15.093 Optimization Methods

Lecture 18: Optimality Conditions and Gradient Methods for Unconstrained Optimization

1 Outline

Slide 1

1. Necessary and sufficient optimality conditions
2. Gradient methods
3. The steepest descent algorithm
4. Rate of convergence
5. Line search algorithms

2 Last Lecture

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Nonlinear Optimization Applications

- Portfolio Selection
- Facility Location (Geometry Problems)
- Traffic Assignment, Routing

3 The general problem

Slide 3

$$f\colon \Re^n \mapsto \Re$$

is a continuous (usually differentiable) function of n variables

$$g_i\colon \Re^n \mapsto \Re, i = 1, \ldots, m, h_j\colon \Re^n \mapsto \Re, j = 1, \ldots, l$$

$$\boxed{\begin{array}{llccc} NLP: & \min & f(\boldsymbol{x}) & & \\ & \text{s.t.} & g_1(\boldsymbol{x}) & \leq & 0 \\ & & \vdots & & \\ & & g_m(\boldsymbol{x}) & \leq & 0 \\ & & h_1(\boldsymbol{x}) & = & 0 \\ & & \vdots & & \\ & & h_l(\boldsymbol{x}) & = & 0 \end{array}}$$

3.1 Local vs Global Minima

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- $\boldsymbol{x} \in \mathcal{F}$ is a *local minimum* of NLP if there exists $\epsilon > 0$ such that $f(\boldsymbol{x}) \leq f(\boldsymbol{y})$ for all $\boldsymbol{y} \in B(\boldsymbol{x}, \epsilon) \cap \mathcal{F}$

- $\boldsymbol{x} \in \mathcal{F}$ is a *global minimum* of NLP if $f(\boldsymbol{x}) \leq f(\boldsymbol{y})$ for all $\boldsymbol{y} \in \mathcal{F}$.

4 Convex Sets and Functions

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- A subset $S \subset \Re^n$ is a *convex set* if

$$\boldsymbol{x}, \boldsymbol{y} \in S \Rightarrow \lambda \boldsymbol{x} + (1-\lambda)\boldsymbol{y} \in S \qquad \forall \lambda \in [0,1]$$

- A function $f(\boldsymbol{x})$ is a *convex function* if

$$f(\lambda \boldsymbol{x} + (1-\lambda)\boldsymbol{y}) \leq \lambda f(\boldsymbol{x}) + (1-\lambda) f(\boldsymbol{y})$$

$$\forall \boldsymbol{x}, \boldsymbol{y} \qquad \forall \lambda \in [0,1]$$

5 Convex Optimization

5.1 Convexity and Minima

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COP is called a *convex optimization problem* if $f(\boldsymbol{x}), g_1(\boldsymbol{x}), \ldots, g_m(\boldsymbol{x})$ are convex functions

This implies that the objective function is convex and the feasible region $\mathcal{F}$ is a convex set.

Implication: If *COP* is a convex optimization problem, then any local minimum will be a global minimum.

6 Optimality Conditions

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Necessary Conds for Local Optima
"If $\bar{x}$ is local optimum then $\bar{x}$ must satisfy ..."

Identifies all candidates for local optima.

Sufficient Conds for Local Optima
"If $\bar{x}$ satisfies ...,then $\bar{x}$ must be a local optimum "

7 Optimality Conditions

7.1 Necessary conditions

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Consider

$$\min_{\boldsymbol{x} \in \Re^n} f(\boldsymbol{x})$$

Zero first order variation along all directions

Theorem
Let $f(\boldsymbol{x})$ be continuously differentiable.
If $\boldsymbol{x}^* \in \Re^n$ is a local minimum of $f(\boldsymbol{x})$, then

$$\nabla f(\boldsymbol{x}^*) = \mathbf{0} \text{ and } \nabla^2 f(x^*) \text{ PSD}$$

7.2 Proof

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Zero slope at local min x^*

- $f(\boldsymbol{x}^*) \leq f(\boldsymbol{x}^* + \lambda \boldsymbol{d})$ for all $\boldsymbol{d} \in \Re^n$, $\lambda \in \Re$
- Pick $\lambda > 0$
$$0 \leq \frac{f(\boldsymbol{x}^* + \lambda \boldsymbol{d}) - f(\boldsymbol{x}^*)}{\lambda}$$
- Take limits as $\lambda \to 0$
$$0 \leq \nabla f(\boldsymbol{x}^*)' \boldsymbol{d}, \qquad \forall \boldsymbol{d} \in \Re^n$$
- Since $\boldsymbol{d}$ arbitrary, replace with $-d \Rightarrow \nabla f(\boldsymbol{x}^*) = \mathbf{0}$.

