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AIRCRAFT COLLISION MODELS

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CHAPTER 1

INTRODUCTION

The threat of midair collisions is one of the most serious problems facing the air traffic control system and has been studied by many researchers. The gas model is one of the models which describe the expected frequency of midair collisions. In this paper, the gas model which has been used, so far, to deal only with simple cases is extended to a generalized form, and some special types of collision models, such as the overtaking model, are deduced from this generalized model. The effects of the probability distributions of aircraft direction and altitude on the frequency of collisions are also analyzed.

The results in this paper can be applied to evaluate the frequency of conflicts as well as that of collisions. In this paper, an aircraft is represented as a circular cylinder, and a collision is described as an overlap of two cylinders. If the size of the cylinder is expanded to the volume of the protected airspace of an aircraft, an overlap of two cylinders means a conflict. Therefore, with a slight modification, the results can be used to analyze the frequency of conflicts.

This flexibility gives the models of this paper an important potential for application to a future air traffic control system. The FAA is currently developing a new type of air traffic control system called AERA (automated en route air traffic control). AERA is expected to reduce the workload of human controllers and expand the capacity of airspace using new computer systems and better communication links. When this system is fully implemented, aircraft will be able to fly under fewer restrictions. However, if many aircraft are flying on random routes, the frequency of potential conflicts the computer system should handle becomes high. Therefore, the frequency of potential conflicts under various circumstances should be calculated in order to estimate the computer workload before full implementation of the system. The models developed in this paper may be helpful in this evaluation.

The consequences of actual collisions are, of course, grave. Fortunately, the average number of such collisions per year has remained relatively small. According to an FAA Report (Report of the FAA Task Force on Aircraft Separation Assurance, Jan. 1979), the average number of midair collisions reported to NTSB from 1974 through 1978 was 33 per year. Most midair collisions have occurred between small general aviation aircraft operating under VFR. However, the report also states that there were 227 near midair collision reports in 1975 alone, and that air carriers were involved in 68 of these cases. (According to the report, a near midair collision is an incident which would probably have resulted in a collision if no action had been taken by either pilot. Closest proximity of less than 500 ft would usually be required for a near midair collision report.) Although the number of conflicts is not available in the report, it is clearly far greater than the number of near midair collisions considering the difference of airspace volumes involved.

The outline of this thesis is as follows. In Chapter 2, we present an overview of two aircraft collision models, the Reich model and the gas model, which have been the most important ones in this field. In Chapter 3, we develop some extensions of the gas model including a generalized two-dimensional gas model, an overtaking model and a three-dimensional gas model. In Chapter 4, we develop an aircraft collision model which does not assume the uniformity of aircraft distribution. The conclusions of this thesis are summarized in Chapter 5.

CHAPTER 2

AIRCRAFT COLLISION MODELS

In this chapter, an overview of the Reich model is first presented, and then the gas model is briefly described. These models have been the most important ones in estimating collision risks.

2.1 The Reich Model

Many models have been developed to estimate aircraft collision rates under various circumstances. Probably the best-known among them is the Reich model which was employed to assess the collision risk to flights over the North Atlantic Ocean as part of efforts to reduce the lateral separation between North Atlantic air routes.[2,3]

In the Reich model, an aircraft is represented by a box, and a collision is described as an overlap of two boxes. The occurrence of this overlap is equivalent to the event that a point enters a box which has dimensions twice as large as those of the original box.

The collision rate is expressed as

$$F_x P_y P_z + F_z P_x P_y + F_y P_z P_x \tag{2-1}$$

where

F_x is the expected frequency per unit of time with which the along-track separation shrinks to less than λ_x.

F_y and F_z are similarly defined.

P_x is the probability that the along-track separation is less than λ_x at a random instant of time.

P_y and P_z are similarly defined.

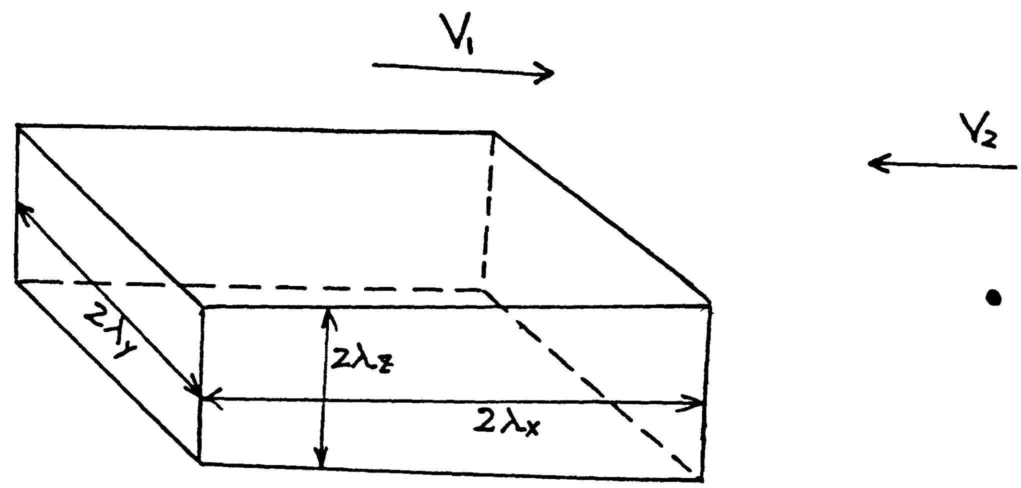


Figure 2-1 Model of Collision

If we are calculating the collision rate between a pair of aircraft assigned to the same flight altitude and flying on parallel flight tracks separated by the lateral separation S_y, (2-1) becomes

$$F_x P_y(S_y) P_z(0) + P_x \left\{ F_z(0) P_y(S_y) + F_y(S_y) P_z(0) \right\} \qquad (2\text{-}2)$$

where

$P_y(S_y)$ is the probability that the across-track separation between a pair of aircraft, nominally spaced at the lateral standard seperation S_y, is less than λ_y.

$P_y(0)$ is the probability that the across-track separation between a pair of aircraft, assigned to the same track, is less than λ_y.

$F_y(S_y)$ is the expected frequency per unit of time with which the across-track separation between a pair of aircraft, nominally spaced at the lateral standard separation S_y, shrinks to less than λ_y.

$F_y(0)$ is the expected frequency per unit of time with which the across-track separation between a pair of aircraft, assigned to the same track, shrinks to less than λ_y.

$P_z(0)$ and $F_z(0)$ are similarly defined for the vertical dimension.

If aircraft on the same track are spaced with the along-track separation S_x,

$$P_x = \frac{2\lambda_x}{S_x}$$

and,

$$F_x = \frac{E[|\dot{x}|]}{S_x} \qquad (2\text{-}3)$$

where

$\dot{X}$ is the relative along-track velocity of a pair of aircraft

$F_z(0)$ and $P_z(0)$ depend upon the vertical dynamics and vertical station keeping of aircraft. Reich presented numerical examples for $F_z(0)$ and $P_z(0)$ in his paper. They are

$$F_z(0) = 40/hr$$

$$P_z(0) = 0.26$$

$P_y(S_y)$ can be calculated if the probability distribution for the lateral deviation of aircraft from the center line of the track is known.

$$P_y(S_y) = \int_{-\infty}^{\infty} \int_{y-\lambda_y}^{y+\lambda_y} p_y(y)\, p_y(S_y - y')\, dy'\, dy \tag{2-4}$$

where

$p_y(y)$ is the probability distribution of lateral deviation of aircraft from the center line of the track.

$p_y(y)$ is estimated through the first Laplacian distribution.

$$p_y(y) = \frac{1}{\sqrt{2}\,\sigma_y} \exp(-\sqrt{2}\,|y - \mu_r|/\sigma_r) \tag{2-5}$$

where

σ_y is the standard devaition, and μ_r is the mean of lateral deviation.

$F_y(S_y)$ is estimated in Reich's paper as

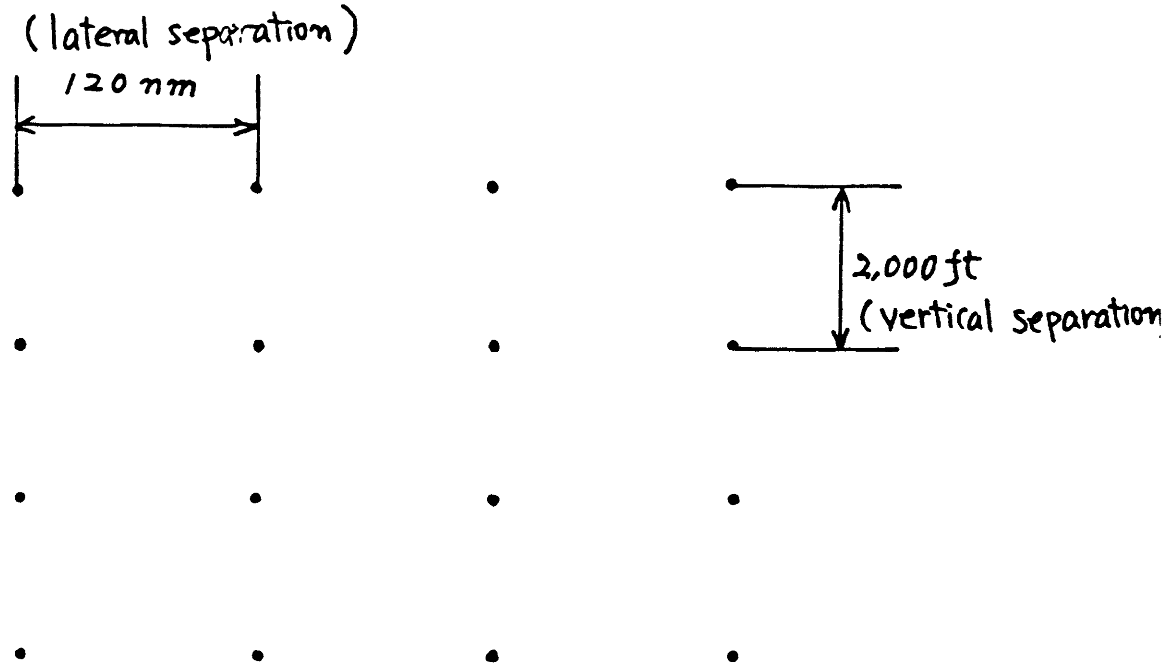
$$F_Y(S_Y) = \frac{E[|\dot{Y}|]}{2\lambda_Y} P_Y(S_Y) \tag{2-6}$$

where

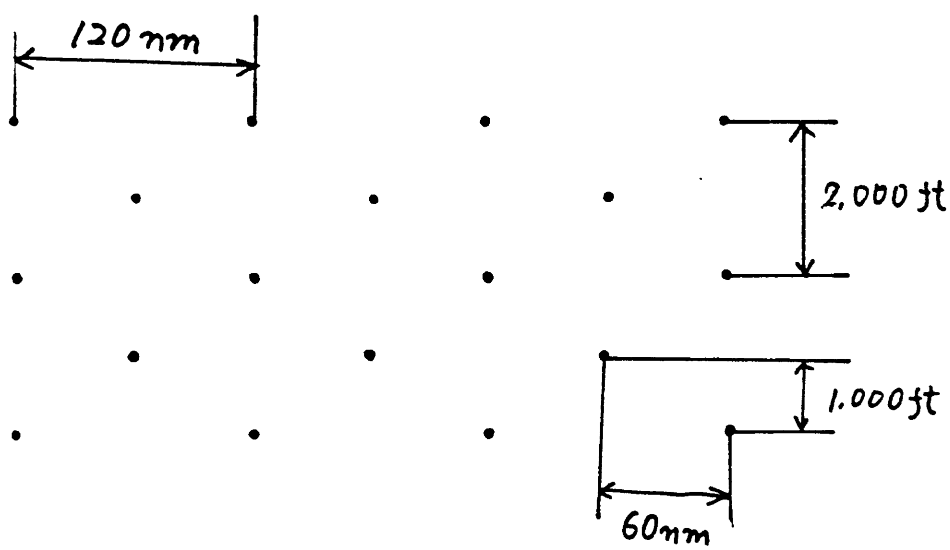
$\dot{Y}$ is the relative across-track velocity of a pair of aircraft.

This completes a brief mathematical description of the Reich model. This model was employed to estimate the collision risk to flights over the North Atlantic Ocean in the 1960's. The number of flights over the North Atlantic Ocean increased dramatically in the 1950's and the early 1960's, and the traffic became seriously congested by the mid-1960's. The lateral separation between the center lines of adjacent air routes over the North Atlantic Ocean was 120 n.m. at that time, and the airlines suggested that the lateral separation be lowered to 90 n.m.. The FAA agreed with the suggestion, and ICAO adopted it. However, airline pilots were strongly against it, and the separation standard tentatively returned to the 120n.m. standard as a result of a hearing. Then the study of the collision risk over the North Atlantic Ocean started. First data for lateral deviation from the center line of the air route were collected through the use of radars on land and on ships. Then the collision risk was determined using the Reich model. The conclusion was that the lateral collision risk at 90n.m. lateral separation was about six times a great as at 120 n.m. separation. It also became obvious that the lateral collision risk far exceeded the risk of collision with one's vertical and longitudinal neighbors. The final solution to this problem was what is called staggered separation which is shown in Figure 2-2.

The Reich model was originally developed to estimated the collision



Conventional Separation



Staggered Separation

Figure 2-2 Conventional Separation and Staggered Separation risk over the ocean. Some more recent studies have attempted to estimate the collision risk to continental flights. Polhemus and Livingston collected data for lateral deviations of aircraft flying in the VOR airways system.[4] They concluded that the deviation distribution varies considerably with position with respect to the VOR. Based on this data, Polhemus calculated the collision rate of aircraft flying on two parallel air tracks within the VOR system.[5]

2.2 The Gas Model

Another well-known collision model is the gas model. The reason for this name is that its basic concept is essentially the same as that of a gas molecular collision model. A brief description of the two dimensional gas model is given below, and the generalized gas model will be developed in Chapter 3.

The two-dimensional gas model assumes that N aircraft operate in an area A. Each aircraft i, travels in a straight line with velocity V_i, and direction distributed uniformly between 0 and 2π. Aircraft are uniformly and independently positioned over A.

The magnitude of the relative velocity of two aircraft i and j, is

$$V_{r_{ij}} = \left(V_i^2 + V_j^2 - 2 V_i V_j \cos\beta \right)^{\frac{1}{2}} \qquad (2\text{-}7)$$

where

$V_{r_{ij}}$ = relative velocity of two aircraft i and j.

β = relative directional angle of the two aircraft.

If an aircraft is represented as a disk with diameter g, the probability of a collision between i and j during a period of time t is given by

$$\frac{2g\, V_{rij}\, t}{A} \tag{2-8}$$

If $E[V_r]$ is the expected relative velocity, the expected number of collisions during a period of time t is

$$\frac{N^2}{2} \frac{2g\, E[V_r]\, t}{A} \tag{2-9}$$

In a three-dimensional problem, (2-9) gives the number of horizontal overlaps during a period of time t assuming that aircraft are flying only horizontally. If the traffic density is uniform in an altitude layer H thick, and an aircraft has the vertical dimension h, then the three-dimensional collision rate can be obtained by multiplying (2-9) by $\frac{2h}{H}$.

This model has been used to estimate the collision rate in terminal area.[6,7] In reference 7, Flanagan and Willis developed a three-dimensional gas model which allowed aircraft to have a vertical velocity component as well. However, this model assumed that directions of aircraft velocity are randomly distributed over three-dimensional angle. This assumption is obviously unrealistic because airplanes do not have the capability to fly vertically like helicopters.

The gas model described above can deal only with simple cases. In the next chapter, the gas model is extended to a generalized form, and the probability distributions of aircraft direction and altitude are analyzed in connection with the collision rate.

CHAPTER 3

SOME EXTENSIONS OF THE GAS MODEL

Some extensions of the gas model are presented in this chapter. First, a generalized two-dimensional gas model is developed. Then, the definition of expected relative velocity is examined, and some special cases are analyzed. The overtaking problem is briefly discussed as a special case of collision. The horizontal overlap rate and the probability of vertical overlap are also discussed. Finally, a three-dimensional gas model is developed.

3.1 Generalized Two-Dimensional Gas Model

This model assumes that N aircraft are flying in the volume with horizontal area A and height H (Figure 3-1).

Horizontally, aircraft are uniformly and independently distributed, and vertically, they are independently but not necessarily uniformly distributed. The horizontal distribution and the vertical distribution are independent of each other. All the distributions in this model are time-invariant. The assumption that the density of aircraft is time-invariant can be justified considering that the collision rate is small and so is the rate of aircraft loss.

Each aircraft travels only horizontally. It is not assumed that each aircraft travels only in a straight line. No collision avoidance maneuver is taken, however. The assumption of no collision avoidance maneuver is obviously not realistic, and leads to an overestimated collision rate. However, as explained in Chapter 1, with a slight modification, the rate of conflicts can be calculated under the same

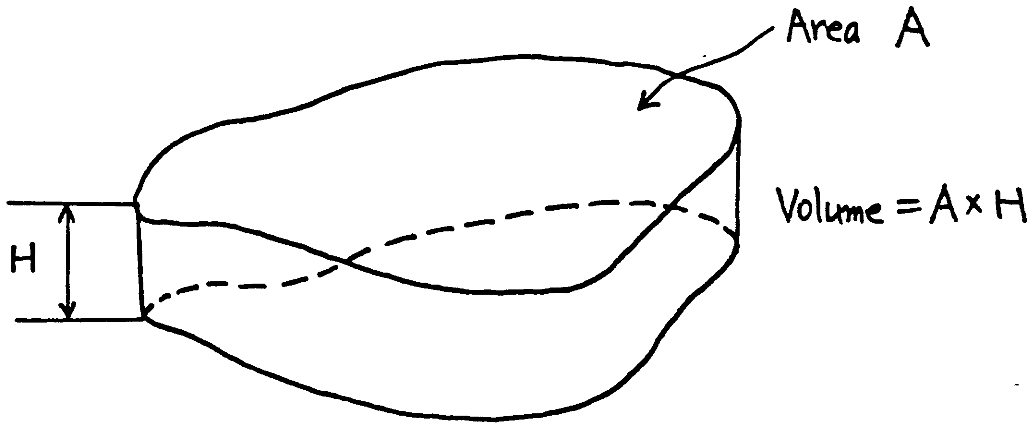


Figure 3-1 Airspace

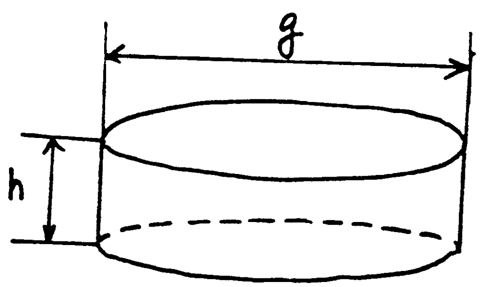


Figure 3-2 Representation of an Aircraft

assumption, and that rate may be helpful in estimating the workload of pilots and air traffic controllers in avoiding conflicts. Therefore, the result with this assumption may be useful both in obtaining a conservative estimate of the collision rate and in estimating the workload of pilots and air traffic controllers in preventing collisions.

