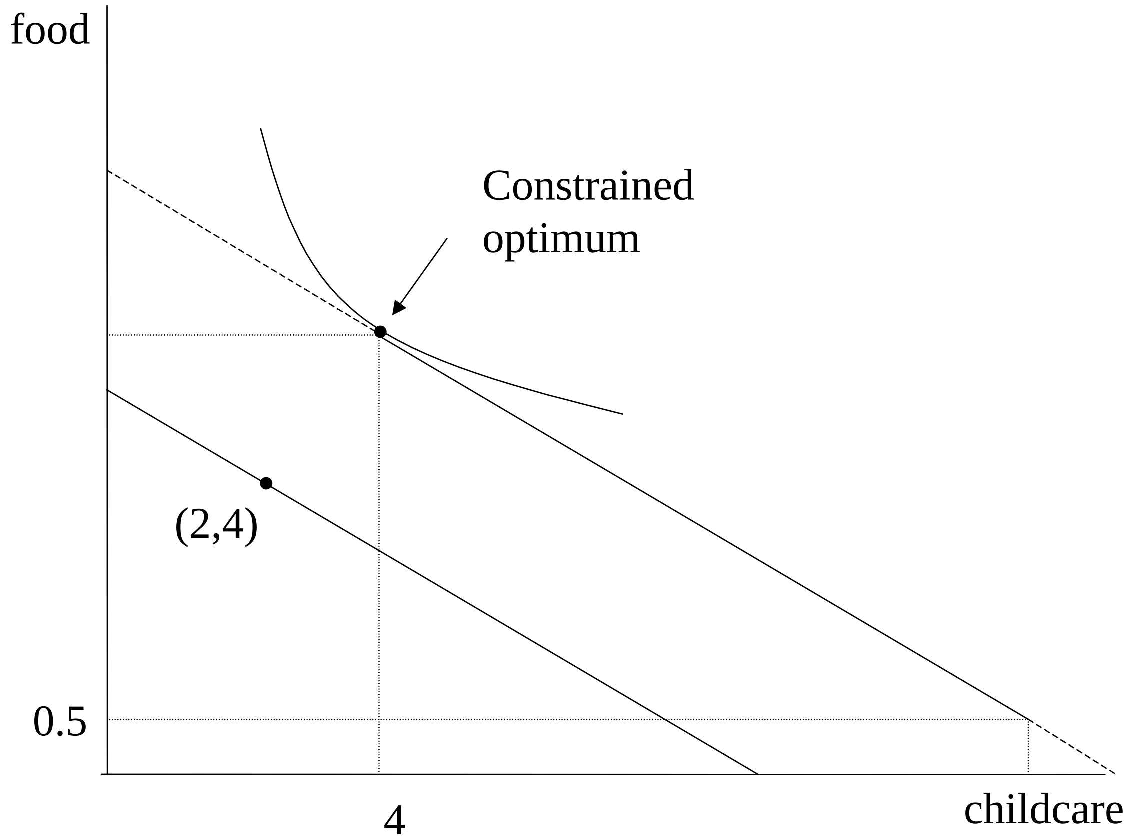
   The choice of food and childcare given the subsidy is:

   $$\begin{aligned} c_S^* &= \frac{1}{5}\frac{30}{2} = 3 \\ f_S^* &= \frac{4}{5}\frac{30}{4} = 6 \end{aligned}$$



3. Suppose now the government decides to give an in-kind transfer to the consumer. The in-kind transfer takes the form of 4 hours of childcare and 0.5 unit of food. Assume that the transfer cannot be re-sold. Draw a carefully labeled graph where you show the pre- and after transfer budget constraint. On this graph indicate the optimal choice for a constrained consumer and the optimal choice for an unconstrained consumer.


food
Unconstrained
optimum
(2,4)
0.5
4
childcare



(a) What are the new consumption levels after the in-kind transfer is given? What is the level of utility attained by the consumer at this consumption level?