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Nonnegative curvature at a local min x^*

- $f(x^* + \lambda d) - f(x^*) = \nabla f(x^*)'(\lambda d) + \frac{1}{2}(\lambda d)' \nabla^2 f(x^*)(\lambda d) + ||\lambda d||^2 R(x^*; \lambda d)$ where $R(x^*; y) \to 0$ as $y \to 0$. Since $\nabla f(x^*) = 0$,
$$= \frac{1}{2}\lambda^2 d' \nabla^2 f(x^*) d + \lambda^2 ||d||^2 R(x^*; \lambda d) \Rightarrow$$
$$\frac{f(\boldsymbol{x}^* + \lambda \boldsymbol{d}) - f(\boldsymbol{x}^*)}{\lambda^2} = \frac{1}{2} d' \nabla^2 f(x^*) d + ||d||^2 R(x^*; \lambda d)$$
If $\nabla^2 f(x^*)$ is not PSD, $\exists \bar{d}$: $\bar{d}\,' \nabla^2 f(x^*) \bar{d} < 0 \Rightarrow f(x^* + \lambda \bar{d}) < f(\bar{x})$, $\forall \lambda$ suff. small QED.

7.3 Example

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$f(x) = \frac{1}{2}x_1^2 + x_1.x_2 + 2x_2^2 - 4x_1 - 4x_2 - x_2^3$
$\nabla f(x) = (x_1 + x_2 - 4,\ x_1 + 4x_2 - 4 - 3x_2^2)$ Candidates $\boldsymbol{x}^* = (4, 0)$ and $\bar{\boldsymbol{x}} = (3, 1)$
$\nabla^2 f(\boldsymbol{x}) = \begin{bmatrix} 1 & 1 \\ 1 & 4 - 6x_2 \end{bmatrix}$
$\nabla^2 f(\boldsymbol{x}^*) = \begin{bmatrix} 1 & 1 \\ 1 & 4 \end{bmatrix}$
PSD

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$\bar{\boldsymbol{x}} = (3, 1)$
$\nabla^2 f(\bar{\boldsymbol{x}}) = \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix}$
Indefinite matrix
$\boldsymbol{x}^*$ is the only candidate for local min

7.4 Sufficient conditions

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Theorem f twice continuously differentiable. If $\nabla f(\boldsymbol{x}^*) = \mathbf{0}$ and $\nabla^2 f(\boldsymbol{x})$ PSD in $B(\boldsymbol{x}^*, \epsilon)$, then $\boldsymbol{x}^*$ is a local minimum.
Proof: Taylor series expansion: For all $\boldsymbol{x} \in B(\boldsymbol{x}^*, \epsilon)$

$$f(\boldsymbol{x}) = f(\boldsymbol{x}^*) + \nabla f(\boldsymbol{x}^*)'(\boldsymbol{x} - \boldsymbol{x}^*)$$
$$+\frac{1}{2}(\boldsymbol{x} - \boldsymbol{x}^*)'\nabla^2 f(\boldsymbol{x}^* + \lambda(\boldsymbol{x} - \boldsymbol{x}^*))(\boldsymbol{x} - \boldsymbol{x}^*)$$

for some $\lambda \in [0, 1]$

$$\Rightarrow f(\boldsymbol{x}) \geq f(\boldsymbol{x}^*)$$

7.5 Example Continued...

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At $x^* = (4, 0)$, $\nabla f(x^*) = 0$ and
$\nabla^2 f(\boldsymbol{x}) = \begin{bmatrix} 1 & 1 \\ 1 & 4 - 6x_2 \end{bmatrix}$
is PSD for $\boldsymbol{x} \in B(\boldsymbol{x}^*, \epsilon)$

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$f(x) = x_1^3 + x_2^2$ and $\nabla f(x) = (3x_1^2, 2x_2)$ $x^* = (0, 0)$
$\nabla^2 f(x) = \begin{bmatrix} 6x_1 & 0 \\ 0 & 2 \end{bmatrix}$ is not PSD in $B(\mathbf{0}, \epsilon)$
$f(-\epsilon, 0) = -\epsilon^3 < 0 = f(\boldsymbol{x}^*)$