Each aircraft is represented as a right circular cylinder, shown in Figure 3-2. A collision is described as an overlap of two cylinders. The occurrence of this overlap is equivalent to the event that a point representing a center of one aircraft enters the cylinder of another aircraft which is twice in length and eight times in volume as large as the original aircraft (Figure 3-3).

The collision rate is expressed as follows.

$$\mathcal{C} = \mathcal{F}_H \times \mathcal{P}_V \tag{3-1}$$

where

$\mathcal{C}$ = the rate of collisions per unit of time

$\mathcal{F}_H$ = the rate of horizontal overlap per unit of time

$\mathcal{P}_V$ = probability that two aircraft overlap vertically given they overlap horizontally

= P (vertical overlap/horizontal overlap)

Horizontal overlap means that, neglecting the vertical axis, the horizontal coordinate of the center of one aircraft enters within a range of g from the horizontal coordinate of the center of another aircraft (Figure 3-4). Vertical overlap is similarly defined.

Let us consider $\mathcal{P}_V$ first.

Because of independence of the horizontal and vertical distribution

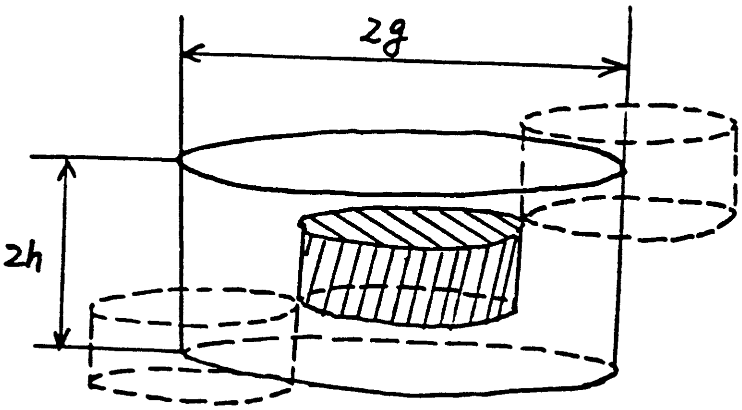


Figure 3-3 Collision Volume

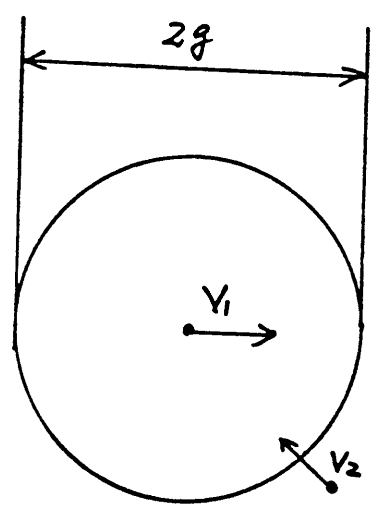


Figure 3-4 Horizontal Overlap

of aircraft,

$$P_v = P(\text{vertical overlap for two aircraft}) \tag{3-2}$$

Let $p_z(z)$ be the probability density function for the altitude of aircraft. $\int_{z_1}^{z_2} p_z(z)\,dz$ is the probability that an aircraft is flying at an altitude between z_1 and z_2. Then,

$$P_v = \int_G^{G+H} p_z(z) \int_{z-h}^{z+h} p_z(u)\,du\,dz \tag{3-3}$$

where G is the altitude of the lowest surface of the airspace under consideration.

Let us calculate P_v for the uniform $p_z(z)$.

$$p_z(z) = \begin{cases} \dfrac{1}{H} & G \le z \le G+H \\ 0 & \text{otherwise} \end{cases}$$

$$P_v = \int_G^{G+H} p_z(z) \int_{z-d}^{z+d} p_z(u)\,du\,dz$$

$$= \frac{2hH - h^2}{H^2}$$

If $H \gg h$, then

$$P_v \doteq \frac{2h}{H} \tag{3-4}$$

This shows that if aircraft are uniformly distributed over an altitude layer H, with H much greater than the aircraft height h, the collision rate is approximately $F_H \times \frac{2h}{H}$.

Next, let us consider the rate of horizontal overlap F_H. As explained earlier, a horizontal overlap takes place when the center of an

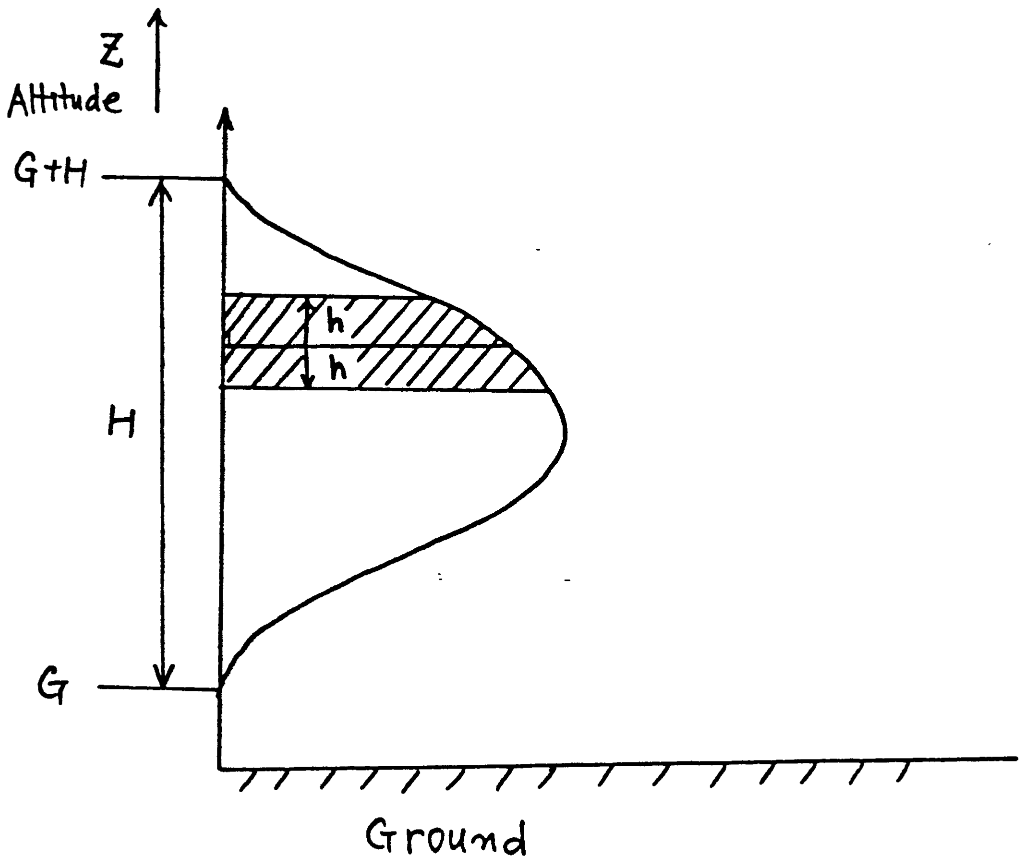


Figure 3-5 Vertical Overlap

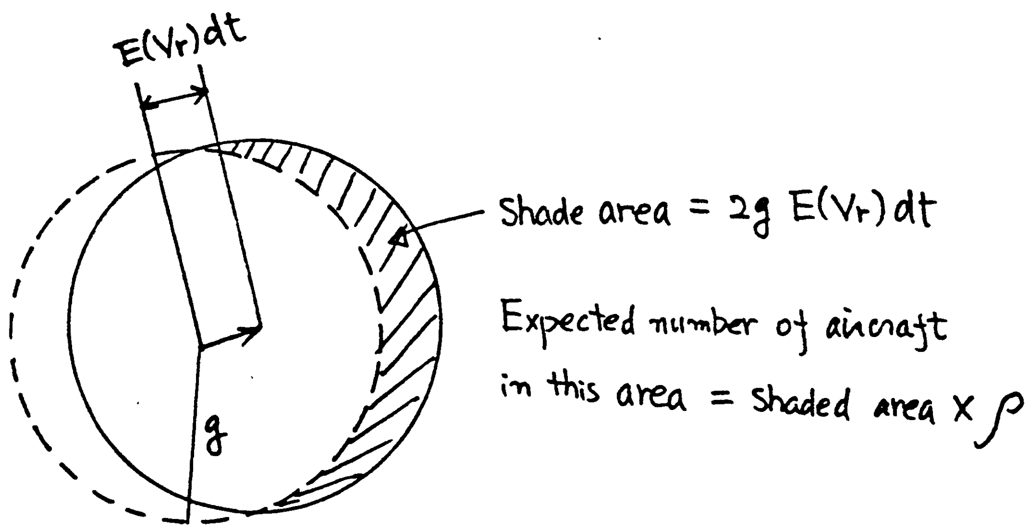


Figure 3-6 Horizontal Overlap in dt

aircraft enters within a range of g from the center of another aircraft. Let $E(V_r)$ be the expected relative velocity. Then, during a very short period of time dt, each aircraft encounters $2g\,E(V_r)dt \times \rho$ overlaps on the average, where ρ is the horizontal density of aircraft.

Since ρ is N/A and there are N aircraft, the total number of horizontal overlaps in dt is

$$\frac{1}{2} \times N \times \left(2g\,E(V_r)dt \times \frac{N}{A}\right)$$

$$= \frac{N^2 g\,E(V_r)\,dt}{A} \tag{3-5}$$

The reason for the multiplier 1/2 in (3-5) is that each overlap would otherwise be counted twice. Then,

$$F_H = \frac{N^2 g\,E(V_r)}{A} \tag{3-6}$$

From (3-1), (3-3) and (3-6), the collision rate is

$$C = \frac{N^2 g\,E(V_r)}{A} \int_G^{G+H} P_z(z) \int_{z-h}^{z+h} P_z(u)\,du\,dz \tag{3-7}$$

where

C = collision rate per unit of time

A = horizontal area of the airspace under consideration

G = altitude of the lowest surface of the airspace

G + H = altitude of the highest surface of the airspace

N = number of aircraft in the airspace

g = horizontal dimension of aircraft

h = vertical dimension of aircraft

$E(V_r)$ = expected relative velocity

$p_z(z)$ = probability density function of altitude of aircraft

If aircraft are uniformly distributed over altitude and $H \gg h$,

$$C \doteq \frac{N^2 g E(V_r)}{A} \frac{2h}{H} \tag{3-8}$$

Below and in Section 3.3, we shall calculate $E(V_r)$ for some special cases. At this point, we shall assume that each aircraft travels in a straight line. This is to simplify the calculation of $E(V_r)$. Suppose then that two aircraft are flying at velocities V_1 and V_2. Then, the relative velocity is

$$V_r = (V_1^2 + V_2^2 - 2V_1V_2 \cos\beta)^{\frac{1}{2}} \tag{3-9}$$

where

V_r = relative velocity

β = relative directional angle of two aircraft

Then,

$$E(V_r) =$$

$$\int_{V_1}\int_{V_2}\int_0^{2\pi} (V_1^2+V_2^2-2V_1V_2\cos\beta)^{\frac{1}{2}} p_\beta(\beta)\, p_V(V_1)\, p_V(V_2)\, d\beta\, dV_1\, dV_2 \tag{3-10}$$

where

$p_\beta(\beta)$ = probability density function of β

$p_V(V)$ = probability density function of V

If directions of aircraft velocity are uniformly distributed between

0 and 2π and the magnitude of velocity is constant,

$$P_\beta(\beta) = \frac{1}{2\pi} \qquad 0 \le \beta < 2\pi$$

$$V = V_0 = \text{constant}$$

Then,

$$E(V_r) = \int_0^{2\pi} (2V_0^2 - 2V_0^2 \cos\beta)^{\frac{1}{2}} \frac{1}{2\pi} \, d\beta \tag{3-11}$$

$$= \frac{4}{\pi} V_0$$

[Numerical Example]

Twenty aircraft are flying in the airspace shown in Figure 3-7. Each aircraft is flying horizontally at 300 kt in a random direction. Aircraft are represented by a circular cylinder with diameter of 150 ft and thickness of 50 ft. Aircraft are uniformly and independently distributed in the airspace both horizontally and vertically.

$$E(V_r) = \frac{4}{\pi} V_0$$

$$= 382 \text{ kt}$$

$$P_v \doteqdot \frac{2h}{H}$$

$$= \frac{1}{100}$$

$$F_H = \frac{N^2 g \, E(V_r)}{A}$$

$$= 0.377 / \text{hr}$$

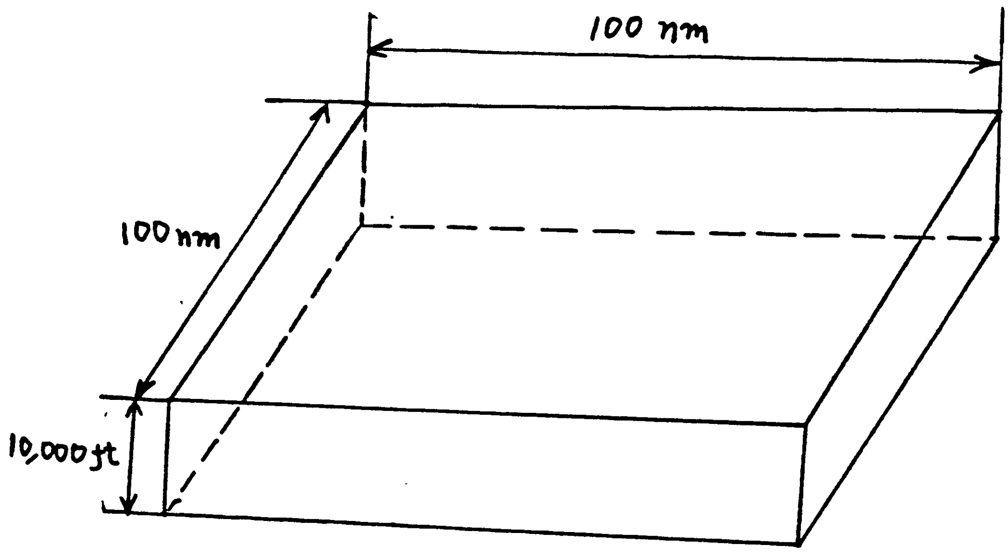


Figure 3-7   Airspace

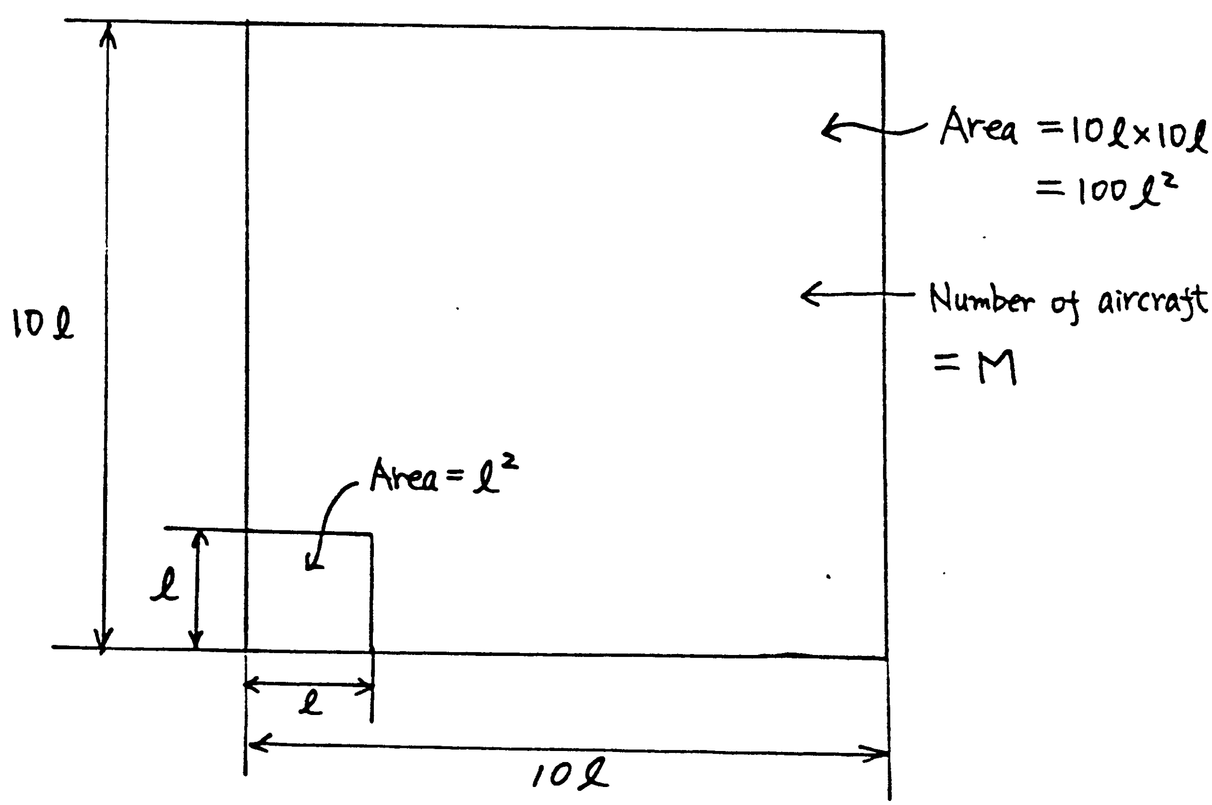


Figure 3-3   Collision Rates in Different Areas

$$C = F_H \times P_V$$
$$= 3.77 \times 10^{-3} /hr$$

This may seem surprisingly large. This is because the expected relative velocity is large in this case.

3.2 Expected Relative Velocity

A brief discussion of the meaning of "expected relative velocity" will be useful at this stage. It seems simple at first to define the expected relative velocity. However, the concept creates some difficulty in estimating the collision rate in the gas model. For simplicity, only two-dimensional problems are dealt with in this section.

It is sometimes maintained not only in the field of air traffic control[6] but also in the field of statistical mechanics[8] that the collision rate is given by the following formula

$$C = \frac{N(N-1)}{2} \frac{2g\,E'(V_r)}{A} \tag{3-12}$$

where $E'(V_r)$ is the expected relative velocity taken over all possible pairs of aircraft. This equation is different from (3-6), which uses $\frac{N^2}{2}$ as its first term. If we are dealing with molecular collisions, N is a very large number and the difference between the two formulae is very small. Therefore, there is no practical problem in that field. But, in the case of aircraft collisions, the difference may be of some significance as the density of aircraft is usually low. We shall discuss the difference between the two formulae below.

