15.093 Optimization Methods

Lecture 21: The Affine Scaling Algorithm

1 Outline

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- History
- Geometric intuition
- Algebraic development
- Affine Scaling
- Convergence
- Initialization
- Practical performance

2 History

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- In 1984, Karmakar at AT&T "invented" interior point method
- In 1985, Affine scaling "invented" at IBM + AT&T seeking intuitive version of Karmarkar's algorithm
- In early computational tests, A.S. far outperformed simplex and Karmarkar's algorithm
- In 1989, it was realised Dikin invented A.S. in 1967

3 Geometric intuition

3.1 Notation

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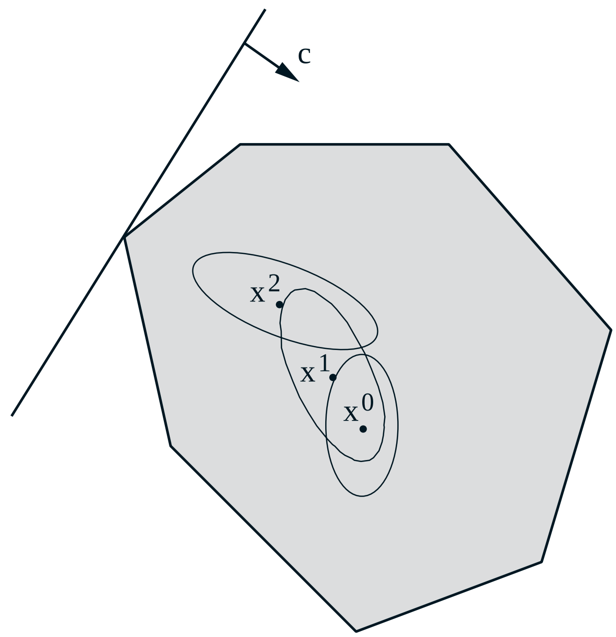
$$\begin{array}{rl} \min & \boldsymbol{c}'\boldsymbol{x} \\ \text{s.t.} & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \geq \boldsymbol{0} \end{array}$$

and its dual

$$\begin{array}{rl} \max & \boldsymbol{p}'\boldsymbol{b} \\ \text{s.t.} & \boldsymbol{p}'\boldsymbol{A} \leq \boldsymbol{c}' \end{array}$$

- $P = \{\boldsymbol{x} \mid \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b},\ \boldsymbol{x} \geq \boldsymbol{0}\}$
- $\{\boldsymbol{x} \in P \mid \boldsymbol{x} > \boldsymbol{0}\}$ the *interior* of P and its elements *interior points*



3.2 The idea

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4 Algebraic development

4.1 Theorem

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$\beta \in (0,1)$, $\boldsymbol{y} \in \Re^n$: $\boldsymbol{y} > \boldsymbol{0}$, and

$$S = \left\{ \boldsymbol{x} \in \Re^n \;\Big|\; \sum_{i=1}^{n} \frac{(x_i - y_i)^2}{y_i^2} \le \beta^2 \right\}.$$

Then, $\boldsymbol{x} > \boldsymbol{0}$ for every $\boldsymbol{x} \in S$
Proof

- $\boldsymbol{x} \in S$
- $(x_i - y_i)^2 \le \beta^2 y_i^2 < y_i^2$
- $|x_i - y_i| < y_i$; $-x_i + y_i < y_i$, and hence $x_i > 0$

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$$\boldsymbol{x} \in S \text{ is equivalent to } \big|\big|\boldsymbol{Y}^{-1}(\boldsymbol{x} - \boldsymbol{y})\big|\big| \le \beta$$

Replace original LP:

$$\begin{array}{ll} \min & \boldsymbol{c}'\boldsymbol{x} \\ \text{s.t.} & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \\ & \big|\big|\boldsymbol{Y}^{-1}(\boldsymbol{x} - \boldsymbol{y})\big|\big| \le \beta. \end{array}$$

$\boldsymbol{d} = \boldsymbol{x} - \boldsymbol{y}$

$$\begin{array}{ll} \min & \boldsymbol{c}'\boldsymbol{d} \\ \text{s.t.} & \boldsymbol{A}\boldsymbol{d} = \boldsymbol{0} \\ & ||\boldsymbol{Y}^{-1}\boldsymbol{d}|| \leq \beta \end{array}$$

4.2 Solution

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If rows of $\boldsymbol{A}$ are linearly independent and $\boldsymbol{c}$ is not a linear combination of the rows of $\boldsymbol{A}$, then