7.6 Characterization of convex functions

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Theorem Let $f(\boldsymbol{x})$ be continuously differentiable.
Then $f(\boldsymbol{x})$ is convex if and only if

$$\nabla f(\boldsymbol{x})'(\overline{\boldsymbol{x}} - \boldsymbol{x}) \leq f(\overline{\boldsymbol{x}}) - f(\boldsymbol{x})$$

7.7 Proof

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By convexity

$$f(\lambda\overline{\boldsymbol{x}} + (1 - \lambda)\boldsymbol{x}) \leq \lambda f(\overline{\boldsymbol{x}}) + (1 - \lambda)f(\boldsymbol{x})$$
$$\frac{f(\boldsymbol{x} + \lambda(\overline{\boldsymbol{x}} - \boldsymbol{x})) - f(\boldsymbol{x})}{\lambda} \leq f(\overline{\boldsymbol{x}}) - f(\boldsymbol{x})$$

As $\lambda \to 0$,

$$\nabla f(\boldsymbol{x})'(\overline{\boldsymbol{x}} - \boldsymbol{x}) \leq f(\overline{\boldsymbol{x}}) - f(\boldsymbol{x})$$

7.8 Convex functions

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Theorem Let $f(\boldsymbol{x})$ be a continuously differentiable convex function. Then $\boldsymbol{x}^*$ is a minimum of f if and only if

$$\nabla f(\boldsymbol{x}^*) = \mathbf{0}$$

Proof: If f convex and $\nabla f(\boldsymbol{x}^*) = \mathbf{0}$

$$f(\boldsymbol{x}) - f(\boldsymbol{x}^*) \geq \nabla f(\boldsymbol{x}^*)'(\boldsymbol{x} - \boldsymbol{x}^*) = 0$$

7.9 Descent Directions

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Interesting Observation

f diff/ble at $\bar{x}$
$\exists d$: $\nabla f(\bar{x})'d < 0 \Rightarrow \forall \lambda > 0$, suff. small, $f(\bar{x} + \lambda d) < f(\bar{x})$
(d: descent direction)

7.10 Proof

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$$f(\bar{x} + \lambda d) = f(\bar{x}) + \lambda \nabla f(\bar{x})^t d + \lambda ||d|| R(\bar{x}, \lambda d)$$

where $R(\bar{x}, \lambda d) \longrightarrow_{\lambda \to 0} 0$

$$\frac{f(\bar{x} + \lambda d) - f(\bar{x})}{\lambda} = \nabla f(\bar{x})^t d + ||d|| R(\bar{x}, \lambda d)$$

$\nabla f(\bar{x})^t d < 0$, $R(\bar{x}, \lambda d) \longrightarrow_{\lambda \to 0} 0 \Rightarrow$
$\forall \lambda > 0$ suff. small $f(\bar{x} + \lambda d) < f(\bar{x})$. QED

8 Algorithms for unconstrained optimization

8.1 Gradient Methods-Motivation

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- Decrease $f(\boldsymbol{x})$ until $\nabla f(\boldsymbol{x}^*) = \mathbf{0}$
- $$f(\bar{\boldsymbol{x}} + \lambda \boldsymbol{d}) \approx f(\bar{\boldsymbol{x}}) + \lambda \nabla f(\bar{\boldsymbol{x}})'\boldsymbol{d}$$
- If $\nabla f(\bar{\boldsymbol{x}})'\boldsymbol{d} < 0$, then for small $\lambda > 0$,

$$f(\bar{\boldsymbol{x}} + \lambda \boldsymbol{d}) < f(\bar{\boldsymbol{x}})$$

9 Gradient Methods

9.1 A generic algorithm

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- $\boldsymbol{x}^{k+1} = \boldsymbol{x}^k + \lambda^k \boldsymbol{d}^k$
- If $\nabla f(\boldsymbol{x}^k) \neq \boldsymbol{0}$, direction $\boldsymbol{d}^k$ satisfies:
$$\nabla f(\boldsymbol{x}^k)' \boldsymbol{d}^k < 0$$
- Step-length $\lambda^k > 0$
- Principal example:
$$\boldsymbol{x}^{k+1} = \boldsymbol{x}^k - \lambda^k \boldsymbol{D}^k \nabla f(\boldsymbol{x}^k)$$
$\boldsymbol{D}^k$ positive definite symmetric matrix

9.2 Principal directions

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- Steepest descent:
$$\boldsymbol{x}^{k+1} = \boldsymbol{x}^k - \lambda^k \nabla f(\boldsymbol{x}^k)$$
- Newton's method:
$$\boldsymbol{x}^{k+1} = \boldsymbol{x}^k - \lambda^k (\nabla^2 f(\boldsymbol{x}^k))^{-1} \nabla f(\boldsymbol{x}^k)$$