The consumer problem is:

$$\begin{aligned} \max U &= c^{\frac{1}{5}} f^{\frac{4}{5}} \\ s.t.\ 2c + 4f &\leq 30 \\ c &\geq 4 \\ f &\geq 0.5 \end{aligned}$$

The proper way to solve this problem is with Kuhn-Tucker conditions as described in the problem above. To simplify our calculations we can just look at the optimal choice in the case of cash transfer and compare it to the constraints that the consumer faces here. Clearly the in-kind transfer forces the consumer to consume more childcare than he would would have under the cash transfer, so we know that the optimal choice is to consume only as much childcare as he has to; the optimal consumption of food and childcare given the in-kind transfer is:

$$\begin{aligned} c_{IK}^{*} &= 4 \\ f_{IK}^{*} &= 5.5 \end{aligned}$$

The level of utility attained is:

$$U_{IK} = 4^{\frac{1}{5}} (5.5)^{\frac{4}{5}} = 5.16$$

(b) What is the minimum expenditure level required to attain the same utility if the consumer were buying all goods on the market?

We can use the expnditure function to answer this question. From other problems we know that the expenditure function for the Cobb-Douglas case is:

$$E = \left(\frac{2}{1/5}\right)^{\frac{1}{5}} \left(\frac{4}{4/5}\right)^{\frac{4}{5}} U$$

So we can plug in the utility level $U_{IK} = 5.16$ and find:

$$E = (10)^{\frac{1}{5}} (5)^{\frac{4}{5}} 5.16 = 29.636$$

(c) What is the cash equivalent of the in-kind transfer? [That is, the cash transfer that the government could give to provide the same level of utility to the consumer as the in-kind transfer?]

The government would have provided the same level of utility by giving to the consumer a cash transfer of 9.636 instead of a value of 10.

4 Food stamps program

You are asked to evaluate the efficiency of the food stamps program. You run an experiment on two otherwise identical groups of benefit recipients. In month 1, you measure the baseline expenditures of both groups. In month 2, you give cash to group $G1$ and food stamps to group $G2$ and again measure expenditures. The total value of the transfer is \$100 per recipient in cash or food stamps.

You observe that within group $G1$ (the cash group), 50% of the recipients increase their food consumption by \$100 and 50% of the recipients increase food consumption by

$60 and consumption of all other goods by $40. [You can also assume that you have a 2nd control group who did not receive either a cash or a stamp transfer in either month. You find that their consumption of food and all other goods is identical in month 1 and 2. So the 'time effect' for the experiment appears to be zero, and you can ignore it.]

You are tempted to conclude that 20% of the transfers to $G2$ is wasted because food stamp recipients who would have spent the money on all other goods are forced to spend it on food instead. You therefore reason that the dead-weight loss of food stamps is $20 per $100. After you read the Whitmore article, you conclude your initial estimate was incorrect.

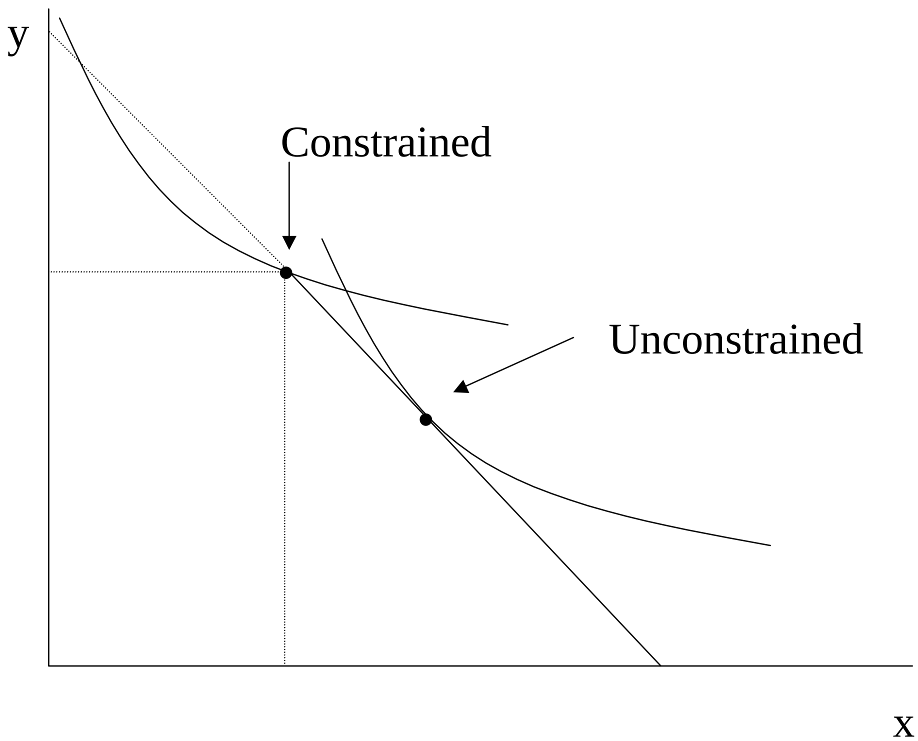
1. Explain why the reasoning above (i.e, that the DWL of food stamps is $20 per $100) is incorrect. Is $20 an overestimate of the DWL or an underestimate or is it indeterminate [explain]?