The argument for (3-12) goes as follows. The probability that one pair of aircraft will meet each other during a very short period of time dt is $\frac{2g\,E'(V_r)\,dt}{A}$. Since there are N aircraft, there are $\frac{N(N-1)}{2}$ possible pairs of aircraft. Then, the expected number of collisions during dt is $\frac{N(N-1)}{2}\,\frac{2g\,E'(V_r)\,dt}{A}$. Therefore, the collision rate is given by (3-12).

Let us assume, at first, that (3-12) gives the correct collision rate.

Consider the situation shown in Figure 3-8. Suppose that there are M aircraft in the large area ($10\ell \times 10\ell$) and the probability distribution for the position of each aircraft is uniform over the area. The expected number of aircraft in the small area ($\ell \times \ell$) is $\frac{M}{100}$. Let us calculate the two collision rates in the large area and in the small area. According to (3-12), the collision rate in the large area C_L is given by

$$C_L = \frac{M(M-1)}{2}\,\frac{2g\,E'(V_r)}{100\,\ell^2} \tag{3-13}$$

Similarly, the collision rate in the small area C_S is

$$C_S = \frac{\frac{M}{100}\left(\frac{M}{100}-1\right)}{2}\,\frac{2g\,E'(V_r)}{\ell^2} \tag{3-14}$$

Since the large area can be sub-divided into 100 small areas, C_L should be 100 times as large as C_S. However,

$$100\,C_S = \frac{M(M-100)}{2}\,\frac{2g\,E'(V_r)}{100\,\ell^2} \tag{3-15}$$

Then,

$$C_L \neq 100\,C_S \tag{3-16}$$

This is a contradiction. The explanation for this contradiction lies in the definition of the expected relative velocity $E'(V_r)$. In (3-13) and (3-14), it is implicitly assumed that $E'(V_r)$ is the same in both areas. But $E'(V_r)$ is a function of N if we are to use (3-12).

$E'(V_r)$ in (3-12) is the expected relative velocity taken over all possible pairs of aircraft. Therefore, if the number of aircraft changes, so does $E'(V_r)$. In order to make this problem clear, let us consider some examples.

Suppose that two aircraft are flying in opposite directions at speed V_0. The probability distribution of their positions is uniform over the area A_1 (Figure 3-9). The number of possible pairs is one, and $E'(V_r)$ is $2V_0$. The collision rate is given by (3-12).

$$C = \frac{N(N-1)}{2} \frac{2g\,E'(V_r)}{A_1}$$

$$= \frac{4g\,V_0}{A_1} \tag{3-17}$$

Next, suppose that four aircraft are flying at speed V_0 in the area $2A_1$. Two of them are flying in the same direction, and the other two are flying in the opposite direction. The probability distribution of their positions is uniform over the area (Figure 3-10).

$E'(V_r)$ is defined as the expected relative velocity taken over all possible pairs aircraft. Viewed from any one of the four aircraft, two other aircraft are moving at speed $2V_0$ and the other one is at rest,

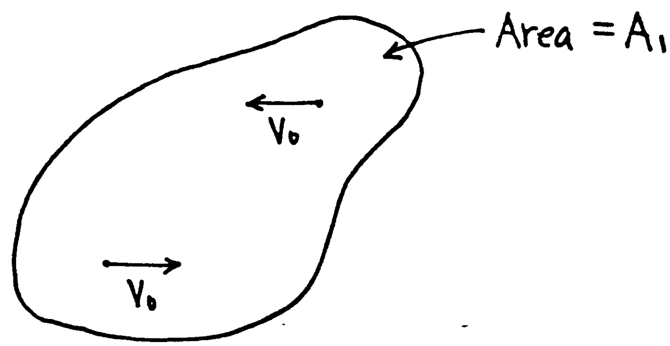


Figure 3-9 Two Aircraft Flying in Opposite Directions

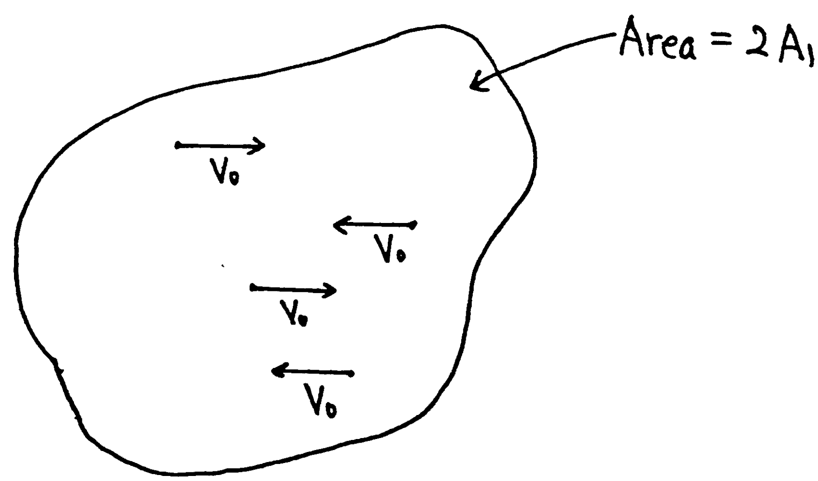


Figure 3-10 Four Aircraft in Area $2A_1$

$$E'(V_r) = \frac{2V_0 + 2V_0 + 0}{3} = \frac{4V_0}{3} \tag{3-18}$$

This value is different from $E'(V_r)$ in (3-17), although the probability distribution of directions of velocity is the same in both cases. The collision rate in this case is

$$C = \frac{N(N-1)}{2} \frac{2g\,E'(V_r)}{2A_1}$$

$$= 6 \times \frac{2g \times \frac{4V_0}{3}}{2A_1} = \frac{8gV_0}{A_1} \tag{3-19}$$

Therefore, the collision rate in the area $2A_1$ is twice as large as the collision rate in the area A_1.

These examples show that $E'(V_r)$, which is defined in (3-12) as the expected relative velocity taken over all possible pairs of aircraft, is a function of the number of aircraft. Next, the relation between $E(V_r)$ in (3-6) and $E'(V_r)$ in (3-12) is analyzed, and the mathematical expression for $E'(V_r)$ in terms of $E(V_r)$ is derived. $E(V_r)$ is defined as follows.

$$E(V_r) \equiv \int_{\vec{V_1}} \int_{\vec{V_2}} |\vec{V_1} - \vec{V_2}|\, P_{\vec{V}}(\vec{V_1})\, P_{\vec{V}}(\vec{V_2})\, d\vec{V_1}\, d\vec{V_2} \tag{3-20}$$

where

$\vec{V_i}$ = velocity vector of aircraft i

$P_{\vec{V}}(\vec{V})$ = probability density function of velocity vector

Let us calculate the average relative speed based on (3-20). Suppose that there are N aircraft, and aircraft i has the velocity vector $\vec{V_i}$. Then, the average relative velocity is

$$\overline{V_r} = \frac{1}{N^2} \sum_{i=1}^{N} \sum_{j=1}^{N} |\vec{V_i} - \vec{V_j}| \tag{3-21}$$

However, the average relative velocity taken over all possible pairs of aircraft, which is defined as $\overline{V_r}'$, is

$$\overline{V_r}' \doteq \frac{1}{N(N-1)} \sum_{i} \sum_{j \neq i} |\vec{V_i} - \vec{V_j}| \tag{3-22}$$

Since $|\vec{V_i} - \vec{V_i}| = 0$, (3-22) becomes

$$\overline{V_r}' = \frac{1}{N(N-1)} \sum_{i=1}^{N} \sum_{j=1}^{N} |\vec{V_i} - \vec{V_j}| \tag{3-23}$$

Then,

$$\overline{V_r}' = \frac{N}{N-1} \overline{V_r} \tag{3-24}$$

Therefore, if the expected number of aircraft is $N (N > 1)$, the relation between $E(V_r)$ and $E'(V_r)$ is as follows.

$$E'(V_r) = \frac{N}{N-1} E(V_r) \tag{3-25}$$

where

$E(V_r)$ = expected relative velocity defined by (3-20)

$E'(V_r)$ = expected relative velocity taken over all possible pairs of aircraft

Therefore, (3-6) and (3-12) used in this way give the same collision rate. The reason for the difference between $E(V_r)$ and $E'(V_r)$ is that when V_r is averaged over all pairs of aircraft, all pairs do not include the pairs consisting of an aircraft paired with itself.

Let us examine the difference between $\overline{V_r}$ and $\overline{V_r}'$ in the examples shown in Figure 3-9 and 3-10. $\overline{V_r}$ in Figure 3-9 is

$$\frac{1}{N^2} \sum_{i=1}^{N} \sum_{j=1}^{N} |\vec{V_i} - \vec{V_j}| = V_0$$

whereas

$$\overline{V_r}' = \frac{1}{N(N-1)} \sum_{i=1}^{N} \sum_{j=1}^{N} |\vec{V_i} - \vec{V_j}| = 2 V_0$$

$\overline{V_r}$ in Figure 3-10 is V_0, and $\overline{V_r}' = \frac{4}{3} V_0$.

The collision rate in Figure 3-9 is

$$C = \frac{N^2}{2} \frac{2g\,\overline{V_r}}{A_1}$$

$$= 2 \times \frac{2g\,V_0}{A_1} = \frac{4g\,V_0}{A_1} \qquad (3\text{-}26)$$

This is the same as (3-17).

The collision rate in Figure 3-10 is

$$C = \frac{N^2}{2} \frac{2g\,\overline{V_r}}{2A_1}$$

$$= 8 \times \frac{2g\,V_0}{2A_1} = \frac{8g\,V_0}{A_1} \qquad (3\text{-}27)$$

This is the same as (3-19).

These examples show that $E'(V_r)$ is a function of the expected number of aircraft. If this fact is recognized, the formula $C = \frac{N(N-1)}{2} \frac{2g\,E'(V_r)}{A}$ offers no problem. However, this fact has been overlooked in the past in using the formula, and it is inconvenient to use $E'(V_r)$ because $E'(V_r)$ is a function of the expected numbers of aircraft. Therefore, from now on the preferred formula $C\,(F_H) = \frac{N^2 g\,E(V_r)}{A}$ will be used (and $C = \frac{N(N-1)}{2} \frac{2g\,E'(V_r)}{A}$ will not be used) to estimate collision rates in this paper.

3.3 Special Cases

The gas model developed by Graham[6] and Flanagan[7] assumes that directions of aircraft velocity are uniformly distributed between 0 and

2π.[see also (3-11)]. However, this assumption is not necessarily a good one because destinations of aircraft are not uniformly distributed.

In this section, the general formula for the collision rate for non-uniform probability distributions of aircraft direction is developed, and the collision rates for constant velocity (i.e., for when the magnitude of velocity is constant) are calculated for some special distributions of aircraft direction. Some numerical examples for cases in which the magnitude of velocities is a random variable are also given.

The general formula for the collision rate is given by (3-7).

$$C = \frac{N^2 g\, E(V_r)}{A} \int_G^{G+H} P_z(z) \int_{z-h}^{z+h} P_z(u)\, du\, dz \qquad (3\text{-}7)$$

$E(V_r)$ is given by the following formula.

$$E(V_r) = \int_{V_1} \int_{V_2} \int_0^{2\pi} \int_0^{2\pi} \left(V_1^2 + V_2^2 - 2V_1 V_2 \cos(\theta_1 - \theta_2) \right)^{\frac{1}{2}} \cdot$$

$$P_\theta(\theta_1)\, P_\theta(\theta_2)\, P_V(V_1)\, P_V(V_2)\, d\theta_1 d\theta_2 dV_1 dV_2 \qquad (3\text{-}28)$$

where

θ_i = directional angle of velocity of aircraft i

V_i = magnitude of velocity of aircraft i

$P_\theta(\theta_i)$ = probability density function of θ_i

$P_V(V_i)$ = probability density function of V_i

The collision rate for any distribution of direction and magnitude of aircraft velocity is given by (3-7) and (3-28).

We calculate first the collision rate for constant velocity (the magnitude of aircraft velocity is constant).

Suppose that the magnitude of aircraft velocity is a constant .

Then,

$$E(V_r) = \int_0^{2\pi}\int_0^{2\pi} \left(2V_0^2 - 2V_0^2\cos(\theta_1-\theta_2)\right)^{\frac{1}{2}} P_\theta(\theta_1)P_\theta(\theta_2)\,d\theta_1 d\theta_2$$

$$= 2V_0 \int_0^{2\pi}\int_0^{2\pi} \left|\sin\frac{\theta_1-\theta_2}{2}\right| P_\theta(\theta_1)P_\theta(\theta_2)\,d\theta_1 d\theta_2 \qquad (3\text{-}29)$$

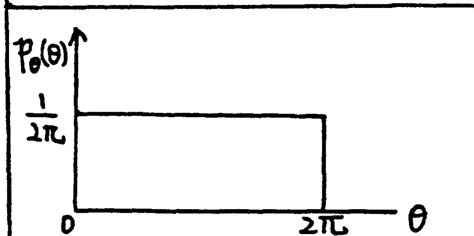
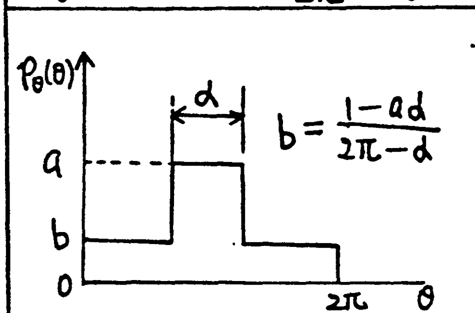
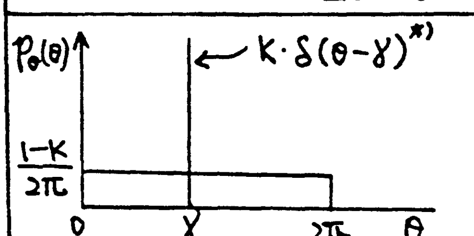
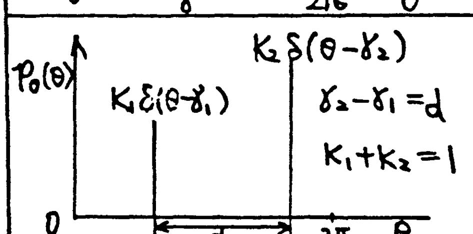
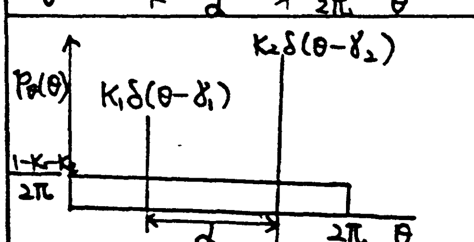
$E(V_r)$ for some special probability density functions of direction is tabulated in Table 3-1. It is assumed that horizontally aircraft are uniformly distributed. The corresponding collision rate is given by (3-7). If positions of aircraft are uniformly distributed over altitude and the height of aircraft is negligibly small compared with the thickness of the altitude layer under consideration, the corresponding collision rate is given by (3-8).

$$C \doteqdot \frac{N^2 g\,E(V_r)}{A}\,\frac{2h}{H} \qquad (3\text{-}8)$$

For distributions other than the uniform vertical distribution, the collision rates can be calculated numerically.

Table 3-1 shows the values of the expected relative velocities for some interesting probability density functions of velocity direction. For example, the third case in Table 3-1 is the one in which a fraction K of aircraft are flying in the same direction, while the directions of the other (1 - K) fraction are uniformly distributed over 0 and 2π. In the fourth case, aircraft are flying only in two directions. Table 3-1 shows $E(V_r)$ only for the cases when the magnitude of the velocity is constant. If the magnitude of velocity is a random variable, the integral of (3-28) can be calculated numerically.

Table 3-1 Expected Relative Velocity ($|v|$ = constant)

<table>
<tr><th>$P_\theta(\theta)$</th><th>$E(V_r)$</th></tr>
<tr><td>
</td><td>$$\frac{4V_0}{\pi}$$</td></tr>
<tr><td>
</td><td>$$8a^2V_0\left(d-2\sin\frac{d}{2}\right)+32abV_0\sin\frac{d}{2}+8b^2V_0\left(2\pi-d-2\sin\frac{d}{2}\right)\equiv V_d$$
$$\left(\begin{array}{l} V_d\rightarrow\frac{4V_0}{\pi}\ \text{when } d\rightarrow0 \text{ and } a \text{ is finite.} \\ V_d\rightarrow\frac{4V_0}{\pi}(1-K^2)\ \text{when } d\rightarrow0 \text{ keeping } ad=K \end{array}\right)$$</td></tr>
<tr><td>
</td><td>$$\frac{4V_0(1-K^2)}{\pi}$$</td></tr>
<tr><td>
</td><td>$$4K_1K_2V_0\sin\frac{d}{2}$$</td></tr>
<tr><td>
</td><td>$$\frac{4V_0}{\pi}(1-K_1-K_2)(1+K_1+K_2)+4K_1K_2V_0\sin\frac{d}{2}$$</td></tr>
</table>

*) $\delta(\theta)$ is the delta function $\int_{-\infty}^{\infty}\delta(\theta)\,d\theta=1$, $\delta(\theta)=\begin{cases}0 & \theta\neq0\\ \infty & \theta=0\end{cases}$

[Numerical Example]

We now illustrate these results through numerical examples: assume that 20 aircraft are flying horizontally in the airspace shown in Figure 3-7. Each aircraft is represented by a circular cylinder with a diameter of 150 ft. and a thickness of 50 ft.

3.3.1 Example 1

Aircraft are uniformly distributed over altitude. All aircraft are flying in the same direction and at the same velocity.

$$E(V_r) = 0$$

$$C \text{ (collision rate)} = 0$$

3.3.2 Example 2 (Overtaking)

Aircraft are uniformly distributed over altitude. All aircraft are flying in the same direction. Fifty percent of the aircraft are flying at 250 kt, and 50% at 350 kt.

$$E(V_r) = \int_{V_1} \int_{V_2} (V_1^2 + V_2^2 - 2V_1V_2)^{\frac{1}{2}} P_v(V_1) P_v(V_2)\, dV_1\, dV_2$$

$$= \frac{1}{4} \times 2 \times (350 - 250) = 50 \text{ kt}$$

$$F_H \text{ (rate of horizontal overlaps)} = \frac{N^2 g E(V_r)}{A}$$

$$= \frac{(20)^2 \times \left(\frac{150}{6080}\right) \times 50}{100 \times 100} = 0.0493 / \text{hr}$$

P_V (probability of vertical overlap) $\doteq \frac{2h}{H}$

$$= \frac{1}{100}$$

C (collision rate) = $F_H \times P_V$

$$= 4.93 \times 10^{-4} /hr$$

= 1 collision/2,027 hours

(The overtaking problem can be treated as a special case of the generalized collision problem. A brief discussion will be presented in the next section.)