9.3 Other directions

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- Diagonally scaled steepest descent
$$\boldsymbol{D}^k = \text{Diagonal approximation to } (\nabla^2 f(\boldsymbol{x}^k))^{-1}$$
- Modified Newton's method
$$\boldsymbol{D}^k = \text{Diagonal approximation to } (\nabla^2 f(\boldsymbol{x}^0))^{-1}$$
- Gauss-Newton method for least squares problems $f(\boldsymbol{x}) = ||g(\boldsymbol{x})||^2$ $\boldsymbol{D}^k = (\nabla g(\boldsymbol{x}^k) \nabla g(\boldsymbol{x}^k)')^{-1}$

10 Steepest descent

10.1 The algorithm

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Step 0 Given $\boldsymbol{x}^0$, set $k := 0$.

Step 1 $\boldsymbol{d}^k := -\nabla f(\boldsymbol{x}^k)$. If $||\boldsymbol{d}^k|| \leq \epsilon$, then stop.

Step 2 Solve $\min_\lambda h(\lambda) := f(\boldsymbol{x}^k + \lambda \boldsymbol{d}^k)$ for the step-length λ^k, perhaps chosen by an exact or inexact line-search.

Step 3 Set $\boldsymbol{x}^{k+1} \leftarrow \boldsymbol{x}^k + \lambda^k \boldsymbol{d}^k$, $k \leftarrow k + 1$. Go to **Step 1**.

10.2 An example

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minimize $f(x_1, x_2) = 5x_1^2 + x_2^2 + 4x_1x_2 - 14x_1 - 6x_2 + 20$

$$\boldsymbol{x}^* = (x_1^*, x_2^*)' = (1,1)'$$

$$f(\boldsymbol{x}^*) = 10$$

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Given $\boldsymbol{x}^k$

$$\boldsymbol{d}^k = -\nabla f(x_1^k, x_2^k) = \begin{pmatrix} -10x_1^k - 4x_2^k + 14 \\ -2x_2^k - 4x_1^k + 6 \end{pmatrix} = \begin{pmatrix} d_1^k \\ d_2^k \end{pmatrix}$$

$$\begin{aligned} h(\lambda) &= f(x^k + \lambda d^k) \\ &= 5(x_1^k + \lambda d_1^k)^2 + (x_2^k + \lambda d_2^k)^2 + 4(x_1^k + \lambda d_1^k)(x_2^k + \lambda d_2^k) - \\ &\quad -14(x_1^k + \lambda d_1^k) - 6(x_2^k + \lambda d_2^k) + 20 \end{aligned}$$

$$\lambda^k = \frac{(d_1^k)^2 + (d_2^k)^2}{2(5(d_1^k)^2 + (d_2^k)^2 + 4d_1^k d_2^k)}$$

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Start at $\boldsymbol{x} = (0, 10)'$

$\varepsilon = 10^{-6}$

k	x_1^k	x_2^k	d_1^k	d_2^k	$\|\|d^k\|\|_2$	λ^k	$f(x^k)$
1	0.000000	10.000000	−26.000000	−14.000000	29.52964612	0.0866	60.000000
2	−2.252782	8.786963	1.379968	−2.562798	2.91071234	2.1800	22.222576
3	0.755548	3.200064	−6.355739	−3.422321	7.21856659	0.0866	12.987827
4	0.204852	2.903535	0.337335	−0.626480	0.71152803	2.1800	10.730379
5	0.940243	1.537809	−1.553670	−0.836592	1.76458951	0.0866	10.178542
6	0.805625	1.465322	0.082462	−0.153144	0.17393410	2.1800	10.043645
7	0.985392	1.131468	−0.379797	−0.204506	0.43135657	0.0866	10.010669
8	0.952485	1.113749	0.020158	−0.037436	0.04251845	2.1800	10.002608
9	0.996429	1.032138	−0.092842	−0.049992	0.10544577	0.0866	10.000638
10	0.988385	1.027806	0.004928	−0.009151	0.01039370	2.1800	10.000156