   $20 is an overestimate of the true deadweight loss, since the $40-worth of food has some value to the constrained consumers so $20 is an upperbound to the true estimate.

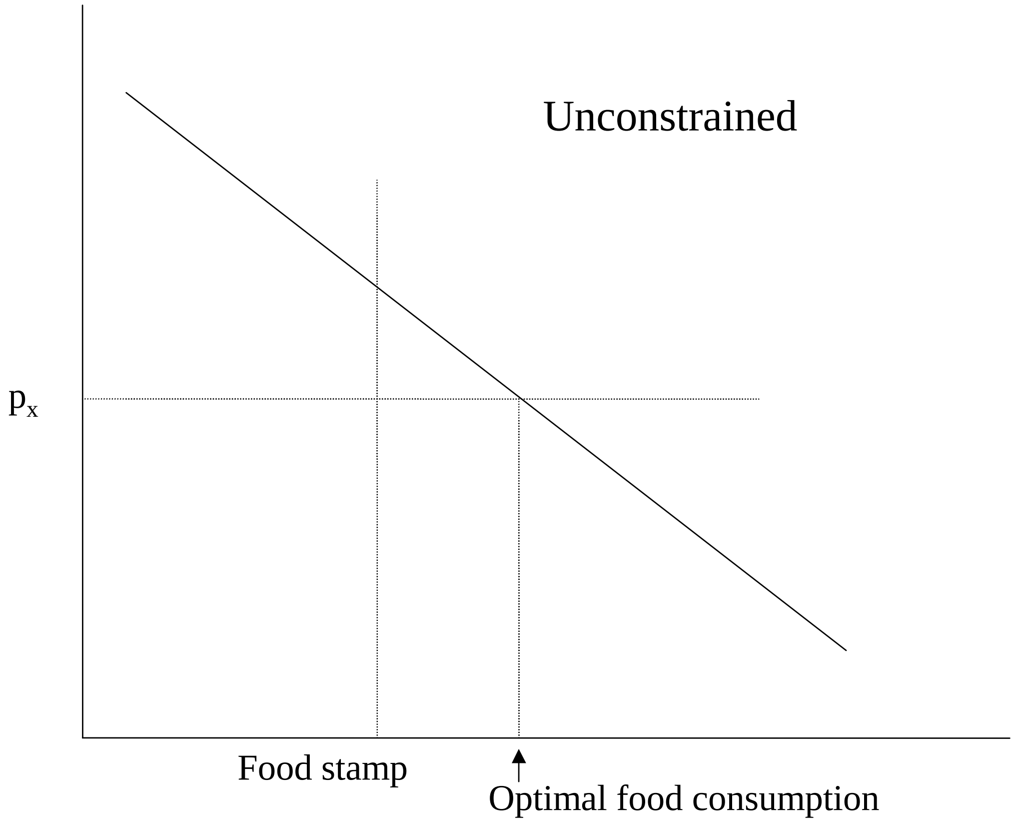
2. Describe qualitatively what information you would need to provide a correct estimate of the true DWL of the food stamp program. Draw a set of diagrams that shows the budget set faced by food stamp recipients for food versus all other goods. Show the indifference curve for the 50% of recipients who would like to spend the full $100 of stamps on food ('unconstrained recipients.'). Show the indifference curve for the 50% of recipients who would like to spend only $60 on food but instead are required to spend $100 ('constrained recipients'). Draw the compensated demand function for food for a hypothetical constrained and unconstrained food stamp recipient. Explain, perhaps using a diagram, how the DWL loss of the food stamp program depends, in part, on the steepness (elasticity) of the compensated demand curve.