3.3.3 Example 3

Aircraft are uniformly distributed over altitude. Directions of aircraft velocity are uniformly distributed over 0 and 2π. Fifty percent of the aircraft are flying at 250 kt, and 50% at 350 kt.

$$E(V_r) = \int_{V_1}\int_{V_2}\int_0^{2\pi} (V_1^2 + V_2^2 - 2V_1V_2\cos\beta)^{\frac{1}{2}} \times \frac{1}{2\pi} \times P_V(V_1)\, P_V(V_2)\, d\beta\, dV_1\, dV_2$$

(β is relative directional angle of velocity)

$$= \frac{1}{4} \times \frac{4 \times 250}{\pi} + \frac{1}{4} \times \frac{4 \times 350}{\pi}$$

$$+ \frac{1}{4} \times 2 \times \frac{1}{2\pi} \times \int_0^{2\pi} (350^2 + 250^2 - 2 \times 350 \times 250 \times \cos\beta)^{\frac{1}{2}}\, d\beta$$

= 389 (this integral is numerically calculated)

$$F_H = \frac{N^2 g\ E(V_r)}{A}$$

$$= \frac{(20)^2 \times \left(\frac{150}{6080}\right) \times 389}{100 \times 100} = 0.384$$

$$P_v \doteqdot \frac{1}{100}$$

$$C = F_H \times P_v$$

$$= 3.84 \times 10^{-3}/\text{hr}$$

3.3.4 Example 4

Aircraft are uniformly distributed over altitude. Directions of aircraft velocity are uniformly distributed over 0 and 2π. Magnitudes of aircraft velocity are uniformly distributed over 250 kt and 350 kt.

$$E(V_r) = \int_{250}^{350}\int_{250}^{350}\int_{0}^{2\pi} (V_1^2 + V_2^2 - 2V_1V_2\cos\beta)^{\frac{1}{2}} \times \frac{1}{2\pi} \times \frac{1}{100} \times \frac{1}{100}\, d\beta\, dV_1\, dV_2$$

= 384 (this integral is numerically calculated)

$$F_H = \frac{N^2 g\cdot E(V_r)}{A}$$

$$= \frac{(20)^2 \times \left(\frac{150}{6080}\right) \times 384}{100 \times 100} = 0.379$$

$$P_V \doteq \frac{1}{100}$$

$$C = F_H \times P_V$$

$$= 3.79 \times 10^{-3}/\text{hr.}$$

3.3.5 Example 5

The probability distribution of aircraft altitude is triangular as shown in Figure 3-11. The other assumptions remain the same as in Example 4.

$$P_V = \int_0^{10,000} p_z(z) \int_{z-50}^{z+50} p_z(u)\,du\,dz$$

$$\doteq 100 \int_0^{10,000} p_z^2(z)\,dz = \frac{1}{75}$$

F_H is the same as in Example 4.

$$C = F_H \times P_V$$

$$= 5.05 \times 10^{-3}/\text{hr.}$$

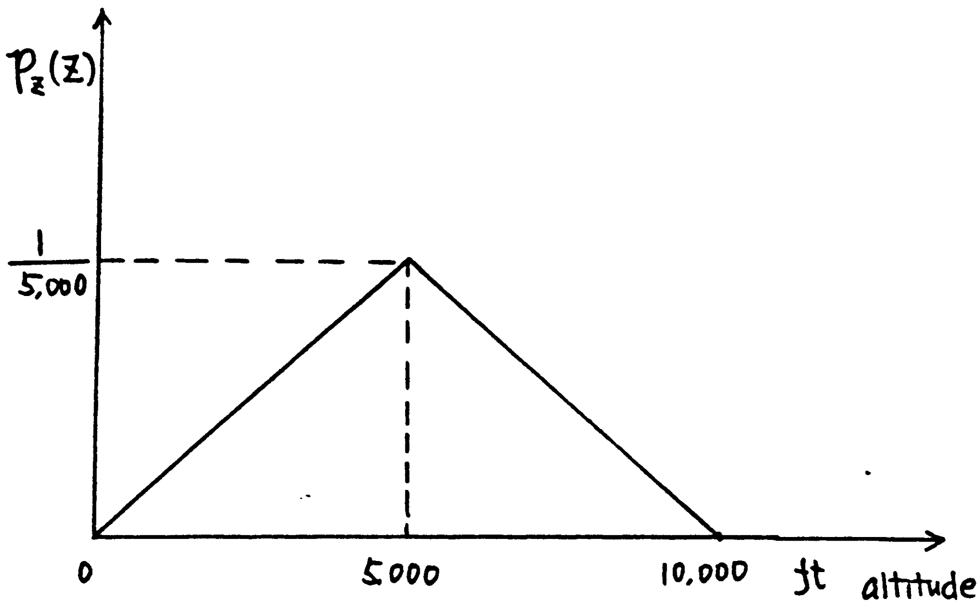


Figure 3-11 Probability Density Function of Aircraft Altitude in Example 5

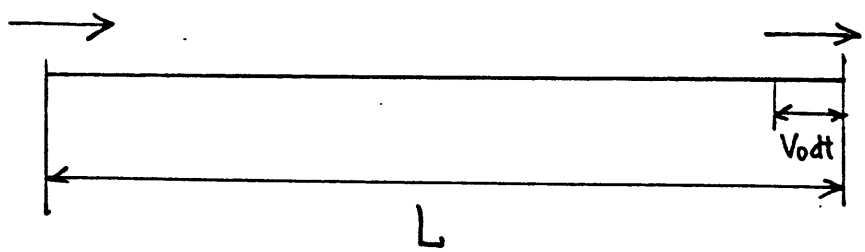


Figure 3-12 Segment of Airway

3.4 Overtaking

The overtaking problem in which aircraft are flying on a single route but not at the same velocity can be treated as a special case of the generalized collision problem. Suppose that aircraft are flying on an airway. For simplicity, the width of the airway is assumed to be zero. Then, the collision rate is the same as the overtaking rate. In this case, the expected relative velocity is

$$E(V_r) = \int_{V_1}\int_{V_2} (V_1^2 + V_2^2 - 2V_1V_2\cos 0)^{\frac{1}{2}} p_v(V_1)\, p_v(V_2)\, dV_1\, dV_2$$

$$= \int_{V_1}\int_{V_2} |V_1 - V_2|\, p_v(V_1)\, p_v(V_2)\, dV_1\, dV_2 \qquad (3\text{-}30)$$

Let us calculate the overtaking rate within the segment L of the airway, assuming N aircraft are uniformly and independently distributed over the segment.

$$T = \frac{N^2}{2L} E(V_r) \qquad (3\text{-}31)$$

where

T = overtaking rate within the segment L of the airway

N = expected number of aircraft within the segment L of the airway

It should be noted that the probability density function of velocity $p_v(V)$ is defined here in the domain of space. In other words, if one aircraft is randomly picked up from the airspace at a certain moment, the probability that its velocity is within V and $V + dV$ is given by $p_v(V)\,dV$. If $p_v(V)$ is uniform between V_1 and V_2 $(V_2 > V_1)$, the expected number of aircraft with velocity range between V_1 and V_1+dV which enter the segment is less than the expected number of aircraft with velocity range between V_2-dV and V_2 which enter the segment. Let $f_V(V)$ be the probability density function for the velocity of each aircraft entering or departing the segment. Let us consider the short portion $V_0\,dt$ from the end of the segment (Figure 3-12). The expected number of aircraft with velocity range V_0 and V_0+dV in this portion is given by

$$N \times \frac{V_0\,dt}{L} \times P_V(V_0)\,dV \tag{3-32}$$

During the period of time dt, these aircraft will depart the segment. Then, the expected total number of aircraft departing the segment ..uring dt is given by

$$\lambda\,dt \equiv \frac{N\,dt}{L}\int_{V_0} V_0\,P_V(V_0)\,dV_0 \tag{3-33}$$

where λ is the expected number of aircraft entering or departing the segment during one unit of time.

Since the expected number of aircraft with velocity range V_0 and V_0+dV which pass the end of the segment during dt is given by (3-32),

$$f_V(V)\,dV = \frac{\frac{NV\,dt}{L}\,P_V(V)\,dV}{\frac{N\,dt}{L}\int_V V\,P_V(V)\,dV}$$

$$f_V(V) = \frac{V\,P_V(V)}{\int_V V\,P_V(V)\,dV} \tag{3-34}$$

Therefore, the overtaking rate in terms of $f_V(V)$ is

$$T = \frac{N^2\left(\int_v v\, p_v(v)\,dv\right)^2}{2L} \int_{v_1}\int_{v_2} \frac{|v_1 - v_2|}{v_1 v_2} f_v(v_1) f_v(v_2)\, dv_1 dv_2$$

$$= \frac{\lambda^2 L}{2} \int_{v_1}\int_{v_2} \frac{|v_1 - v_2|}{v_1 v_2} f_v(v_1) f_v(v_2)\, dv_1 dv_2 \qquad (3\text{-}35)$$

When the overtaking rate is calculated, we should recognize which probability density function is given. If $p_v(v)$ is given, (3-30) and (3-31) should be used. On the other hand, (3-35) gives the overtaking rate when $f_v(v)$ is employed. $p_v(v)$ is defined in the domain of space, whereas $f_v(v)$ is defined in the domain of time. In order to illustrate the result of this section, a numerical example is presented below.

[Numerical Example]

The probability density function of velocity defined in the domain of space, $p_v(v)$ is uniform from 200 kt to 300 kt.

$$p_v(v) = \begin{cases} \frac{1}{100} & 200 \leq v \leq 300 \\ 0 & \text{otherwise} \end{cases}$$

Aircraft are uniformly distributed along an airway, and the expected number of aircraft is 1/10 n.m.

From (3-31), the overtaking rate within a 100 n.m.-segment of the airway is

$$T = \frac{(10)^2}{2 \times 100} \int_{200}^{300}\int_{200}^{300} |v_1 - v_2| \times \frac{1}{100} \times \frac{1}{100}\, dv_1 dv_2$$

$$= 16.7/\text{hr}.$$

Next, the overtaking rate is calculated using (3-35).

$$f_V(v) = \frac{v\, P_V(v)}{\int_V v\, P_V(v)\, dv}$$

$$= \frac{V \times \frac{1}{100}}{\int_{200}^{300} V \times \frac{1}{100}\, dV} = \frac{V}{25{,}000} \qquad (200 \le V \le 300)$$

From (3-33),

$$\lambda = \frac{N}{L} \int_{200}^{300} \frac{V}{100}\, dV = 25/\text{hr.}$$

From (3-35),

$$T = \frac{(25)^2 \times 100}{2} \int_{200}^{300} \int_{200}^{300} \frac{|V_1 - V_2|}{V_1 V_2} \times \frac{V_1}{25{,}000} \times \frac{V_2}{25{,}000}\, dV_1 dV_2$$

$$= 16.7/\text{hr.}$$

This value agrees with the value by (3-31).

3.5 Probability Density Function for the Direction of Aircraft Which Maximizes the Collision Rate

Assuming that aircraft are flying horizontally and that the density of aircraft is uniform, the probability density function which maximizes the collision rate is the uniform probability density function between 0 and 2π. In other words, collisions happen most frequently when destinations of aircraft are uniformly distributed, assuming the density of aircraft is uniform. The proof of this statement is presented below.

Since the collision rate is proportional to $E(V_r)$, the probability density function for aircraft direction which maximizes $E(V_r)$ gives the maximum collision rate.

Let θ_1, θ_2 be the directional angles of velocity of two aircraft and β the angle between the two vectors (Figure 3-13).

$P_\theta(\theta)$ and $P_\beta(\beta)$ are the probability density functions for θ and β. $P_\beta(\beta)$ is given by

$$P_\beta(\beta) = \int_0^{2\pi-\beta} P_\theta(\theta) P_\theta(\theta+\beta) d\theta + \int_{2\pi-\beta}^{2\pi} P_\theta(\theta) P_\theta(\theta+\beta-2\pi) d\theta \qquad (3\text{-}36)$$

For convenience of calculation, $P'_\theta(\theta)$ is also defined as follows

$$P'_\theta(\theta) = \begin{cases} P_\theta(\theta) & 0 \le \theta < 2\pi \\ P_\theta(\theta-2\pi) & 2\pi \le \theta < 4\pi \end{cases} \qquad (3\text{-}37)$$

Then, $P_\beta(\beta)$ is

$$P_\beta(\beta) = \int_0^{2\pi} P'_\theta(\theta) P'_\theta(\theta+\beta) d\theta \qquad (3\text{-}38)$$

$E(V_r)$ is given in terms of $P_\beta(\beta)$ by

$$E(V_r) = \int_{V_1}\int_{V_2}\int_0^{2\pi} (V_1^2+V_2^2-2V_1V_2\cos\beta)^{\frac{1}{2}} \cdot P_\beta(\beta) P_V(V_1) P_V(V_2)\, d\beta\, dV_1 dV_2 \qquad (3\text{-}10)$$

Let us examine $P_\beta(\beta)$. Since $P'_\theta(\theta)$ has a cycle of 2π, it can be expressed as a Fourier series.

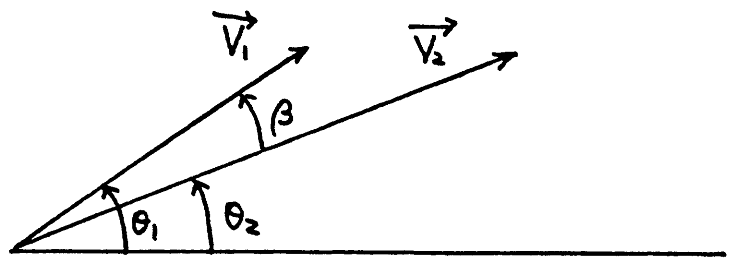


Figure 3-13 θ and β

$$P_\theta'(\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta) \qquad 0 \le \theta < 4\pi \tag{3-39}$$

where

$$a_n = \frac{1}{\pi} \int_0^{2\pi} P_\theta'(\theta) \cos n\theta \, d\theta \qquad n = 0, 1, 2, \cdots$$

$$b_n = \frac{1}{\pi} \int_0^{2\pi} P_\theta'(\theta) \sin n\theta \, d\theta \qquad n = 1, 2, \cdots \tag{3-40}$$

$$a_0 = \frac{1}{\pi}$$

From (3-39) and (3-40),

$$P_\theta'(\theta) = \frac{1}{2\pi} + \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta) \qquad 0 \le \theta < 4\pi \tag{3-41}$$

Then,

$$P_\beta(\beta) = \int_0^{2\pi} P_\theta'(\theta) P_\theta'(\theta + \beta) \, d\theta$$

$$= \frac{1}{2\pi} + \pi \sum_{n=1}^{\infty} (a_n^2 + b_n^2) \cos n\theta \qquad 0 \le \beta < 2\pi \tag{3-42}$$

From (3-10) and (3-42),

$$E(V_r) = \int_{V_1} \int_{V_2} P_V(V_1) P_V(V_2) \, dV_1 dV_2 \int_0^{2\pi} (V_1^2 + V_2^2 - 2V_1V_2 \cos\beta)^{\frac{1}{2}} \cdot$$

$$\left(\frac{1}{2\pi} + \pi \sum_{n=1}^{\infty} (a_n^2 + b_n^2) \cos n\beta \right) d\beta$$

$$= \int_{V_1}\int_{V_2} P_v(V_1)P_v(V_2)\,dV_1\,dV_2 \left\{ \frac{1}{2\pi}\int_0^{2\pi}(V_1^2+V_2^2-2V_1V_2\cos\beta)^{\frac{1}{2}}d\beta \right.$$

$$\left. + \pi\sum_{n=1}^{\infty}(a_n^2+b_n^2)\int_0^{2\pi}(V_1^2+V_2^2-2V_1V_2\cos\beta)^{\frac{1}{2}}\cos n\beta\,d\beta \right. \qquad (3\text{-}43)$$

It can be shown that $\int_0^{2\pi}(V_1^2+V_2^2-2V_1V_2\cos\beta)^{\frac{1}{2}}\cos n\beta\,d\beta$ is always negative for any set of positive values of n, V_1 and V_2. The proof is given in Appendix A. Then, $E(V_r)$ is maximum when a_n and b_n are zeroes for all n. This means that $E(V_r)$ is maximum when $P_\theta(\theta) = \frac{1}{2\pi}$.

From (3-37) and (3-40),

$$P_\theta(\theta) = \frac{1}{2\pi} + \sum_{n=1}^{\infty}(a_n\cos n\theta + b_n\sin n\theta) \qquad 0 \le \theta < 2\pi \quad (3\text{-}44)$$

If $a_n = b_n = 0$,

$$P_\theta(\theta) = \frac{1}{2\pi} \qquad (3\text{-}45)$$

This result may be contrary to our intuition. It would seem that if 50% of aircraft were flying in the same direction and 50% in exactly the opposite direction, the expected relative speed would be maximum. This is not true, however, because among each 50% of the aircraft flying in the same direction, the relative velocity is zero.

If the magnitude of velocity is constant V_0, the expected relative velocity in this case is V_0, whereas the expected relative velocity for the uniform distribution of direction is $\frac{4V_0}{\pi}$. Therefore, the collision rate becomes maximum when directions of aircraft velocity are uniformly distributed between 0 and 2π.

(NOTE) If directions of aircraft are discrete, the probability density function of $p_\theta(\theta)$ consists of impulses. For example, when 50% of aircraft are flying in the direction $\theta = 0$ and 50% in the direction $\theta = \pi$, the probability density function consists of one impulse at $\theta = 0$ and the other at $\theta = \pi$. In this case, $p_\theta(\theta)$ cannot be expressed as a Fourier series. However, even if such cases are considered, the uniform continuous (not discrete) probability distribution maximizes the expected relative velocity. In other words, the uniform probability distribution (the probability density function is $\frac{1}{2\pi}$ between 0 and 2π) maximizes the expected relative velocity for any continuous and discrete probability distributions. The proof is given in Appendix B.