- optimal solution $\boldsymbol{d}^*$:

$$\boldsymbol{d}^* = -\beta \frac{\boldsymbol{Y}^2(\boldsymbol{c} - \boldsymbol{A}'\boldsymbol{p})}{\big|\big|\boldsymbol{Y}(\boldsymbol{c} - \boldsymbol{A}'\boldsymbol{p})\big|\big|}, \quad \boldsymbol{p} = (\boldsymbol{A}\boldsymbol{Y}^2\boldsymbol{A}')^{-1}\boldsymbol{A}\boldsymbol{Y}^2\boldsymbol{c}.$$

- $\boldsymbol{x} = \boldsymbol{y} + \boldsymbol{d}^* \in P$
- $\boldsymbol{c}'\boldsymbol{x} = \boldsymbol{c}'\boldsymbol{y} - \beta\big|\big|\boldsymbol{Y}(\boldsymbol{c} - \boldsymbol{A}'\boldsymbol{p})\big|\big| < \boldsymbol{c}'\boldsymbol{y}$

4.2.1 Proof

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- $\boldsymbol{A}\boldsymbol{Y}^2\boldsymbol{A}'$ is invertible;if not, there exists some $\boldsymbol{z} \neq \boldsymbol{0}$ such that $\boldsymbol{z}'\boldsymbol{A}\boldsymbol{Y}^2\boldsymbol{A}'\boldsymbol{z} = 0$
- $\boldsymbol{w} = \boldsymbol{Y}\boldsymbol{A}'\boldsymbol{z}$; $\boldsymbol{w}'\boldsymbol{w} = 0 \;\Rightarrow\; \boldsymbol{w} = \boldsymbol{0}$
- Hence $\boldsymbol{A}'\boldsymbol{z} = \boldsymbol{0}$ contradiction
- Since $\boldsymbol{c}$ is not a linear combination of the rows of $\boldsymbol{A}$, $\boldsymbol{c} - \boldsymbol{A}'\boldsymbol{p} \neq \boldsymbol{0}$ and $\boldsymbol{d}^*$ is well defined
- $\boldsymbol{d}^*$ feasible

$$\boldsymbol{Y}^{-1}\boldsymbol{d}^* = -\beta \frac{\boldsymbol{Y}(\boldsymbol{c} - \boldsymbol{A}'\boldsymbol{p})}{\big|\big|\boldsymbol{Y}(\boldsymbol{c} - \boldsymbol{A}'\boldsymbol{p})\big|\big|} \Rightarrow ||\boldsymbol{Y}^{-1}\boldsymbol{d}^*|| = \beta$$

  $\boldsymbol{A}\boldsymbol{d}^* = \boldsymbol{0}$, since $\boldsymbol{A}\boldsymbol{Y}^2(\boldsymbol{c} - \boldsymbol{A}'\boldsymbol{p}) = \boldsymbol{0}$

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$$\begin{aligned} \boldsymbol{c}'\boldsymbol{d} &= (\boldsymbol{c}' - \boldsymbol{p}'\boldsymbol{A})\boldsymbol{d} \\ &= (\boldsymbol{c}' - \boldsymbol{p}'\boldsymbol{A})\boldsymbol{Y}\boldsymbol{Y}^{-1}\boldsymbol{d} \\ &\geq -\big|\big|\boldsymbol{Y}(\boldsymbol{c} - \boldsymbol{A}'\boldsymbol{p})\big|\big| \cdot ||\boldsymbol{Y}^{-1}\boldsymbol{d}|| \\ &\geq -\beta\big|\big|\boldsymbol{Y}(\boldsymbol{c} - \boldsymbol{A}'\boldsymbol{p})\big|\big|. \end{aligned}$$

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$$\begin{aligned} \boldsymbol{c}'\boldsymbol{d}^* &= (\boldsymbol{c}' - \boldsymbol{p}'\boldsymbol{A})\boldsymbol{d}^* \\ &= -(\boldsymbol{c}' - \boldsymbol{p}'\boldsymbol{A})\beta \frac{\boldsymbol{Y}^2(\boldsymbol{c} - \boldsymbol{A}'\boldsymbol{p})}{\big|\big|\boldsymbol{Y}(\boldsymbol{c} - \boldsymbol{A}'\boldsymbol{p})\big|\big|} \\ &= -\beta \frac{\big(\boldsymbol{Y}(\boldsymbol{c} - \boldsymbol{A}'\boldsymbol{p})\big)'\big(\boldsymbol{Y}(\boldsymbol{c} - \boldsymbol{A}'\boldsymbol{p})\big)}{\big|\big|\boldsymbol{Y}(\boldsymbol{c} - \boldsymbol{A}'\boldsymbol{p})\big|\big|} \\ &= -\beta\big|\big|\boldsymbol{Y}(\boldsymbol{c} - \boldsymbol{A}'\boldsymbol{p})\big|\big|. \end{aligned}$$