   We need to know the shape of the hicksian demand curve between $60 and $100 of expenditure on food. More realistically, we find the derivatives of the Marshallian demand curve with respect to the food price and income and we apply the Slutsky theorem.

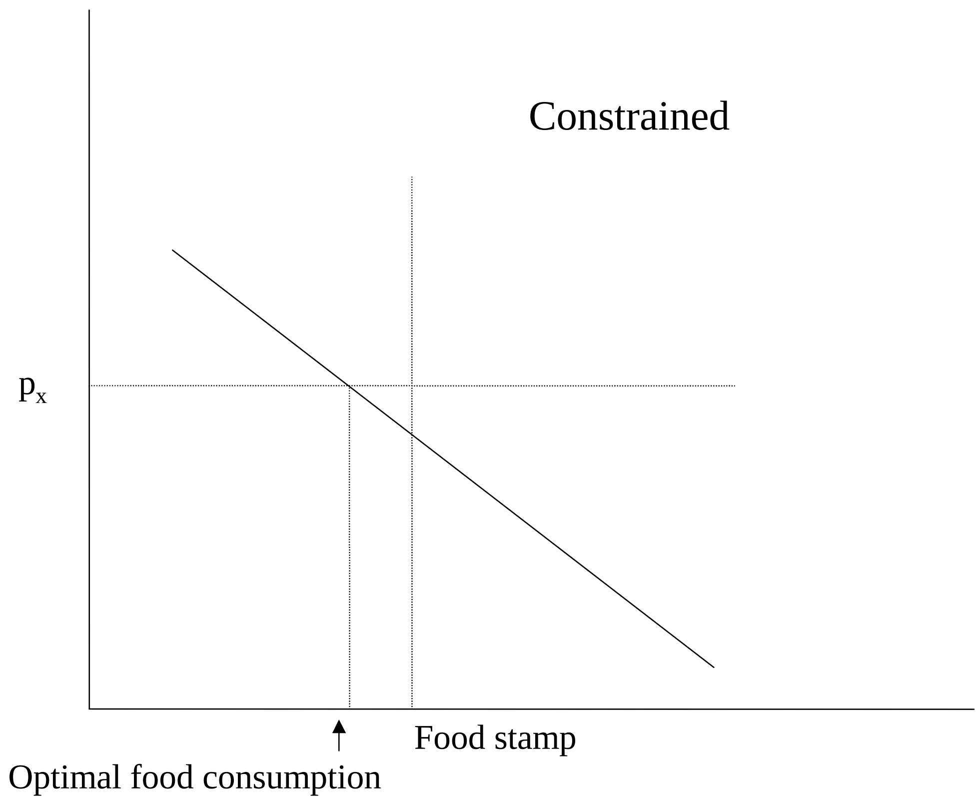
   On the indifference curve map we can indicate constrained and unconstrained consumers as follows:



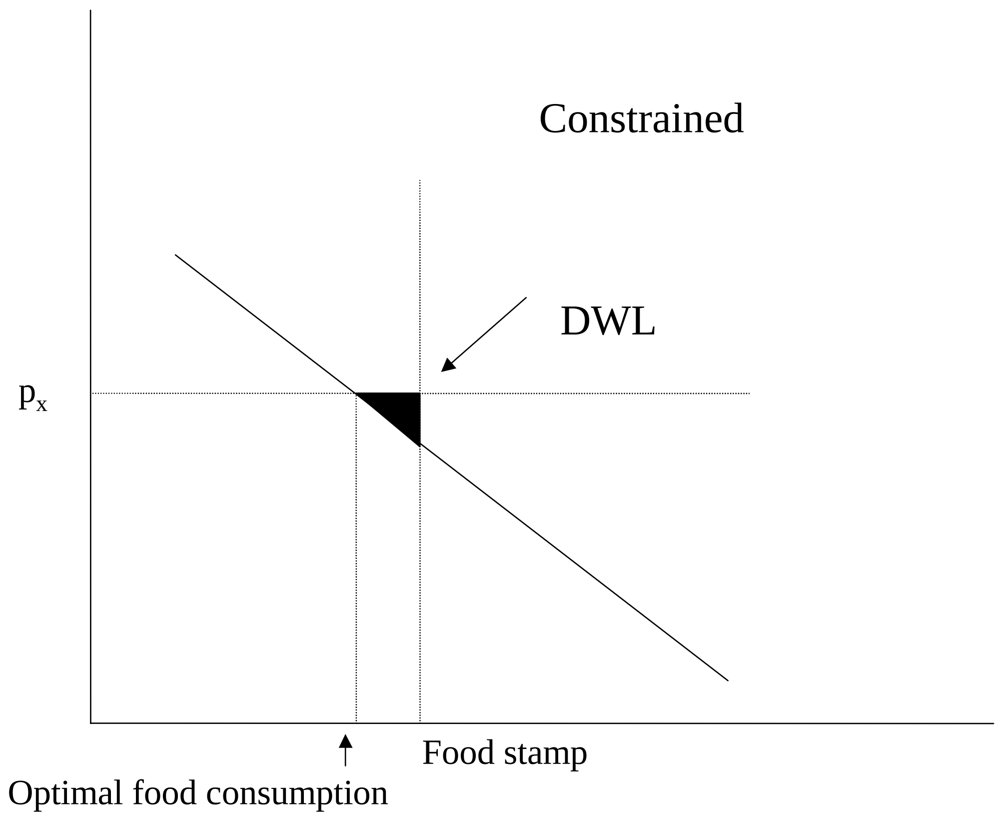
The following graph represents the hicksian (compesated) demand for that 50% of consumers that receive food stamps, but are not constrained by the transfer because their optimal consumption of food is greater than the amount of food stamps.



The following graph represent the compensated demand for food for the 50% of consumers that would rather substitute some of the food stamps for other goods. In this graph I indicate the amount of the food stamps and the consumption of food that the consumer would choose if he was given income I_0 instead of the food stamps.



If we could observe or obtain a precise estimate of the compensated demand schedule then we would be able to calculate the DWL as the area in black in the graph below. This is the cash that the government would have saved if it decided to give the constrained consumers cash that is equivalent to the food stamps.



3. You later find out that recipients in $G2$ sell 40% of their food stamps on the black market for 85% their value. Describe qualitatively how you incorporate this information in your DWL calculations. What does this fact imply about the marginal utility of food consumption for the food stamp recipients that use the black market?

   The fact that some recipients use the black market indicates that the marginal utility that the recipient receives from the last dollar of food stamp (and probably from part of the inframarginal units) is lower than 85% of its monetary value. The question actually suggested an amount of food stamps sold on the black market that is greater than the "forced consumption" of food. That is my mistake (Matilde), so apologize if this created confusion. A more reasonable statement would have been: imagine that the 50% constrained consumers sell 20% their stamps on the black market. This means that 20% of the 40% forced consumption is worth to the consumers less than 85% of its market value. The possibility of selling stamps on the black market should induce us to reduce the estimate of the dead-weight loss, because the black market partially relaxes the constraint on food consumption. More precisely, the availability of a black market menas that we cna ignore any part of the black triangle that lies below $0.85p_x$.

4. The government introduces an Electronic Benefit Transfer (EBT) system: food stamp recipients now receive electronic debit cards instead of stamps. Because debit cards cannot be used without identification and cannot be resold without detection, the EBT system entirely eliminates fraud. Hence, it shuts down the black market. Is the introduction of EBT likely to affect the DWL of the food stamp program? If yes, will it raise or lower it. Explain. [Assume that recipients who did not previously use the black market are indifferent between stamps and EBT.]

   In the same spirit as question 3, introducing EBT would induce us to increase our estimate of the DWL since it would effectively prevent the constrained consumers from selling the stamps.

5 Waldfogel and Christmas

1. The gift giver gets utility equal to $U = I$ from giving cash, , while the utility from giving a non-cash gift is equal to $U = N + \alpha I$. Therefore the giver gives cash if and only if $I > N + \alpha I$, namely that $I(1 - \alpha) > N$.