3.6 Probability of Vertical Overlap

The collision rate is given by (3-1) and (3-2).

$$C = F_H \times P_V \tag{3-1}$$

where

C = collision rate

F_H = rate of horizontal overlap

P_V = probability of vertical overlap (3-2)

The maximization of F_H was discussed in Section 3.5. In this section, P_V is briefly discussed. P_V is given by (3-3) (see Figure 3-5).

$$P_V = \int_G^{G+H} p_z(z) \int_{z-h}^{z+h} p_z(u)\,du\,dz \tag{3-3}$$

where

$p_z(z)$ = probability density function of altitude of aircraft

h = height of aircraft

G = lowest altitude of the airspace under consideration

$G+H$ = highest altitude of the airspace under consideration

H = thickness of the airspace

If $H \gg h$ and $p_z(z)$ can be regarded approximately constant within an altitude range 2h, P_V can be approximated by

$$P_V \doteqdot 2h \int_G^{G+H} p_z^2(z)\,dz \tag{3-46}$$

Through this approximation, it can be shown that the probability of vertical overlap becomes minimum when aircraft are uniformly distributed over altitude.

$$\int_G^{G+H} p_z^2(z)\,dz - \int_G^{G+H} \left(\frac{1}{H}\right)^2 dz$$

$$= \int_G^{G+H} \left(p_z(z) + \frac{1}{H}\right)\left(p_z(z) - \frac{1}{H}\right) dz \tag{3-47}$$

If $p_z(z) \geq \frac{1}{H}$,

$$\int \left(p_z(z) + \frac{1}{H}\right)\left(p_z(z) - \frac{1}{H}\right) dz \geq \int \frac{2}{H}\left(p_z(z) - \frac{1}{H}\right) dz \tag{3-48}$$

If $p_z(z) < \frac{1}{H}$,

$$\int \left(p_z(z)+\frac{1}{H}\right)\left(\frac{1}{H}-p_z(z)\right) dz < \int \frac{2}{H}\left(\frac{1}{H}-p_z(z)\right) dz$$

$$\int \left(p_z(z)+\frac{1}{H}\right)\left(p_z(z)-\frac{1}{H}\right) dz > \int \frac{2}{H}\left(p_z(z)-\frac{1}{H}\right) dz \tag{3-49}$$

From (3-47), (3-48) and (3-49),

$$\int_G^{G+H} p_z^2(z)\,dz - \int_G^{G+H} \left(\frac{1}{H}\right)^2 dz$$
$$\geq \frac{2}{H}\int_G^{G+H} \left(p_z(z)-\frac{1}{H}\right) dz = 0 \tag{3-50}$$

Then,

$$\int_G^{G+H} p_z^2(z)\,dz \geq \int_G^{G+H} \left(\frac{1}{H}\right)^2 dz \tag{3-51}$$

Therefore, P_V is minimum when $p_z(z) = \frac{1}{H}$.

It is obvious that the probability of overlap is maximum $(P_V = 1)$ when the vertical distribution of aircraft is concentrated within the altitude range h. (All aircraft are flying with altitude between z_0 and $z_0 + h$.)

The result that the probability of overlap is minimum when the distribution of aircraft altitude is uniform contrasts with the result in Section 3.5 that the horizontal overlap rate is maximum when the distribution of aircraft velocity direction is uniform. However, this is not a surprising result, intuitively.

3.7 Collision Rate between VFR Aircraft and Aircraft on an Airway

So far, collision rates have been calculated for one type of aircraft. In other words, it has been assumed that all aircraft have the same probability distribution with respect to velocity, direction and space. In this section, a special case of collision between two different types of aircraft is analyzed. Consider the situation described below.

Assume that there are two types of aircraft. Type 1 aircraft are similar to the aircraft which have been analyzed so far. In this section, it is further assumed that type 1 aircraft are uniformly and independently distributed in the airspace. (In Section 3.1, uniformity is assumed only horizontally.) Type 1 aircraft can be regarded as "VFR aircraft". Type 2 aircraft are flying on an airway at constant velocity, and in a direction parallel to the airway. They are flying at a constant separation distance ℓ from each other (see Figure 3-14).

The expected relative velocity of type 1 and type 2 aircraft, $E(V_{r12})$, is then given by

$$E(V_{r12}) = \int_{V_1} P_v(V_1)\,dV_1 \int_0^{2\pi} (V_1^2+V_2^2-2V_1V_2\cos\theta)^{\frac{1}{2}} P_\theta(\theta)\,d\theta \qquad (3\text{-}52)$$

where

V_1 = velocity of type 1 aircraft

V_2 = velocity of type 2 aircraft = constant

θ = angle of direction of type 1 aircraft

(θ = 0 in the direction of type 2 aircraft)

$P_v(V_1)$ = probability density function of V_1

$P_\theta(\theta)$ = probability density function of θ

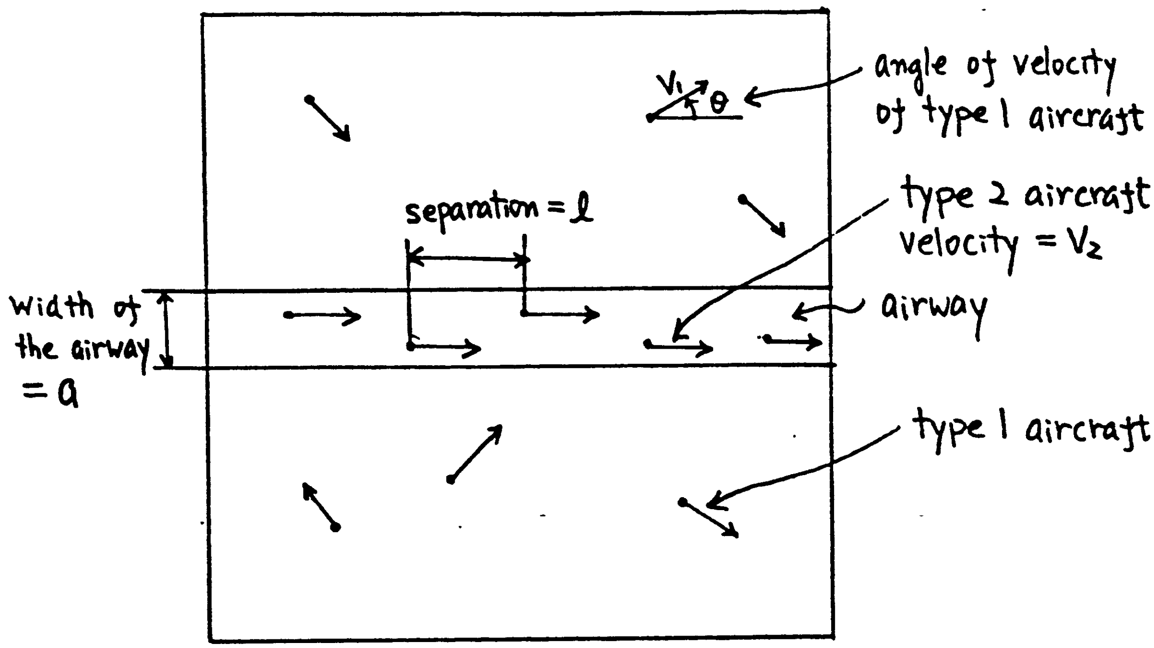


Figure 3-14 Airspace

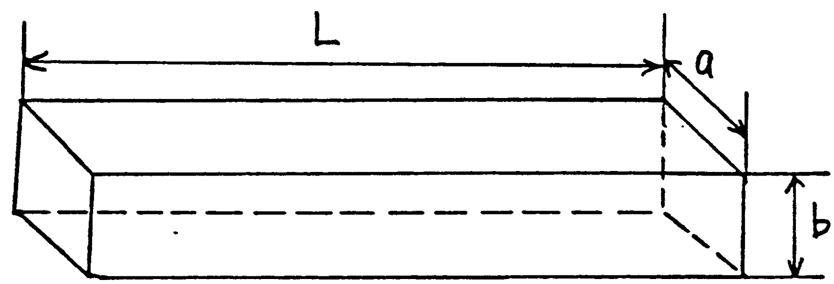


Figure 3-15 Segment of Airway

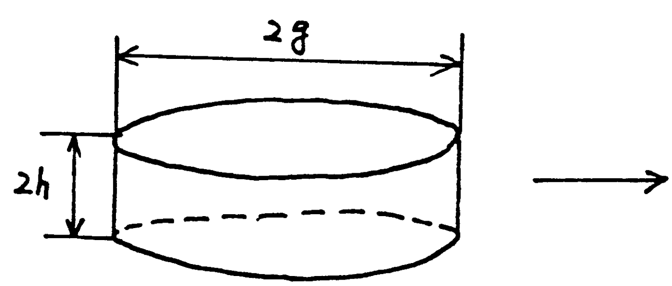


Figure 3-16 Collision Volume

The width of the airway is a, and the thickness of the airway (the altitude layer in which type 2 aircraft are flying) is b, as shown in Figure 3-15.

Each aircraft is represented as a circular cylinder with diameter g and height h as before. The occurrence of a collision is equivalent to the event that the center of an aircraft enters the cylinder of another aircraft which is twice in length and eight times in volume as large as the original cylinder, as explained in Section 3.1. Therefore, the number of collisions one type 2 aircraft is expected to have during one unit of time is the density of type 1 aircraft multiplied by the volume which the cylinder moving at the expected relative velocity generates during one unit of time (Figure 3-16).

The volume generated by the movement of the cylinder during one unit of time is given by

$$4gh\,E(V_{r12}) \tag{3-53}$$

Then, the number of collisions one type 2 aircraft is expected to have during one unit of time is

$$4gh\,E(V_{r12})\,\rho \tag{3-54}$$

where ρ is the expected number of type 1 aircraft in one unit of volume.

Since there are $\frac{L}{\ell}$ type 2 aircraft within the segment L of the airway, the collision rate is

$$C = \frac{4ghL\rho\,E(V_{r12})}{\ell} \tag{3-55}$$

It should be noted that no assumption has been made with regard to the probability distribution of the lateral and vertical position of type 2 aircraft within the airway. The rate of collisions between type 1 aircraft and type 2 aircraft is thus independent of the probability distribution of position of type 2 aircraft on the cross-section of the airway and the dimension of the airway (C is independent of a and b). The reason is that no matter where a type 2 aircraft is flying, the expected density of aircraft 1 and the expected relative velocity are constant.

[Numerical Example]

Velocity of type 1 aircraft, V_1 , is constant 300 kt.

Velocity of type 2 aircraft, V_2 , is constant 300 kt.

Horizontal dimension of aircraft, g, is 150 ft.

Vertical dimension of aircraft, h, is 50 ft.

Length of the segment of the airway, L, is 100 nm.

Separation of type 2 aircraft, ℓ , is 10 n.m.

Density of type 1 aircraft is the same as in Figure 3-7.

$$(\rho = 20/(10{,}000\,\text{nm}^2 \times 10{,}000\,\text{ft}))$$

Directions of type 1 aircraft are uniformly distributed between 0 and 2π. Then

$$E(V_{r12}) = \frac{4}{\pi} \times 300 \text{ kt}$$

$$= 382 \text{ kt}$$

$$C = \frac{4ghL\rho E(V_{r12})}{\ell}$$

$$= 3.77 \times 10^{-3}/\text{hr.}$$

3.8 Collision Rate at the Intersection of Two Airways

In this section, we shall discuss the collision rate at the intersection of two airways. Consider the two intersecting airways shown in Figure 3-17. Both airways have the same width a and the same height b. The probability distribution of aircraft position on the cross-section is uniform. At first, it is assumed that aircraft on the same airway are flying at the same velocity and with the same separation. The case in which separations between aircraft are given by Poisson processes will be discussed later. The separations on airway 1 and 2 are B_1 and B_2, respectively. The velocities of aircraft on airways 1 and 2 are V_1 and V_2, respectively. The angle between the two airways is α.

Consider one aircraft on airway 2 which enters the intersection shown as the shaded area in Figure 3-17. The expected number of aircraft on airway 1 which the above aircraft encounters during one unit of time is given by

$$\frac{2g\,E(V_r)}{B_1 a} \times \frac{2h}{b}. \tag{3-56}$$

The first term of (3-56) represents the rate of horizontal overlap, and the second term is the probability of vertical overlap, assuming that a and b are sufficiently large compared with g and h which are the horizontal and vertical dimensions of aircraft. Since $E(V_r)$

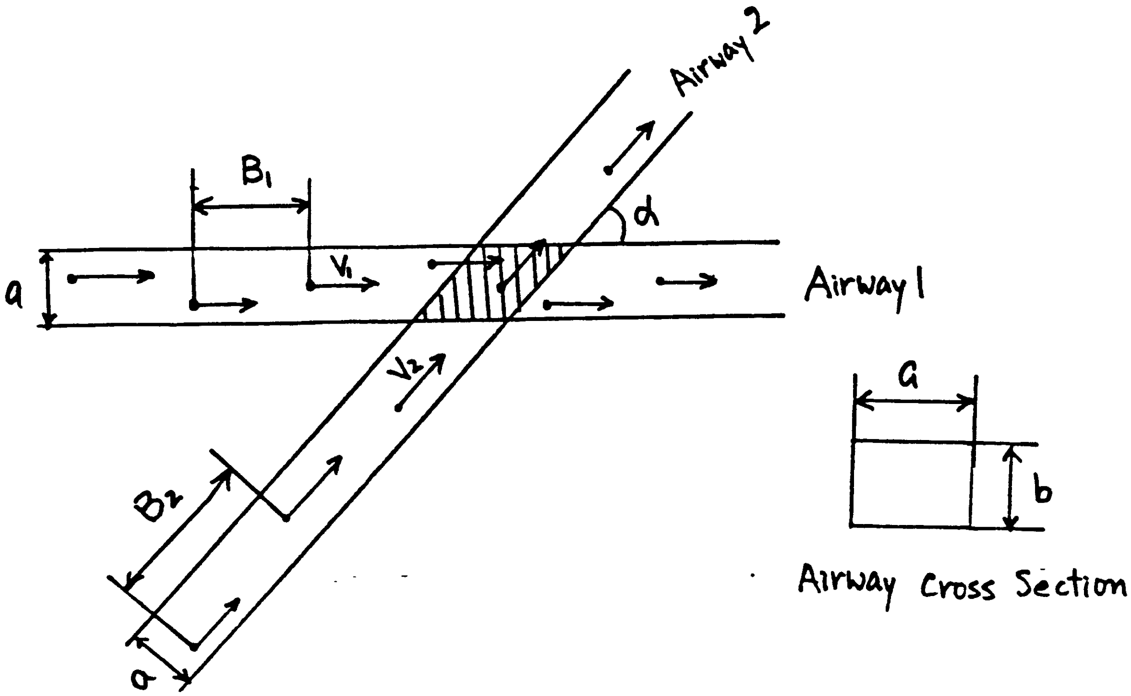


Figure 3-17 Intersection of Two Airways

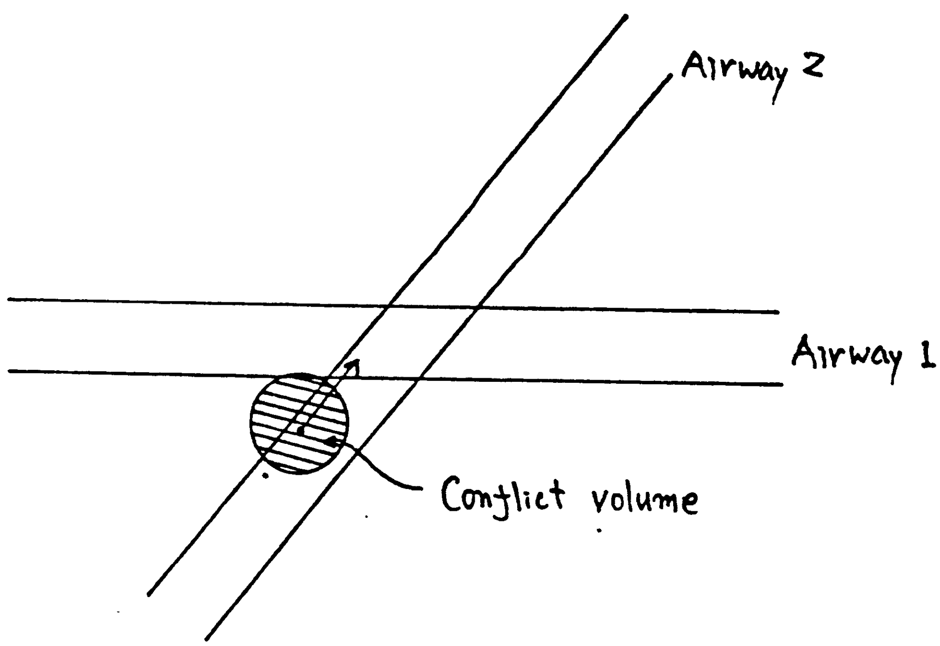


Figure 3-18 Conflict at the Intersection

is constant and given by $\left(V_1^2+V_2^2-2V_1V_2\cos\alpha\right)^{\frac{1}{2}}$, (3-56)

becomes

$$\frac{4gh\left(V_1^2+V_2^2-2V_1V_2\cos\alpha\right)^{\frac{1}{2}}}{a\,b\,B_1} \tag{3-57}$$

The expected number of aircraft 2 in the intersection is given by

$$\frac{1}{a\quad B_2}\times\frac{a^2}{\sin\alpha} = \frac{a}{B_2\sin\alpha} \tag{3-58}$$

From (3-57) and (3-58), the collision rate is given by

$$\frac{4gh\left(V_1^2+V_2^2-2V_1V_2\cos\alpha\right)^{\frac{1}{2}}}{a\,b\,B_1}\times\frac{a}{B_2\sin\alpha}$$

$$= \frac{4gh\left(V_1^2+V_2^2-2V_1V_2\cos\alpha\right)^{\frac{1}{2}}}{b\,B_1B_2\sin\alpha} \tag{3-59}$$

This is the collision rate with fixed separations. Next, we shall discuss the situation in which separations between aircraft are given by Poisson processes. Aircraft on airway 1 enter the airway with velocity V_1 according to a Poisson process with intensity λ_1. V_2 and λ_2 are similarly defined. Then, the average separations on airway 1 and 2 are given by $\frac{V_1}{\lambda_1}$ and $\frac{V_2}{\lambda_2}$. Since the two processes are independent, the collision rate is given by (3-59) substituting $\frac{V_1}{\lambda_1}$ and $\frac{V_2}{\lambda_2}$ for B_1 and B_2. Then, the collision rate is

$$\frac{4gh\lambda_1\lambda_2\left(V_1^2+V_2^2-2V_1V_2\cos\alpha\right)^{\frac{1}{2}}}{b\,V_1V_2\sin\alpha} \tag{3-60}$$

More gnerally, the collision rate at the intersection is given by the following formula.

$$C = \frac{4gh\left(V_1^2+V_2^2-2V_1V_2\cos\alpha\right)^{\frac{1}{2}}}{b\,E(B_1)\,E(B_2)\sin\alpha} \tag{3-61}$$

where

$E(B_1)$ = the expected separation distance between two consecutive aircraft on airway 1

$E(B_2)$ is similarly defined.

It is assumed that the processes of aircraft flows on the two airways are independent of each other.

Section 3.7 and this section have dealt with the rate of collisions between two different types of aircraft. The general formula for these cases will be derived in the next section.