- $\boldsymbol{c}'\boldsymbol{x} = \boldsymbol{c}'\boldsymbol{y} + \boldsymbol{c}'\boldsymbol{d}^* = \boldsymbol{c}'\boldsymbol{y} - \beta\big|\big|\boldsymbol{Y}(\boldsymbol{c} - \boldsymbol{A}'\boldsymbol{p})\big|\big|$

4.3 Interpretation

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- $\boldsymbol{y}$ be a nondegenerate BFS with basis $\boldsymbol{B}$
- $\boldsymbol{A} = [\boldsymbol{B} \;\; \boldsymbol{N}]$
- $\boldsymbol{Y} = \text{diag}(y_1, \ldots, y_m, 0, \ldots, 0)$ and $\boldsymbol{Y}_0 = \text{diag}(y_1, \ldots, y_m)$, then $\boldsymbol{AY} = [\boldsymbol{BY}_0 \;\; \boldsymbol{0}]$

$$\begin{aligned} \boldsymbol{p} &= (\boldsymbol{AY}^2\boldsymbol{A}')^{-1}\boldsymbol{AY}^2\boldsymbol{c} \\ &= (\boldsymbol{B}')^{-1}\boldsymbol{Y}_0^{-2}\boldsymbol{B}^{-1}\boldsymbol{BY}_0^2\boldsymbol{c}_B \\ &= (\boldsymbol{B}')^{-1}\boldsymbol{c}_B \end{aligned}$$

- Vectors $\boldsymbol{p}$ dual estimates
- $\boldsymbol{r} = \boldsymbol{c} - \boldsymbol{A}'\boldsymbol{p}$ becomes reduced costs:

$$\boldsymbol{r} = \boldsymbol{c} - \boldsymbol{A}'(\boldsymbol{B}')^{-1}\boldsymbol{c}_B$$

- Under degeneracy?

4.4 Termination

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$\boldsymbol{y}$ and $\boldsymbol{p}$ be primal and dual feasible solutions with

$$\boldsymbol{c}'\boldsymbol{y} - \boldsymbol{b}'\boldsymbol{p} < \epsilon$$

$\boldsymbol{y}^*$ and $\boldsymbol{p}^*$ be optimal primal and dual solutions. Then,

$$\begin{aligned} \boldsymbol{c}'\boldsymbol{y}^* &\le \boldsymbol{c}'\boldsymbol{y} < \boldsymbol{c}'\boldsymbol{y}^* + \epsilon, \\ \boldsymbol{b}'\boldsymbol{p}^* - \epsilon &< \boldsymbol{b}'\boldsymbol{p} \le \boldsymbol{b}'\boldsymbol{p}^* \end{aligned}$$

4.4.1 Proof

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- $\boldsymbol{c}'\boldsymbol{y}^* \le \boldsymbol{c}'\boldsymbol{y}$
- By weak duality, $\boldsymbol{b}'\boldsymbol{p} \le \boldsymbol{c}'\boldsymbol{y}^*$
- Since $\boldsymbol{c}'\boldsymbol{y} - \boldsymbol{b}'\boldsymbol{p} < \epsilon$,

$$\boldsymbol{c}'\boldsymbol{y} < \boldsymbol{b}'\boldsymbol{p} + \epsilon \le \boldsymbol{c}'\boldsymbol{y}^* + \epsilon$$

$$\boldsymbol{b}'\boldsymbol{p}^* = \boldsymbol{c}'\boldsymbol{y}^* \le \boldsymbol{c}'\boldsymbol{y} < \boldsymbol{b}'\boldsymbol{p} + \epsilon$$

5 Affine Scaling

5.1 Inputs

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- $(\boldsymbol{A},\ \boldsymbol{b},\ \boldsymbol{c})$;
- an initial primal feasible solution $\boldsymbol{x}^0 > \boldsymbol{0}$
- the optimality tolerance $\epsilon > 0$
- the parameter $\beta \in (0,1)$

5.2 The Algorithm

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1. (Initialization) Start with some feasible $\boldsymbol{x}^0 > \boldsymbol{0}$; let $k = 0$.
2. (Computation of dual estimates and reduced costs) Given some feasible $\boldsymbol{x}^k > \boldsymbol{0}$, let

$$\begin{aligned}\boldsymbol{X}_k &= \operatorname{diag}(x_1^k, \ldots, x_n^k),\\ \boldsymbol{p}^k &= (\boldsymbol{A}\boldsymbol{X}_k^2\boldsymbol{A}')^{-1}\boldsymbol{A}\boldsymbol{X}_k^2\boldsymbol{c},\\ \boldsymbol{r}^k &= \boldsymbol{c} - \boldsymbol{A}'\boldsymbol{p}^k.\end{aligned}$$