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k	x_1^k	x_2^k	d_1^k	d_2^k	$\|\|d^k\|\|_2$	λ^k	$f(x^k)$
11	0.999127	1.007856	−0.022695	−0.012221	0.02577638	0.0866	10.000038
12	0.997161	1.006797	0.001205	−0.002237	0.00254076	2.1800	10.000009
13	0.999787	1.001920	−0.005548	−0.002987	0.00630107	0.0866	10.000002
14	0.999306	1.001662	0.000294	−0.000547	0.00062109	2.1800	10.000001
15	0.999948	1.000469	−0.001356	−0.000730	0.00154031	0.0866	10.000000
16	0.999830	1.000406	0.000072	−0.000134	0.00015183	2.1800	10.000000
17	0.999987	1.000115	−0.000332	−0.000179	0.00037653	0.0866	10.000000
18	0.999959	1.000099	0.000018	−0.000033	0.00003711	2.1800	10.000000
19	0.999997	1.000028	−0.000081	−0.000044	0.00009204	0.0866	10.000000
20	0.999990	1.000024	0.000004	−0.000008	0.00000907	2.1803	10.000000
21	0.999999	1.000007	−0.000020	−0.000011	0.00002250	0.0866	10.000000
22	0.999998	1.000006	0.000001	−0.000002	0.00000222	2.1817	10.000000
23	1.000000	1.000002	−0.000005	−0.000003	0.00000550	0.0866	10.000000
24	0.999999	1.000001	0.000000	−0.000000	0.00000054	0.0000	10.000000

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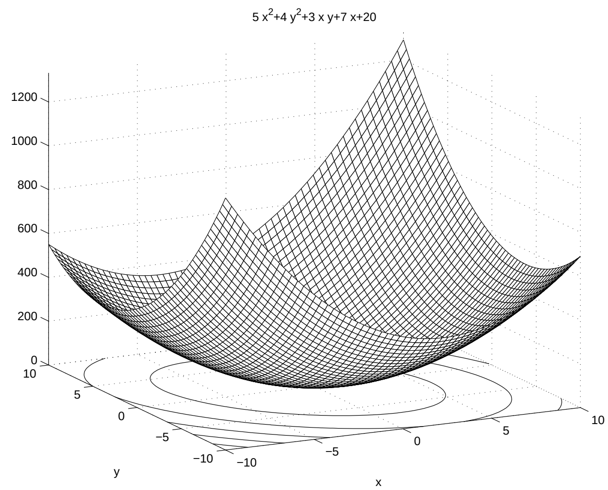

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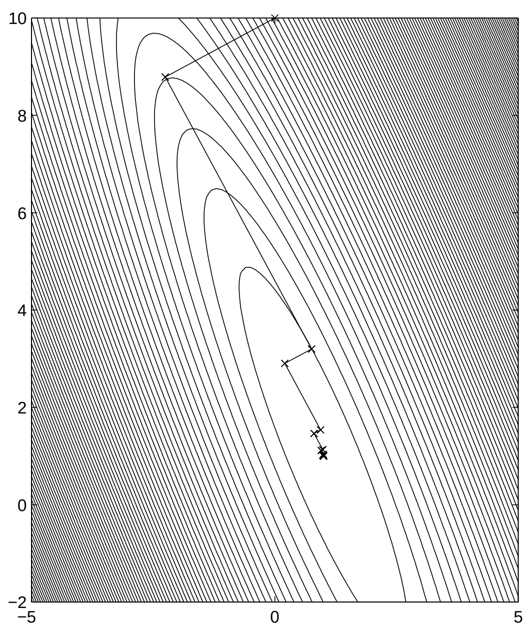

10.3 Important Properties

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- $f(\boldsymbol{x}^{k+1}) < f(\boldsymbol{x}^k) < \cdots < f(\boldsymbol{x}^0)$ (because $\boldsymbol{d}^k$ are descent directions)
- Under reasonable assumptions of $f(\boldsymbol{x})$, the sequence $\boldsymbol{x}^0, \boldsymbol{x}^1, \ldots,$ will have at least one cluster point $\bar{\boldsymbol{x}}$
- Every cluster point $\bar{\boldsymbol{x}}$ will satisfy $\nabla f(\bar{\boldsymbol{x}}) = \boldsymbol{0}$
- *Implication*: If $f(\boldsymbol{x})$ is a convex function, $\bar{\boldsymbol{x}}$ will be an optimal solution

11 Global Convergence Result

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Theorem:
$f : R^n \to R$ is continuously diff/ble on $\mathcal{F} = \{x \in R^n : \; f(x) \leq f(x^0)\}$ closed, bounded set
Every cluster point $\bar{x}$ of $\{x_k\}$ satisfies $\nabla f(\bar{x}) = 0$.