[NOTE]

In Reference 9, Dunlay developed a conflict model which is similar to the model in this section. However, there is some difference between the two models. First, the Dunlay model assumes that the width of an airway is equal to zero, which can be treated as a special case of the model in this section. The second difference concerns the volumes involved in a collision and in a conflict. A collision is described as an event in which the volume of an aircraft overlaps the volume of another aircraft. A conflict happens when an aircraft penetrates the outer surface of the protected airspace of another aircraft. Therefore, as long as the density of aircraft is uniform in airspace, the rate of conflicts can be obtained from a collision model simply by making the appropriate change for the volume involved. In the case of the intersection problem, however, the density of aircraft is not uniform in airspace. Therefore, the collision model in this section cannot be directly applied to the rate of conflicts at the intersection because of the boundary problem (Figure 3-18). However, the rate of conflicts can be deduced from the collision model in this section.

Consider a portion of the airspace shaped like a parallelogram with the length of a side $\frac{L}{\sin\alpha}$ shown in Figure 3-19. Aircraft fly only in the two directions parallel to the sides of the airspace. The probability distribution of aircraft in the airspace is uniform. L is chosen so large that the diameter of the conflict volume can be regarded as negligibly small. The conflict rate in this case is directly calculated from (3-61), substituting the dimensions of conflict for g and h. Let us consider the cases shown in Figure 3-20. First all the aircraft flying in one direction are concentrated on an airway parallel to the aircraft direction.

Since the number of conflicts each aircraft on the airway is expected to have during one unit of time is equal to that of each aircraft flying in the direction in the original case (Figure 3-19), the conflict rates in both cases are the same. Next, all the aircraft in the other direction are concentrated on another airway parallel to the aircraft direction. Since the number of conflicts each aircraft on the second airway is expected to have during one unit of time is the same as in the previous case, the conflict rate of the last case is the same as that of the original case. Then, the conflict rate at the intersection can be obtained from (3-61) because the conflict rate of the

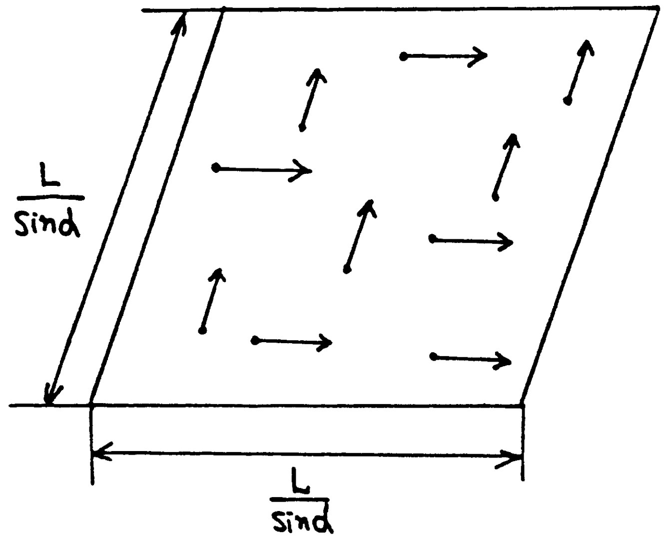


Figure 3-19 Parallelogram Airspace

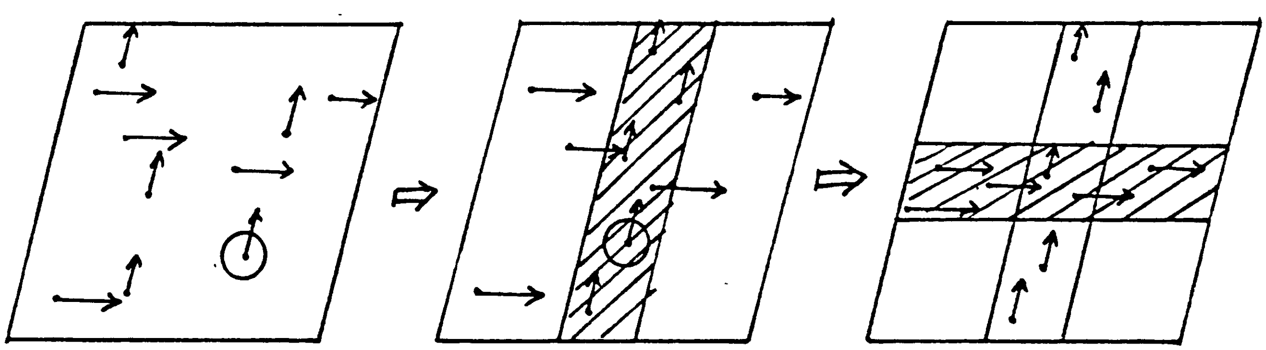


Figure 3-20 Equivalent Cases

original case is directly calculated from (3-61). It should be noted that no assumption has been made with regard to the probability distribution of cross-airway position of aircraft. Therefore, (3-61) gives the conflict rate as well as the collision rate for any probability distribution of cross-airway position of aircraft. The result of the Dunlay model can be derived as a special case of (3-61).

3.9 Three-Dimensional Gas Model

So far, it has been assumed that each aircraft travels only horizontally. In this section, the limitation is removed. The three-dimensional gas model presented here assumes that N aircraft are flying in the airspace volume B. Aircraft are uniformly and independently distributed in the airspace. The vertical velocity and the horizontal velocity of an aircraft are assumed independent of each other. No collision avoidance maneuver is taken. Each aircraft is represented as a right circular cylinder, as shown in Figure 3-2. It is assumed that this cylinder does not tilt even if its velocity has a vertical component.

A collision takes place when the center of one aircraft enters the cylinder of another aircraft shown in Figure 3-3. Therefore, the number of collisions an aircraft is expected to have during one unit of time is the density of aircraft multiplied by the volume which the cylinder moving at the relative velocity generates during one unit of time.

Let V_r be the relative velocity. V_{rv} is the vertical relative velocity or the vertical component of V_r . V_{rh} is the horizontal relative velocity or the horizontal component of V_r .

The volume the cylinder generates can be divided into two parts. One part is generated by V_{rv}, and the other by V_{rh}. The V_{rv} part is generated by the movement of the disk, as shown in Figure 3-22. The V_{rh} part is generated by the movement of the half cylinder, as shown in Figure 3-23. Then, the number of collisions an aircraft is expected to have during one unit of time is given by

$$\left(\pi g^2 E(|V_{rv}|) + 4gh\, E(|V_{rh}|)\right) \times \frac{N}{B} \tag{3-62}$$

Since there are N aircraft, the total collision rate is

$$C = \frac{N^2}{2B}\left(\pi g^2 E(|V_{rv}|) + 4gh\, E(|V_{rh}|)\right) \tag{3-63}$$

where

N = number of aircraft

B = volume of the airspace

g = horizontal dimension of aircraft

h = vertical dimension of aircraft

$E(|V_{rv}|)$ = expected vertical relative velocity

$E(|V_{rh}|)$ = expected horizontal relative velocity .

This is the formula for the rate of collisions between the same type of aircraft. When there are two different types of aircraft, the rate of collisions between different types of aircraft is similarly derived. The rate of collisions between two different types of aircraft is

$$C_{12} = \frac{N_1 N_2}{B}\left(\pi g^2 E(|V'_{rv}|) + 4gh\, E(|V'_{rh}|)\right) \tag{3-64}$$

where

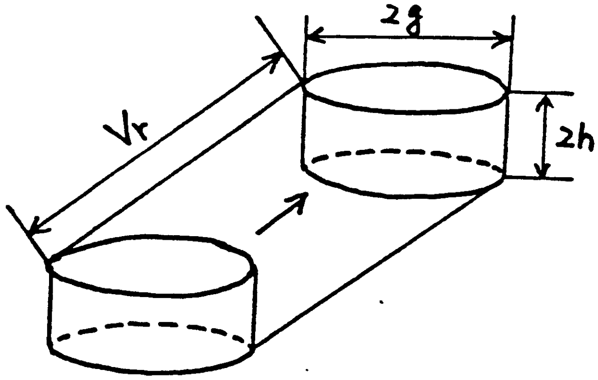


Figure 3-21 Collision Volume

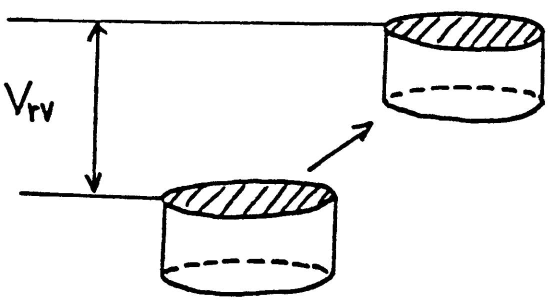


Figure 3-22 Collision Volume by Vertical Relative Velocity

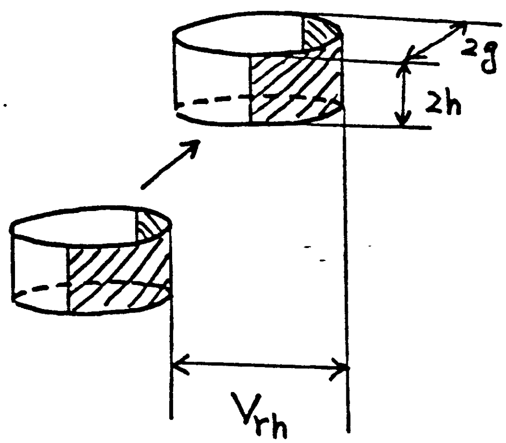


Figure 3-23 Collision Volume by Horizontal Relative Velocity

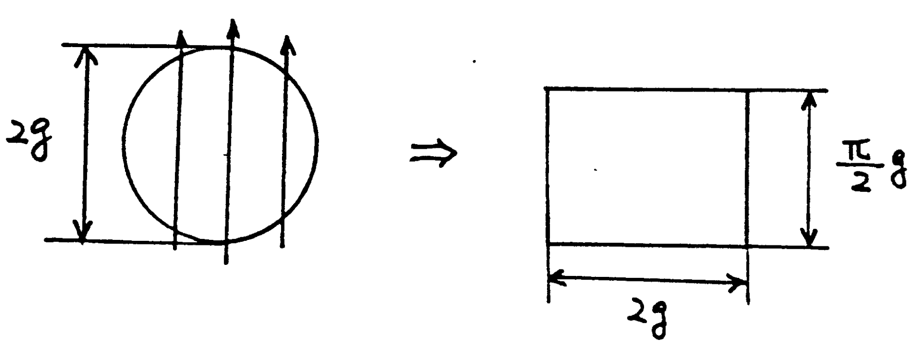


Figure 3-24 Average Time to pass the Disk

C_{12} = rate of collisions between two different types of aircraft

N_1 = number of type 1 aircraft

N_2 = number of type 2 aircraft

$E(|V'_{rv}|)$ = expected vertical relative velocity of type 1 and type 2 aircraft

$E(|V'_{rh}|)$ = expected horizontal relative velocity of type 1 and type 2 aircraft

[Numerical Example] [Rate of Collisions between the Same Type of Aircraft]

The conditions are the same as in the example of Section 3.1 except that aircraft velocity has a vertical component, as well. The probability distribution of vertical velocity is uniform between + 30 kt (climbing) and -30 kt (descending).

$$E(|V_{rh}|) = \frac{4}{\pi} \times 300 \text{ kt} = 382 \text{ kt}$$

$$E(|V_{rv}|) = \frac{60}{3} \text{ kt} = 20 \text{ kt}$$

$$\frac{N^2}{2B} \times 4gh\, E(|V_{rh}|) = \frac{N^2}{2AH} 4gh\, E(|V_{rh}|)$$

$$= \frac{N^2 g\, E(|V_{rh}|)}{A} \times \frac{2h}{H} = 3.77 \times 10^{-3}/\text{hr.}$$

$$\frac{N^2}{2B} \pi g^2 E(|V_{rv}|) = \frac{N^2}{2AH} \pi g^2 E(|V_{rv}|)$$

$$= \frac{N^2 E(|V_{rv}|)}{2H} \times \frac{\pi g^2}{A} = 4.7 \times 10^{-4}/\text{hr}$$

$$C = 4.24 \times 10^{-3}/\text{hr}$$

This value is greater than the value of the two-dimensional model due to the contribution of the expected vertical relative velocity.

This example shows that (3-63) can also be expressed as follows:

$$C = \frac{N^2 g E(|V_{rh}|)}{A} \times \frac{2h}{H} + \frac{N^2 E(|V_{rv}|)}{2H} \times \frac{\pi g^2}{A} \tag{3-65}$$

where

A = horizontal area of the airspace

H = height of the airspace (Figure 3-1)

This means

C = (horizontal overlap rate) x (probability of vertical overlap)

+ (vertical overlap rate) x (probability of horizontal overlap)

(3-66)

This way of presenting C corresponds to Reich's formula (2-1). However, it should be noted that this relationship relies on the assumption that the cylinder representing an aircraft does not tilt.

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CHAPTER 4

TERMINAL AREA COLLISION MODEL

The basic assumption of the gas model is that, horizontally, aircraft are uniformly and independently distributed and the expected relative velocity is constant in airspace. This assumption may be a good one if the airspace under consideration is distant from airports. If, however, we are considering the airspace near an airport, this assumption is a poor one. Generally speaking, the farther the airspace is from an airport, the lower the density of aircraft is likely to be. The expected relative velocity may also not be constant with distance from the airport. Furthermore, the density of aircraft and the expected relative velocity may change considerably with the time of the day. In this section, a general model for airspace close to an airport is developed, and the collision rates are calculated for a few special cases.

The rate of collisions between the same type of aircraft associated with the three-dimensional gas model is given by (3-63). (3-63) assumes that aircraft are uniformly and independently distributed and the expected relative velocities are constant. Let ρ, $\overline{V_{rv}}$ and $\overline{V_{rh}}$ be the expected density of aircraft, the expected vertical relative velocity and the expected horizontal relative velocity. Then, (3-63) becomes,

$$C = \frac{B\rho^2}{2}\left(\pi g^2 \overline{V_{rv}} + 4gh\overline{V_{rh}}\right) \qquad (4\text{-}1)$$

where

C = collision rate (expected number of collisions during one unit of time)

B = volume of the airspace

g = horizontal dimension of aircraft

h = vertical dimension of aircraft

In section 3.8, ρ, $\overline{V_{rv}}$ and $\overline{V_{rh}}$ were assumed to be constant. However, these are generally functions of the space coordinates and of time. Let x, y and z be the space coordinates shown in Figure 4-1, and let t be time. Then,

$$\begin{aligned} \rho &= \rho(x, y, z, t) \\ \overline{V_{rv}} &= \overline{V_{rv}}(x, y, z, t) \\ \overline{V_{rh}} &= \overline{V_{rh}}(x, y, z, t) \end{aligned} \tag{4-2}$$

Since ρ, $\overline{V_{rv}}$ and $\overline{V_{rh}}$ are almost constant in a very small volume dB, the expected number of collisions from time t to time t is given by

$$\frac{1}{2}\int_{t_1}^{t_2}\int_z\int_y\int_x \rho^2(x,y,z,t)\left\{\pi g^2\overline{V_{rv}}(x,y,z,t) + 4gh\overline{V_{rh}}(x,y,z,t)\right\}dx\,dy\,dz\,dt \tag{4-3}$$

(4-3) gives the expected number of collisions near an airport between t_1 and t_2.

If there are two different types of aircraft, the formula for the expected number of collisions between the different types of aircraft is similarly derived from (3-64). The expected number of collisions between two different types of aircraft between t_1 and t_2 is

$$\int_{t_1}^{t_2}\int_z\int_y\int_x \rho_1(x,y,z,t)\rho_2(x,y,z,t)\left\{\pi g^2\overline{V_{rv}'}(x,y,z,t) + 4gh\overline{V_{rh}'}(x,y,z,t)\right\}dx\,dy\,dz\,dt \tag{4-4}$$

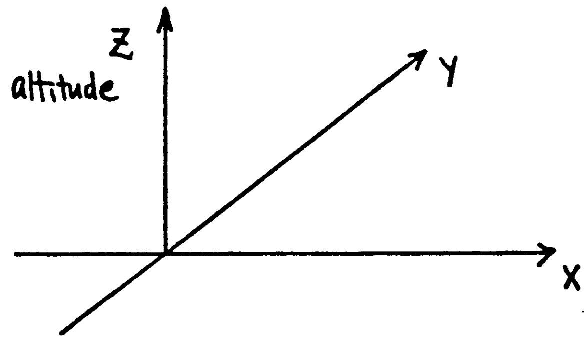

Figure 4-1 Coordinate System

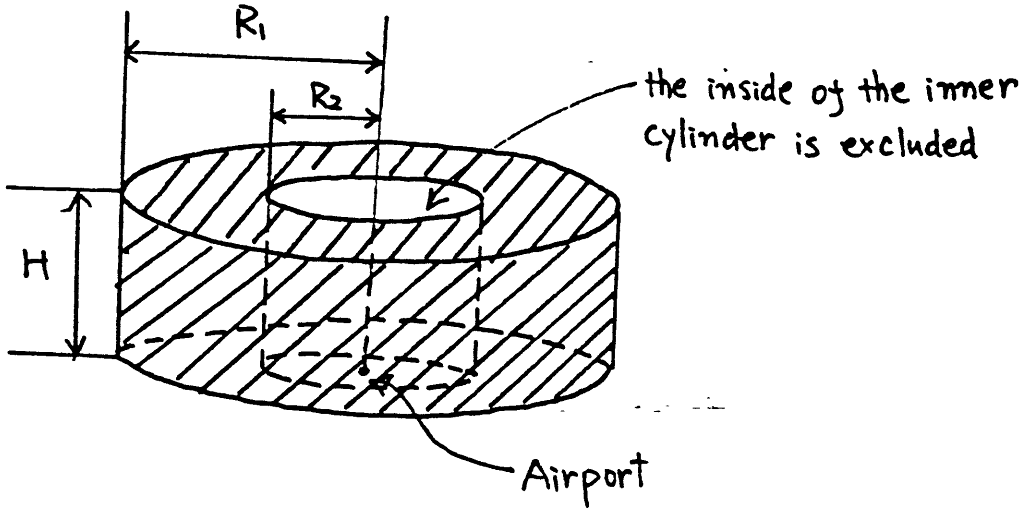

Figure 4-2 Airspace

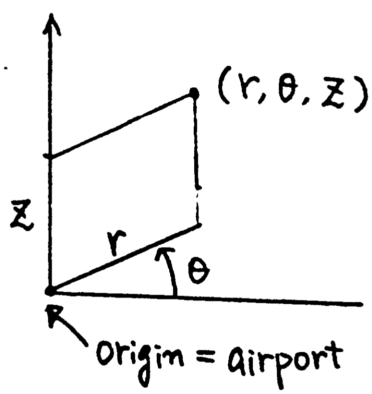

Figure 4-3 Cylindrical Coordinate System

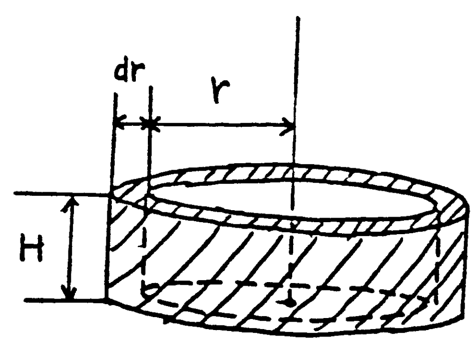

Figure 4-4 Airspace

where

$\rho_1(x, y, z, t)$ = density of type 1 aircraft

$\rho_2(x, y, z, t)$ = density of type 2 aircraft

$\overline{V'_{rv}}(x, y, z, t)$ = expected vertical relative velocity of type 1 and type 2 aircraft

$\overline{V'_{rh}}(x, y, z, t)$ = expected horizontal relative velocity of type 1 and type 2 aircraft

In order to calculate these integrals, the densities of aircraft and the expected relative velocities should be calculated first. We now illustrate the above concepts through analysis of some special cases. First a few special cases for the rate of collisions between the same type of aircraft are presented. One example of collisions between different types of aircraft will be shown later.