3. (Optimality check) Let $\boldsymbol{e} = (1,1,\ldots,1)$. If $\boldsymbol{r}^k \geq \boldsymbol{0}$ and $\boldsymbol{e}'\boldsymbol{X}_k\boldsymbol{r}^k < \epsilon$, then stop; the current solution $\boldsymbol{x}^k$ is primal ϵ-optimal and $\boldsymbol{p}^k$ is dual ϵ-optimal.
4. (Unboundedness check) If $-\boldsymbol{X}_k^2\boldsymbol{r}^k \geq \boldsymbol{0}$ then stop; the optimal cost is $-\infty$.
5. (Update of primal solution) Let

$$\boldsymbol{x}^{k+1} = \boldsymbol{x}^k - \beta\frac{\boldsymbol{X}_k^2\boldsymbol{r}^k}{||\boldsymbol{X}_k\boldsymbol{r}^k||}.$$

5.3 Variants

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- $||\boldsymbol{u}||_\infty = \max_i |u_i|, \quad \gamma(\boldsymbol{u}) = \max\{u_i \mid u_i > 0\}$
- $\gamma(\boldsymbol{u}) \leq ||\boldsymbol{u}||_\infty \leq ||\boldsymbol{u}||$
- *Short-step* method.
- *Long-step* variants

$$\boldsymbol{x}^{k+1} = \boldsymbol{x}^k - \beta\frac{\boldsymbol{X}_k^2\boldsymbol{r}^k}{||\boldsymbol{X}_k\boldsymbol{r}^k||_\infty}$$

$$\boldsymbol{x}^{k+1} = \boldsymbol{x}^k - \beta\frac{\boldsymbol{X}_k^2\boldsymbol{r}^k}{\gamma(\boldsymbol{X}_k\boldsymbol{r}^k)}$$

6 Convergence

6.1 Assumptions

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Assumptions A:

(a) The rows of the matrix $\boldsymbol{A}$ are linearly independent.

(b) The vector $\boldsymbol{c}$ is not a linear combination of the rows of $\boldsymbol{A}$.

(c) There exists an optimal solution.

(d) There exists a positive feasible solution.

Assumptions B:

(a) Every BFS to the primal problem is nondegenerate.

(b) At every BFS to the primal problem, the reduced cost of every nonbasic variable is nonzero.

6.2 Theorem

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If we apply the long-step affine scaling algorithm with $\epsilon = 0$, the following hold:

(a) For the Long-step variant and under Assumptions A and B, and if $0 < \beta < 1$, $\boldsymbol{x}^k$ and $\boldsymbol{p}^k$ converge to the optimal primal and dual solutions

(b) For the second Long-step variant, and under Assumption A and if $0 < \beta < 2/3$, the sequences $\boldsymbol{x}^k$ and $\boldsymbol{p}^k$ converge to some primal and dual optimal solutions, respectively

7 Initialization

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$$\begin{array}{lrcl}
\min & \boldsymbol{c}'\boldsymbol{x} & + & M x_{n+1} \\
\text{s.t.} & \boldsymbol{A}\boldsymbol{x} & + & (\boldsymbol{b} - \boldsymbol{A}\boldsymbol{e})x_{n+1} = \boldsymbol{b} \\
& (\boldsymbol{x}, x_{n+1}) \geq \boldsymbol{0} & &
\end{array}$$

8 Example

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$$\begin{array}{lrcrcl}
\max & x_1 & + & 2x_2 & & \\
\text{s.t.} & x_1 & + & x_2 & \leq & 2 \\
& -x_1 & + & x_2 & \leq & 1 \\
& x_1, x_2 \geq 0 & & & &
\end{array}$$

9 Practical Performance

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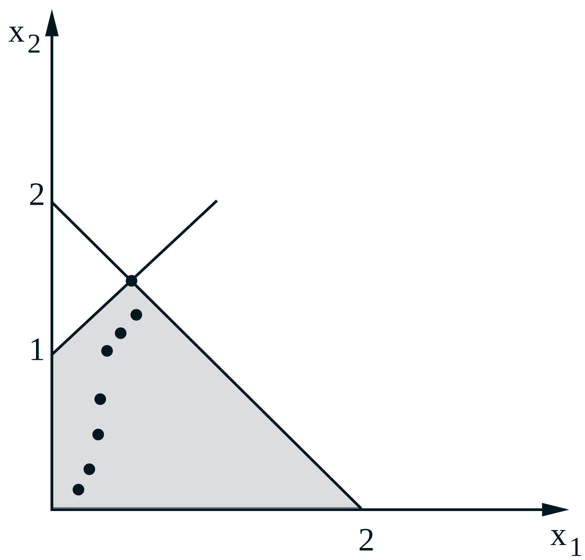
- Excellent practical performance, simple
- Major step: invert $\boldsymbol{A}\boldsymbol{X}_k^2\boldsymbol{A}'$
- Imitates the simplex method near the boundary


x2
2
1
2
x1