2. The dead weight loss to the gift recipient is equal to $I(1 - \alpha)$, while the giver satisfaction is given by N. The deadweight loss to the recipient is informative about giver surplus because the giver only gives the non-cash gift if $N > I(1 - \alpha)$.

3. The dead weight loss is a *lower bound* on the giver's surplus of non-cash giving. The giver only gifts the non-cash gift if her surplus exceeds the loss to the recipient.

4. You could allow α to vary by relationship between the gift giver and the recipient. People are probably better at choosing gifts for 'significant others'. In terms of the model, this means that α is likely to be higher when giving to significant others. If α is higher, then the deadweight loss is lower, and the decision rule tells us that

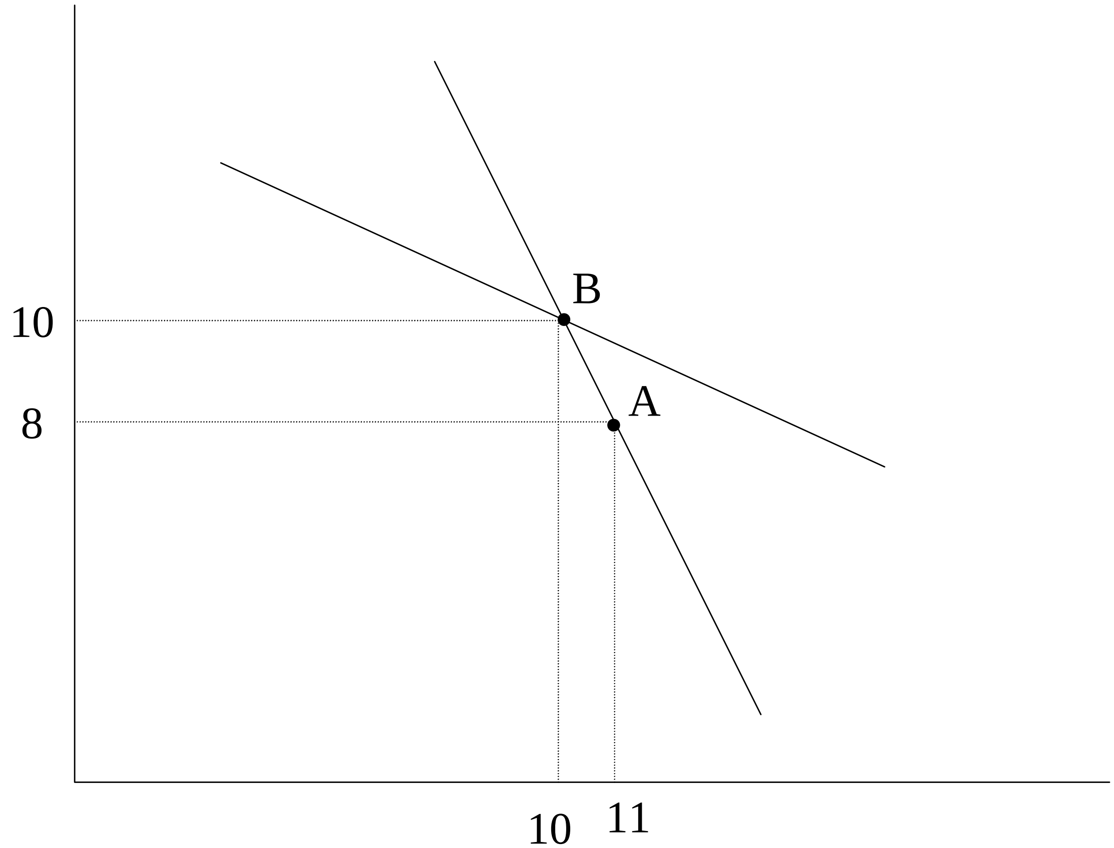
the person is more likely to give a non-cash gift. This could explain why distant relatives are likely to give cash. Holding N constant, distant relatives recognize that their α is likely to be relatively low. So, although they enjoy giving non-cash gifts, their yield is likely to be so low that they will give cash instead.