4.1 Special Cases: Collisions between the same type of aircraft

4.1.1 Case 1

Aircraft are approaching an airport at constant horizontal velocity V_0 . The aircraft arrival at the airport is a Poisson process with intensity λ. The airspace under consideration is shown in Figure 4-2. The airspace is the circular cylinder with diameter R_1 and height H from which the inner circular cylinder with diameter R_2 and height H is excluded.

Each aircraft is supposed to fly in the exact direction of the airport. However, due to the imperfect precision of the navigational instruments, the real direction may deviate from the supposed direction. Let the probability density function of this deviation angle be uniform from $-\gamma$ to $+\gamma$. Steady state is assumed.

For convenience of calculation, the cylindrical coordinate system is chosen. (Figure 4-3) The aircraft density is assumed to be a function of r and independent of θ and Z.

Consider a ring with width of dr. (Figure 4-4)

Assuming that all aircraft are flying through the ring, the expected number of aircraft entering the ring during a short period of time dt is λdt. dr is chosen as $V'dt$, where V' is the average velocity component in the direction of the airport. Since the expected number of aircraft leaving the ring during dt is equal to the expected number of aircraft in the ring,

$$\lambda dt = \rho \times 2\pi r H \times dr$$

$$= 2\pi H V' r \rho \, dt$$

$$\rho = \frac{\lambda}{2\pi H V' r} \tag{4-5}$$

$$V' = \int_{-\gamma}^{\gamma} V_0 \cos\beta \times \frac{1}{2\gamma} \, d\beta$$

$$= \frac{V_0 \sin\gamma}{\gamma} \tag{4-6}$$

From (4-5) and (4-6).

$$\rho = \frac{\lambda\gamma}{2\pi H V_0 \sin\gamma \; r}$$

$\overline{V_{rh}}$ can be obtained from Table 3-1 for our case.

$$\overline{V_{rh}} = \frac{4V_0}{\gamma^2}(\gamma - \sin\gamma)$$

From (4-3), the collision rate is then given by

$$C = \frac{1}{2}\int_{R_2}^{R_1} \rho^2 \times 4gh\,\overline{V_{rh}} \times 2\pi r H \, dr$$

$$C = \frac{4\lambda^2 g h(\gamma - \sin\gamma)}{\pi H V_0 \sin^2\gamma} \int_{R_2}^{R_1} \frac{1}{r}\, dr$$

$$= \frac{4\lambda^2 g h(\gamma - \sin\gamma) \log \frac{R_1}{R_2}}{\pi H V_0 \sin^2\gamma} \tag{4-7}$$

Numerical Example

λ = 10/hr

g = 150 ft

h = 50 ft

H = 5000 ft

V_0 = 200 kt = 1,216,000 ft/hr

γ = 5° = 0.0873 rad.

R_1 = 100 nm

R_2 = 50 nm

$$C = 1.59 \times 10^{-6}/\text{hr.}$$

4.1.2 Case 2

Aircraft are approaching and departing from an airport at a constant horizontal velocity V_0 . The arrival and departure processes are two independent Poisson processes with intensity λ_a and λ_d . The airspace under consideration is the same as in Case 1. Each aircraft is flying in the direction straight to or away from the airport. Steady state is assumed. It is also assumed that all aircraft are to go through the airspace of interest. $\underset{\smile}{\rho}$ is constant for a given r.

From (4-5),

$$\rho = \frac{\lambda_a + \lambda_d}{2\pi H V_0 r} \tag{4-8}$$

From (3-30), (because all aircraft fly at the same velocity V_0) $p_v(v)$ is equal to $f_v(v)$ in this case, where $p_v(v)$ and $f_v(v)$ are the probability density functions of velocity defined in the domain of space and time, respectively. Then,

$$\overline{V_{rh}} = \frac{2\lambda_a\lambda_d}{(\lambda_a+\lambda_d)^2} \times 2V_0$$

$$= \frac{4\lambda_a\lambda_d V_0}{(\lambda_a+\lambda_d)^2} \tag{4-9}$$

From (4-3), (4-8) and (4-9),

$$C = \frac{1}{2}\int_{R_2}^{R_1} \rho^2 \times 4gh\overline{V_{rh}} \times 2\pi r H \, dr$$

$$= \frac{4\lambda_a\lambda_d g h}{\pi H V_0} \log \frac{R_1}{R_2} \tag{4-10}$$

[Numerical Example]

$\lambda_a = \lambda_d = 5/\text{hr}$

$g = 150$ ft

$h = 50$ ft

$H = 5{,}000$ ft

$V_0 = 200$ kt $= 1{,}216{,}000$ ft/hr

$R_1 = 100$ n.m.

$R_2 = 50$ n.m.

$$C = 2.72 \times 10^{-5}/\text{hr}$$

4.1.3. Case 3

Aircraft are approaching an airport with horizontal velocity V, which now is a random variable. The arrival process is Poisson with intensity λ. The probability density functions of V in the space domain and time domain are $p_V(v)$ and $f_V(V)$. (see section 3.4) The other assumptions are the same as in Case 2.

The relation between $p_V(V)$ and $f_V(V)$ is given by

$$f_V(v) = \frac{V\, p_V(V)}{\int_V V\, p_V(v)\, dV} \tag{3-34}$$

$$p_V(v) = \frac{\frac{f_V(v)}{V}}{\int_V \frac{f_V(V)}{V}\, dV} \tag{4-11}$$

The expected number of aircraft entering the ring in Figure 4-4 during a period of time dt is λ dt. Since it takes $\frac{dr}{V}$ for aircraft at velocity V to go through the ring, the expected number of aircraft in the ring is

$$\int_V \lambda \frac{dr}{V} f_V(v)\, dV = \lambda\, dr \int_V \frac{f_V(v)}{V}\, dV$$

$$\rho = \frac{\lambda \int_V \frac{f_V(v)}{V}\, dV}{2\pi H r} \tag{4-12}$$

$$\overline{V_{rh}} = \int_{V_1}\int_{V_2} |V_1 - V_2|\, p_V(V_1)\, p_V(V_2)\, dV_1\, dV_2 \tag{4-13}$$

From (4-3) and (4-12),

$$C = \frac{1}{2}\int_{R_2}^{R_1} 4\rho^2 g h\, \overline{V_{rh}} \times 2\pi r H\, dr$$

$$= \frac{\lambda^2 g h\, \overline{V_{rh}} \left(\int_V \frac{f_V(v)}{V}\right)^2}{\pi H} \log \frac{R_1}{R_2} \tag{4-14}$$

, where $\overline{V_{rh}}$ is given by (4-13) and (4-11).

[Numerical Example]

λ = 10/hr

$f_v(v)$ is uniform from 195 kt to 205 kt.

g = 150 ft

h = 50 ft

H = 5,000 ft

R_1 = 100 n.m.

R_2 = 50 n.m.

$$f_v(v) = \frac{1}{10} \qquad (195 \leq V \leq 205)$$

$$P_v(v) = \frac{\frac{1}{10V}}{\int_{195}^{205} \frac{1}{10V} dV} \doteqdot \frac{20}{V} \doteqdot \frac{1}{10}$$

$$\overline{V_{rh}} = \int_{195}^{205} \int_{195}^{205} |V_1 - V_2| P_v(V_1) P_v(V_2) dV_1 dV_2$$

$$= \frac{10}{3}$$

$$C = 4.54 \times 10^{-7} / \text{hr}$$

The differences between numerical results of Cases 1, 2 and 3 are mainly due to the expected relative velocities.

Next, one special case of collisions between different types of aircraft is presented below.

4.2 Special Case: Collisions between two different types of aircraft

There are two types of aircraft. Type 1 aircraft are the same as the aircraft treated in the previous examples. Their position and velocity are independently distributed in airspace. They can be regarded as VFR aircraft. Type 2 aircraft are flying at constant velocity with constant separation on an airway which runs near the airport. Their direction of travel is parallel to the airway. The coordinate system is defined as shown in Figure 4-5. It is assumed that the density of type 1 aircraft is constant on the cross-section of the airway at a given x. It is also assumed that the probability distribution of velocity vector (a velocity vector's components are the magnitude and the direction of the velocity vector.) remains unchanged on the cross-section. These assumptions can be justified considering that the cross-section of the airway may not be large enough for these distributions to change. The separation of two consecutive type 2 aircraft is ℓ. Then from (4-4), the rate of collisions between type 1 and type 2 aircraft within the segment L of the airway is given by

$$C_{12} = \int_0^L \frac{1}{ab\ell} \times \rho_1(x) \times \left\{ \pi g^2 \overline{V'_{rv}}(x) + 4gh\overline{V'_{rh}}(x) \right\} ab\, dx$$

$$= \frac{1}{\ell} \int_0^L \rho_1(x) \left\{ \pi g^2 \overline{V'_{rv}}(x) + 4gh\overline{V'_{rh}}(x) \right\} dx \qquad (4\text{-}15)$$

This formula gives the rate of collisions between type 1 and type 2 aircraft.

[Numerical Example]

The assumptions for type 1 aircraft are the same as in Case 1 of Section 4.1.1. The airway runs through the airspace into the airport. Since the width of the airway is small compared with R , the direction of aircraft 2 is almost directly to the airport. The separation and

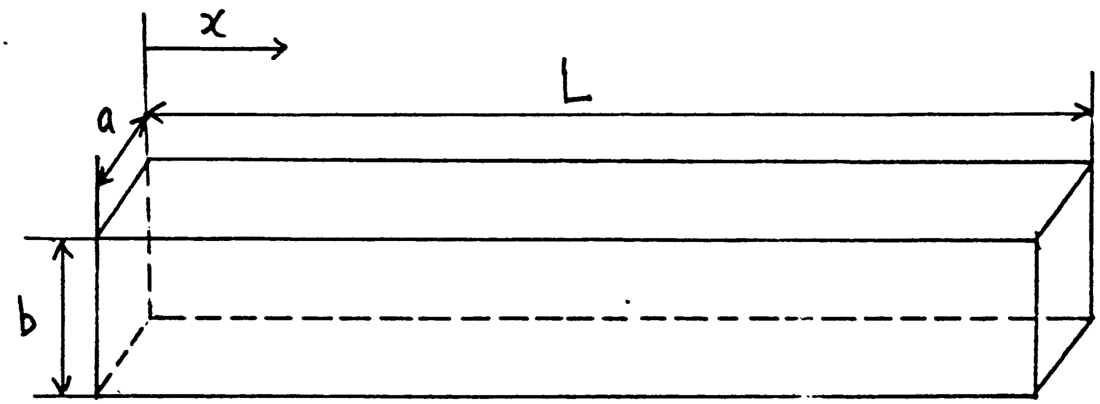

Figure 4-5 Airway

the velocity of aircraft 2 are 10 n.m. and 300 kt. The other numerical values are the same as in the numerical example of Section 4.1.1.

$$\rho_1 = \frac{\lambda \gamma}{2\pi H V_0 \sin\gamma \, r}$$

$$= \frac{0.00969}{r} \Big/ nm^3$$

$$\overline{V'_{rv}} = 0$$

$$\overline{V'_{rh}} \doteq 100 \text{ kt}$$

$$C_{12} = 5.45 \times 10^{-5} / hr$$

4.3 The Upper and Lower Bounds on the Collision Rate

So far, we have illustrated the concepts of (4-3) and (4-4) through special cases. However, these special cases include some poor assumptions, such as the one that aircraft fly only horizontally and the probability distribution of altitude of type 1 aircraft is uniform in the neighborhood of an airport. If a more realistic collision rate is necessary, (4-3) or (4-4) should be directly used. One way to estimate the integrals is to divide the airspace into small sub-spaces in which the densities of aircraft and the expected relative velocities are almost constant, and compute the integrals numerically.

However, this is difficult to do. It also takes much time to collect the data needed to estimate these distributions. In addition, as the density of aircraft and the probability distribution of velocity are closely connected with each other, the consistency of the data should be examined. Considering this situation, the following method to evaluate the collision rate may be useful.

It is difficult to find the distributions of ρ, $\overline{V_{rv}}$ and $\overline{V_{rh}}$ in (4-3) in real situations. However, it may not be so difficult to estimate or observe the maxima and minima of ρ, $\overline{V_{rv}}$ and $\overline{V_{rh}}$ in a given airspace. Let the maxima and minima of ρ, $\overline{V_{rv}}$ and $\overline{V_{rh}}$ be ρ_{max}, ρ_{min}, $\overline{V_{rvmax}}$, $\overline{V_{rvmin}}$, $\overline{V_{rhmax}}$ and $\overline{V_{rhmin}}$. Then, the upper bound on the rate of collisions between the same type of aircraft, , is given by

$$C \leq \frac{1}{2} \rho_{max}^2 (\pi g^2 \overline{V_{rvmax}} + 4gh \overline{V_{rhmax}}) B \tag{4-16}$$

where B is the volume of the airspace.

The lower bound on C is calculated as follows:

$$\bar{\rho} \equiv \frac{1}{B} \int_z \int_y \int_x \rho \, dx\,dy\,dz \tag{4-17}$$

If $\rho \geq \bar{\rho}$,

$$\rho^2 - \bar{\rho}^2 = (\rho + \bar{\rho})(\rho - \bar{\rho}) \geq 2\bar{\rho}(\rho - \bar{\rho})$$

If $\rho < \bar{\rho}$,

$$(\rho + \bar{\rho})(\bar{\rho} - \rho) < 2\bar{\rho}(\bar{\rho} - \rho)$$

$$(\rho + \bar{\rho})(\rho - \bar{\rho}) > 2\bar{\rho}(\rho - \bar{\rho})$$

Therefore,

$$\int_z \int_y \int_x (\rho^2 - \bar{\rho}^2)\,dx\,dy\,dz \geq 2\bar{\rho} \int_z \int_y \int_x (\rho - \bar{\rho})\,dx\,dy\,dz = 0 \tag{4-18}$$

Then,

$$C \geqq \frac{1}{2}\left(\pi g^2 \overline{V_{rv}}_{min} + 4gh\overline{V_{rh}}_{max}\right)\int_z\int_y\int_x \rho^2\,dx\,dy\,dz$$

$$\geqq \frac{1}{2}\bar{\rho}^2\left(\pi g^2 \overline{V_{rv}}_{min} + 4gh\overline{V_{rh}}_{max}\right) B \qquad (4\text{-}19)$$

The upper and lower bounds on the rate of collisions between the same type of aircraft is thus given by (4-16) and (4-19).

[Numerical Example]

The airspace is 100 n.m. x 100 n.m. x 1 n.m.

ρ, $\overline{V_{rv}}$ and $\overline{V_{rh}}$ are estimated as follows:

$$\bar{\rho} = 0.002 \text{ aircraft/nm}^3$$

$$\rho \leqq 2\bar{\rho}$$

$$10\text{ kt} \leqq \overline{V_{rv}} \leqq 20\text{ kt}$$

$$50\text{ kt} \leqq \overline{V_{rh}} \leqq 100\text{ kt}$$

$$g = 150\text{ ft} = 0.0247\text{ nm}$$

$$h = 50\text{ ft} = 0.0082\text{ nm}$$

$$C \leqq \frac{1}{2}(0.004)^2 \times (\pi \times 0.0247^2 \times 20 + 4 \times 0.0247 \times 0.082 \times 100) \times 10{,}000$$

$$= 9.55 \times 10^{-3}/\text{hr}$$

$$C \geqq \frac{1}{2}(0.002)^2 \times (\pi \times 0.0247^2 \times 10 + 4 \times 0.0247 \times 0.082 \times 50) \times 10{,}000$$

$$= 1.19 \times 10^{-3}/\text{hr}$$

$$1.19 \times 10^{-3}/\text{hr} \leqq C \leqq 9.55 \times 10^{-3}/\text{hr}$$

(4-16) and (4-19) indicate that the most important factor in estimating the collisions rate is the density of aircraft. If ρ max is estimated ten times are large as $\bar{\rho}$, the upper bound on C is greater than or equal to the lower bound multiplied by 100. Therefore, if the airspace is to be sub-divided in order to obtain a better estimate, the part of the airspace in which ρ changes considerably should be sub-divided into many parts.

The upper and lower bounds on the rate of collisions between two different types of aircraft, C_{12}, are given in a similar way by

$$C_{12} \leqq \rho_{1max}\,\rho_{2max}\left(\pi g^2\,\overline{V'_{rv}}_{max} + 4gh\,\overline{V'_{rh}}_{max}\right)B$$

$$C_{12} \geqq \rho_{1min}\,\rho_{2min}\left(\pi g^2\,\overline{V'_{rv}}_{min} + 4gh\,\overline{V'_{rh}}_{min}\right)B \qquad (4\text{-}20)$$

CHAPTER 5
CONCLUSION

The gas model developed in the past had dealt only with the rate of collisions between random aircraft with a uniform probability distribution for aircraft directions. In this thesis, the gas model has been extended to a generalized form which can provide estimates of the collision rate for any probability distribution of aircraft directions and magnitude of aircraft velocity. This generalized model deals with the rate of collisions between the same type of aircraft. (All aircraft have the same probability distributions of velocity and density.) An aircraft collision model which estimates the rate of collisions between different types of aircraft was also developed. It was shown that these generalized gas models can deal with many other types of collision problems including the problems of the overtaking rate, the rate of collisions between random aircraft and aircraft on an airway and the rate of collisions at the intersection of airways. The rate of collisions between aircraft on parallel airways which the Reich model dealt with can also be obtained by a generalized gas model.

It was also proved that the uniform probability distribution of aircraft direction maximizes the collision rate and that the uniform probability distribution of aircraft position minimizes the collision rate.

In the process of developing a generalized gas model, the definition of the expected relative velocity which has been occasionally misused was made clear, and the values of the expected relative velocities for some interesting probability density functions of velocity direction were calculated.

The most generalized formulae for collision rates were discussed in Chapter 4, and the collision rates near an airport for some special cases were calculated. The collision rate cannot be calculated if the exact probability distributions of velocity and density of aircraft are not given. A method which can provide the upper bound and the lower bound of the collision rate when the exact probability distributions cannot be estimated was also discussed in Chapter 4.

All these results can be applied to conflict problems with slight modifications as explained in Chapter 1. However, it should be noted that if the protected airspace of an aircraft is large enough for the density of aircraft to change considerably, the formula for the rate of conflicts becomes more complex. This can be understood considering the fact that the occurrence of a conflict is described as the event that an aircraft penetrates the outer surface of the protected airspace of another aircraft, and that the density of aircraft on the outer surface may not be the same as at the center of the protected airspace.

In this thesis, all the numerical results were calculated under a number of simplified assumptions. If data regarding the density of aircraft and the probability distribution of the aircraft velocity vector are collected, the generalized gas model developed in this thesis can be applied to real ATC problems. This model can be employed to estimate the workload of pilots and airtraffic controllers in preventing collisions because since this model assumes no collision avoidance maneuver the collision rate provided by this model is equivalent to the frequency of actions by pilots or controllers needed to avoid actual collisions. Furthermore, this model may be helpful to evaluate the computer workload of AERA (a new type of air traffic control system the FAA is currently developing) in preventing conflicts if the necessary data are collected.

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APPENDIX A

$$I \equiv \int_0^{2\pi} (V_1^2 + V_2^2 - 2V_1V_2\cos\beta)^{\frac{1}{2}} \cos n\beta \, d\beta \qquad \text{(A-1)}$$

Case 1: $V_1 = V_2$

$$I = \sqrt{2}\, V_1 \int_0^{2\pi} (1-\cos\beta)^{\frac{1}{2}} \cos n\beta \, d\beta$$

$$= \sqrt{2}\, V_1 \int_0^{2\pi} \sqrt{2} \sin\frac{\beta}{2} \cos n\beta \, d\beta$$

$$= - \frac{8\, V_1}{(2n+1)(2n+1)} < 0 \qquad \text{(A-2)}$$

Case 2: $V_2 \neq V_2$

$$I = \frac{1}{\sqrt{V_1^2 + V_2^2}} \int_0^{2\pi} \left(1 - \frac{2V_1V_2}{V_1^2+V_2^2} \cos\beta\right)^{\frac{1}{2}} \cos n\beta \, d\beta \qquad \text{(A-3)}$$

$$V_1^2 + V_2^2 - 2V_1V_2 = (V_1 - V_2)^2 > 0 \quad (V_1 \neq V_2) \qquad \text{(A-4)}$$

Then,

$$\frac{2V_1V_2}{V_1^2 + V_2^2} < 1 \qquad \text{(A-5)}$$

$$a \equiv \frac{2V_1V_2}{V_1^2 + V_2^2}$$

$$\sqrt{V_1^2+V_2^2}\ I = \int_0^{2\pi} (1 - a\cos\beta)^{\frac{1}{2}} \cos n\beta \, d\beta \qquad \text{(A-6)}$$

By the generalized binominal expansion,

$$(1 - a\cos\beta)^{\frac{1}{2}} = 1 - \frac{1}{2} a\cos\beta - \frac{1}{8}(a\cos\beta)^2 - \frac{1\cdot 3\cdot \cdots \cdot(2K-3)}{K!\,2^K}(a\cos\beta)^K - \cdots\cdots \qquad \text{(A-7)}$$

Then,

$$\int_0^{2\pi} (1 - a\cos\beta)^{\frac{1}{2}} \cos n\beta \, d\beta$$

$$= \int_0^{2\pi} \cos n\beta \, d\beta - \frac{1}{2}\int_0^{2\pi} \cos n\beta \cos\beta \, d\beta - \cdots$$

$$\cdots\cdots - \frac{1\cdot 3\cdot \cdots\cdots (2K-3)}{K!\,2^K} a^K \int_0^{2\pi} \cos^K\beta \cos n\beta \, d\beta$$

$$= -\frac{1}{2} a \int_0^{2\pi} \cos\beta \cos n\beta \, d\beta - \cdots\cdots - \frac{1\cdot 3\cdot \cdots\cdot (2K-3)}{K!\,2^K} a^K \int_0^{2\pi} \cos^K\beta \cos n\beta \, d\beta \qquad \text{(A-8)}$$

In order to examine $\int_0^{2\pi} \cos^K\beta \cos n\beta \, d\beta$, the value of $\int_0^{2\pi} \cos^{\ell}\beta \, d\beta$ is first computed.

$$\int_0^{2\pi} \cos^{2P}\beta \, d\beta = \frac{(2P-1)!! \times 2\pi}{(2P)!!} \qquad P = 0, 1, 2, \cdots\cdots \qquad \text{(A-9)}$$

where

$$(2p)!! = 2p(2p-2)\cdots\cdots 4\cdot 2$$

$$(2p-1)!! = (2p-1)(2p-3)\cdots\cdots 5\cdot 3\cdot 1$$

$$0!! = (-1)!! = 1$$

$$\int_0^{2\pi} \cos^{2p+1}\beta \, d\beta = 0 \qquad p = 0, 1, 2, \cdots \tag{A-10}$$

Next, the value of $\int_0^{2\pi} \cos^K\beta \cos n\beta \, d\beta$ is computed.

$$\int_0^{2\pi} \cos^K\beta \cos n\beta \, d\beta = \frac{K}{K+n} \int_0^{2\pi} \cos^{K-1}\beta \cos(n-1)\beta \, d\beta \tag{A-11}$$

If $K > n$,

$$\int_0^{2\pi} \cos^K\beta \cos n\beta \, d\beta = \frac{K}{K+n} \cdot \frac{K-1}{(K-1)+(n-1)} \cdots\cdots \frac{K-n+1}{K-n+2} \int_0^{2\pi} \cos^{K-n} \tag{A-12}$$

This value is given by (A-9) and (A-10), and is non-negative.

If K = N,

$$\int_0^{2\pi} \cos^k\beta \cos n\beta \, d\beta = \frac{K}{K+n} \, \frac{K-1}{(K-1)+(n-1)} \cdots\cdots \frac{1}{2} \times 2\pi \tag{A-13}$$

If K < N,

$$\int_0^{2\pi} \cos^k\beta \cos n\beta \, d\beta = 0 \tag{A-14}$$

From (A-12), (A-13) and (A-14),

$$\int_0^{2\pi} \cos^k\beta \cos n\beta \, d\beta \geq 0 \qquad \begin{array}{l} K = 1, 2, \cdots \\ n = 1, 2, \cdots \end{array} \tag{A-15}$$

From (A-6), (A-8) and (A-15),

$$\int_0^{2\pi} (V_1^2 + V_2^2 - 2V_1V_2 \cos\beta)^{\frac{1}{2}} \cos n\beta \, d\beta < 0 \tag{A-16}$$

So far, it has been proved that the integral (A-1) is always negative. Furthermore, its numerical value can be obtained through the above formulae. Let's calculate the integral for $a = 0.5$ and $n = 2$.

From (A-8), (A-9), (A-10), (A-12), (A-13) and (A-14),

$$\int_0^{2\pi} (1 - 0.5\cos\beta)^{\frac{1}{2}} \cos 2\beta \, d\beta$$

$$= -\frac{1}{2} \times 0.5 \times \int_0^{2\pi} \cos\beta \cos 2\beta \, d\beta - \frac{1}{8} \times (0.5)^2 \times \int_0^{2\pi} \cos^2\beta \cos 2\beta \, d\beta$$

$$- \frac{1}{16} \times (0.5)^3 \times \int_0^{2\pi} \cos^3\beta \cos 2\beta \, d\beta - \frac{5}{128} \times (0.5)^4 \times \int_0^{2\pi} \cos^4\beta \cos 2\beta \, d\beta - \cdots$$

$$= -0.0491 - 0.038 - 0.005 - \cdots\cdots$$

$$= -0.0534 - d$$

This integral has also been numerically calculated through the computer, and its value is -0.0535, which agrees well with the above result.

APPENDIX B

$P_\theta(\theta)$ can usually be expanded as a Fourier series. But, if $P_\theta(\theta)$ consists of impulses (delta functions), it cannot be expanded as a Fourier series.

Assume $P_\theta(\theta) = \delta(\theta)$, where $\delta(\theta)$ is the delta function.

$$\delta(\theta) = \begin{cases} 0 & \theta \neq 0 \\ \infty & \theta = 0 \end{cases}$$

$$\int_{-\infty}^{\infty} \delta(\theta)\, d\theta = 1 \tag{B-1}$$

Let $F(\theta)$ be the Fourier expansion of $\delta(\theta)$.

$$F(\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta) \tag{B-2}$$

where

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \delta(\theta) \cos n\theta \, d\theta = \frac{1}{\pi} \tag{B-3}$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \delta(\theta) \sin n\theta \, d\theta = 0 \tag{B-4}$$

From (B-2), (B-3) and (B-4),

$$F(\theta) = \frac{1}{2\pi} + \frac{1}{\pi} \sum_{n=1}^{\infty} \cos n\theta \tag{B-5}$$

$F(\theta)$ does not converge, and is not equal to $\delta(\theta)$. Therefore, it would seem that the proof given in 3.5 is not valid in this case. However, in fact, the result of 3.5 can be applied even if $P_\theta(\theta)$ consists of impulses. The proof is given below.

Let's consider a function $g(\theta)$ which can be expanded as a Fourier series. $g(\theta)$ has a cycle of 2π. Then,

$$g(\theta) = \frac{A_0}{2} + \sum_{n=1}^{\infty}(A_n \cos n\theta + B_n \sin n\theta) \qquad \text{(B-6)}$$

where

$$A_n = \frac{1}{\pi}\int_{-\pi}^{\pi} g(\theta)\cos n\theta \, d\theta \qquad \text{(B-7)}$$

$$B_n = \frac{1}{\pi}\int_{-\pi}^{\pi} g(\theta)\sin n\theta \, d\theta \qquad \text{(B-8)}$$

From (B-5) and (B-7),

$$\begin{aligned}\int_{-\pi}^{\pi} F(\theta)g(\theta)\,d\theta &= \frac{1}{2\pi}\int_{-\pi}^{\pi} g(\theta)\,d\theta + \frac{1}{\pi}\sum_{n=1}^{\infty}\int_{-\pi}^{\pi} g(\theta)\cos n\theta\,d\theta \\ &= \frac{A_0}{2} + \frac{1}{\pi}\sum_{n=1}^{\infty} A_n \qquad \text{(B-9)}\end{aligned}$$

From (B-6) and (B-9),

$$\int_{-\pi}^{\pi} F(\theta)g(\theta)\,d\theta = g(0) \qquad \text{(B-10)}$$

Since $F(\theta)$ and $g(\theta)$ have a cycle of 2π,

$$\int_{-\pi}^{\pi} F(\theta+\alpha_0)g(\theta)\,d\theta = \int_{-\pi}^{\pi} F(\theta)g(\theta-\alpha_0)\,d\theta \qquad \text{(B-11)}$$

From (B-10) and (B-11),

$$\int_{-\pi}^{\pi} F(\theta+\alpha_0)g(\theta)\,d\theta = g(-\alpha_0) \qquad \text{(B-12)}$$

From the definition of $\delta(\theta)$,

$$\int_{-\pi}^{\pi} \delta(\theta+\alpha_0)g(\theta)\,d\theta = g(-\alpha_0) \qquad \text{(B-13)}$$

From (B-12) and (B-13)

$$\int_{-\pi}^{\pi} \delta(\theta+\alpha_0)g(\theta)\,d\theta = \int_{-\pi}^{\pi} F(\theta+\alpha_0)g(\theta)\,d\theta \qquad \text{(B-14)}$$

From (B-13),

$$\int_{-\pi}^{\pi}\delta(\theta-\alpha_1)\delta(\theta-\alpha_2+\gamma)d\theta = \delta(\alpha_1-\alpha_2+\gamma) \qquad \text{(B-15)}$$

From (B-5),

$$\int_{-\pi}^{\pi}F(\theta)F(\theta+\gamma)d\theta = \int_{-\pi}^{\pi}\left(\frac{1}{2\pi}+\frac{1}{\pi}\sum_{n=1}^{\infty}\cos n\theta\right)\left(\frac{1}{2\pi}+\frac{1}{\pi}\sum_{m=1}^{\infty}\cos m(\theta+\gamma)\right)$$

$$= \frac{1}{2\pi}+\frac{1}{\pi}\sum_{n=1}^{\infty}\cos n\gamma \qquad \text{(B-16)}$$

From (B-5) and (B-16),

$$\int_{-\pi}^{\pi}F(\theta)F(\theta+\gamma)d\theta = F(\gamma) \qquad \text{(B-17)}$$

Since $F(\theta)$ has a cycle of 2π,

$$\int_{-\pi}^{\pi}F(\theta-\alpha_1)F(\theta-\alpha_2+\gamma)d\theta = \int_{-\pi}^{\pi}F(\theta)F(\theta+\alpha_1-\alpha_2+\gamma)d\theta \qquad \text{(B-18)}$$

From (B-17) and (B-18)

$$\int_{-\pi}^{\pi}F(\theta-\alpha_1)F(\theta-\alpha_2+\gamma)d\theta = F(\alpha_1-\alpha_2+\gamma) \qquad \text{(B-19)}$$

From (B-14), (B-15) and (B-19)

$$\int_{-\pi}^{\pi}g(\gamma)d\gamma\int_{-\pi}^{\pi}\delta(\theta-\alpha_1)\delta(\theta-\alpha_2+\gamma)d\theta$$

$$= \int_{-\pi}^{\pi}g(\gamma)d\gamma\int_{-\pi}^{\pi}F(\theta-\alpha_1)F(\theta-\alpha_2+\gamma)d\theta \qquad \text{(B-20)}$$

Now, let's return to the probability density function of aircraft velocity $P_\theta(\theta)$. For convenience of calculation, let's assume that $P_\theta(\theta)$ is defined for $-\pi\le\theta<\pi$. $P'_\theta(\theta)$ is also defined for $-\pi\le\theta<3\pi$.

$$P_\theta'(\theta) = \begin{cases} P_\theta(\theta) & -\pi \le \theta < \pi \\ P_\theta(\theta - 2\pi) & \pi \le \theta < 3\pi \end{cases} \qquad \text{(B-21)}$$

If the direction of aircraft velocity takes discrete values, consists of impulses. So does $P_\theta'(\theta)$. $P_\theta'(\theta)$ can be expressed as below.

$$P_\theta'(\theta) = \sum_{i=1}^{N} K_i \,\delta(\theta - d_i)$$

The expected relative velocity is,

$$E(V_r) = \int_{V_1}\int_{V_2} P_v(V_1) P_v(V_2) dV_1 dV_2 \int_{-\pi}^{\pi} (V_1^2 + V_2^2 - 2V_1V_2\cos\beta) d\beta \times$$

$$\int_{-\pi}^{\pi} P_\theta'(\theta) P_\theta'(\theta + \beta) d\theta$$

$$= \sum_{i=1}^{N}\sum_{j=1}^{N} K_i K_j \int_{V_1}\int_{V_2} P_v(V_1) P_v(V_2) dV_1 dV_2 \times$$

$$\int_{-\pi}^{\pi} (V_1^2 + V_2^2 - 2V_1V_2\cos\beta) d\beta \int_{-\pi}^{\pi} \delta(\theta - d_i)\delta(\theta - d_j + \beta) d\theta \qquad \text{(B-22)}$$

From (B-20) and (B-22)

$$E(V_r) = \int_{V_1}\int_{V_2} P_v(V_1) P_v(V_2) dV_1 dV_2 \times$$

$$\int_{-\pi}^{\pi} (V_1^2 + V_2^2 - 2V_1V_2\cos\beta) d\beta \int_{-\pi}^{\pi} H(\theta) H(\theta + \beta) d\theta \qquad \text{(B-23)}$$

where $H(\theta)$ is the Fourier expansion of $P_\theta'(\theta)$ which is the same as that of $P_\theta(\theta)$.

Without a numerical example, it may be difficult to understand what this proof means. One numerical example is given below.

Suppose that 50 percent of aircraft are flying in the direction $\theta = 0$, and 50 percent in the direction $\theta = -\pi$. The absolute value of velocity is constant Vo. In this case, $P_\theta(\theta)$ is given by,

$$P_\theta(\theta) = \frac{1}{2}\delta(\theta) + \frac{1}{2}\delta(\theta+\pi)$$

Let $H(\theta)$ be the Fourier expansion of $P_\theta(\theta)$.

$$H(\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos n\theta + b_n \sin n\theta)$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} P_\theta(\theta)\cos n\theta \, d\theta = \begin{cases} 1 & n = 2m, \ m = 0, 1, 2, \cdots \\ 0 & n = 2m+1 \end{cases}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} P_\theta(\theta)\sin n\theta \, d\theta = 0$$

Then,

$$H(\theta) = \frac{1}{2\pi} + \frac{1}{\pi}\sum_{m=1}^{\infty}\cos 2m\theta \neq P_\theta(\theta)$$

In this section, it has been proved that (3-39) is valid even if $P_\theta(\theta) \neq H(\theta)$. Let's check this result for our example. In this example, the right-hand side of (3-39) is,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}(2V_0^2 - 2V_0^2\cos\beta)^{\frac{1}{2}} d\beta + \pi\sum_{n=1}^{\infty}(a_n^2 + b_n^2)\int_{-\pi}^{\pi}(2V_0^2 - 2V_0^2\cos\beta)^{\frac{1}{2}}\cos n\beta \, d\beta$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} 2V_0 \left|\sin\frac{\beta}{2}\right| d\beta + \frac{1}{\pi}\sum_{m=1}^{\infty}\int_{-\pi}^{\pi} 2V_0\left|\sin\frac{\beta}{2}\right|\cos 2m\beta \, d\beta$$

$$= \frac{4V_0}{\pi} + \frac{4V_0}{\pi} \sum_{m=1}^{\infty} \left(\frac{1}{4m+1} - \frac{1}{4m-1} \right)$$

$$= \frac{4V_0}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = V_0$$

(From Leibniz's formula, $\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = \frac{\pi}{4}$)

This shows that even if the Fourier expansion of $p_\theta(\theta)$ does not converge and is not equal to $p_\theta(\theta)$, (3-39) is still valid in this example. The proof given in this section guarantees this relation in the general case in which $p_\theta(\theta)$ consists of impulses. This means that the uniform distribution makes the collision rate maximum even if $p_\theta(\theta)$ which contains impulses is considered. (The proof in this appendix is easily extended to the case in which $p_\theta(\theta)$ consists of impulses and a continuous function.)