5. A gift certificate probably doesn't give the same pleasure as giving a purely non-cash gift, but probably provides more pleasure than giving money. So assume it gives pleasure n where $N > n > 0$. But with a gift-certificate, the deadweight loss should be lower – you prefer cash to a CD, but you'd prefer to choose your own CD rather than receive Barry Manilow's Greatest Hits Live – so α is probably higher with gift certificates. Denoting the α for gift certificates as a, someone would give a gift certificate rather than a gift if $n + aI > N + \alpha I$ and would also prefer giving a gift certificate to cash if $n + aI > I$.

6 Rational choice

In the first period, $p_x = 2$ and $p_y = 1$ and the consumer buys 11 units of x and 8 units of y. In the second period, $p_x = 1$ and $p_y = 2$ and the consumer buys 10 units of x and 10 units of y. Prove that this set of choices is not consistent with rational utility maximization of preferences that satisfies all 5 axioms of consumer theory.

There are a few ways you can answer this question and all these are equally acceptable.



Call A the bundle $(11, 8)$ and B the bundle $(10, 10)$.

1. If indifference curves are convex then in order to justify this consumer's choice they would have to cross, which violates non-satiation and transitivity

2. Alternatively you could say that since these are interior solutions and under the assumption that preferences are well behaved, then you should have:

$$\frac{U_x}{U_y} = \frac{p_x}{p_y}$$

so

$$\begin{aligned} \left.\frac{U_x}{U_y}\right|_A &= 2 \\ \left.\frac{U_x}{U_y}\right|_B &= \frac{1}{2} \end{aligned}$$

This is inconsistent with diminishing marginal rate of substitution since as the consumption of x declines the MRS declines.

3. You can also use a revealed preferences argument: By non-satiation, the consumer must strictly prefer C= $(11, 9.5)$ to A. But C is in the consumer's budget set in the second period, and the consumer chose B; so the consumer weakly prefers B to C. By transitivity, the consumer strictly prefers B to A. However, in the first period the consumer reveals that she weakly prefers A to B (since both bundles cost the same and she chose A). This is a contradiction, so the observed choices are inconsistent with revealed preference.

7 Short questions

For each of the following questions state whether it is true, false or uncertain and explain your answer. No points will be given without explanation.

1. The effect of an in-kind transfer on consumer welfare is indeterminate because a constraint can be beneficial if well chosen.

   FALSE: an in-kind transfer will always make the consumer weakly worse off compared to a cash transfer of the same value.

2. The Lagrange multiplier in the utility maximization problem gives the shadow price of additional goods.

   FALSE: the Lagrange multiplier in a problem of utility maximization subject to budget constraint gives the shadow value of income, that is the increase in utility due to a marginal increase in income.

3. A good is only likely to be Giffen if expenditures on it are initially large relative to income.

   TRUE: A good is Giffen when the negative income effect dominates the price effect. Therefore for the negative income effect to be large enough expenditure on this good has to be relatively large.

4. In response to widespread student malnutrition at MIT, President Vest establishes an in-kind food transfer program which gives each student two slices of pizza per day valued at \$1 each. Every day after eating his two free slices, Fred buys a third slice from the MIT truck, also at \$1 per slice. Fred would have been better off if President Vest had given him \$2 per day to spend on whatever he liked.

FALSE: from Fred's behavior we can infer that his choice is not constrained by the in-kind transfer. This in-kind transfer is equivalent to a cash transfer of equal value.

5. Orange juice sells for \$2 per gallon and gasoline sells for \$1 per gallon. Although we don't know how to measure utility, we do know that if a consumer buys both goods, she receives twice as much utility from orange juice as from gasoline.

   FALSE: all we know that in an interior solution:

$$\frac{U_x}{U_y} = \frac{p_x}{p_y}$$

   that is marginal utility of orange juice is two times marginal utility of gasoline.

6. A consumer with convex, 'well-behaved' indifference curves is indifferent between two bundles of X and Y: $(4,1)$ and $(2,9)$. She therefore prefers the bundle $(3,8)$ to either of the first two.

   TRUE: convex indifference curves imply that $\frac{1}{2}X + \frac{1}{2}Y$, which is the bundle $(3,5)$, is preferred to X and Y. By non-satiation $(3,8)$ is preferred to $(3,5)$ and by transitivity $(3,8)$ is preferred to $(4,1)$ and $(2,9)$.