11.1 Work Per Iteration

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Two computation tasks at each iteration of steepest descent:

- Compute $\nabla f(\boldsymbol{x}^k)$ (for quadratic objective functions, it takes $O(n^2)$ steps) to determine $\boldsymbol{d}^k = -\nabla f(\boldsymbol{x}^k)$

- Perform line-search of $h(\lambda) = f(\boldsymbol{x}^k + \lambda \boldsymbol{d}^k)$ to determine $\lambda^k = \arg\min_\lambda h(\lambda) = \arg\min_\lambda f(\boldsymbol{x}^k + \lambda \boldsymbol{d}^k)$

12 Rate of convergence of algorithms

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Let $z_1, \ldots, z_n, \ldots \to z$ be a convergent sequence. We say that the order of convergence of this sequence is p^* if

$$p^* = \sup \left\{ p : \limsup_{k \to \infty} \frac{|z_{k+1} - z|}{|z_k - z|^p} < \infty \right\}$$

Let

$$\beta = \limsup_{k \to \infty} \frac{|z_{k+1} - z|}{|z_k - z|^{p^*}}$$

The larger p^*, the faster the convergence

12.1 Types of convergence

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1. $p^* = 1$, $0 < \beta < 1$, then linear (or geometric) rate of convergence
2. $p^* = 1$, $\beta = 0$, super-linear convergence
3. $p^* = 1$, $\beta = 1$, sub-linear convergence
4. $p^* = 2$, quadratic convergence

12.2 Examples

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- $z_k = a^k$, $0 < a < 1$ converges linearly to zero, $\beta = a$

- $z_k = a^{2^k}$, $0 < a < 1$ converges quadratically to zero

- $z_k = \frac{1}{k}$ converges sub-linearly to zero

- $z_k = \left(\frac{1}{k}\right)^k$ converges super-linearly to zero

12.3 Steepest descent

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- $z_k = f(\boldsymbol{x}^k)$, $z = f(\boldsymbol{x}^*)$, where $\boldsymbol{x}^* = \arg\min f(\boldsymbol{x})$
- Then an algorithm exhibits *linear convergence* if there is a constant $\delta < 1$ such that

$$\frac{f(\boldsymbol{x}^{k+1}) - f(\boldsymbol{x}^*)}{f(\boldsymbol{x}^k) - f(\boldsymbol{x}^*)} \leq \delta \ ,$$

  for all k sufficiently large, where $\boldsymbol{x}^*$ is an optimal solution.

12.3.1 Discussion

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$$\frac{f(\boldsymbol{x}^{k+1}) - f(\boldsymbol{x}^*)}{f(\boldsymbol{x}^k) - f(\boldsymbol{x}^*)} \leq \delta < 1$$

- If $\delta = 0.1$, every iteration adds another digit of accuracy to the optimal objective value.

- If $\delta = 0.9$, every 22 iterations add another digit of accuracy to the optimal objective value, because $(0.9)^{22} \approx 0.1$.

13 Rate of convergence of steepest descent

13.1 Quadratic Case

13.1.1 Theorem

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Suppose $f(\boldsymbol{x}) = \frac{1}{2}\,\boldsymbol{x}'\boldsymbol{Q}\boldsymbol{x} - \boldsymbol{c}'\boldsymbol{x}$
$\boldsymbol{Q}$ is psd

$\lambda_{\max}$ = largest eigenvalue of $\boldsymbol{Q}$
$\lambda_{\min}$ = smallest eigenvalues of $\boldsymbol{Q}$

Linear Convergence Theorem: If $f(\boldsymbol{x})$ is a quadratic function and $\boldsymbol{Q}$ is psd, then

$$\frac{f(\boldsymbol{x}^{k+1}) - f(\boldsymbol{x}^*)}{f(\boldsymbol{x}^k) - f(\boldsymbol{x}^*)} \leq \left(\frac{\left(\frac{\lambda_{\max}}{\lambda_{\min}}\right) - 1}{\left(\frac{\lambda_{\max}}{\lambda_{\min}}\right) + 1} \right)^2$$

13.1.2 Discussion

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$$\frac{f(x^{k+1}) - f(x^*)}{f(x^k) - f(x^*)} \leq \left(\frac{\left(\frac{\lambda_{\max}}{\lambda_{\min}}\right) - 1}{\left(\frac{\lambda_{\max}}{\lambda_{\min}}\right) + 1} \right)^2$$

- $\kappa(\boldsymbol{Q}) := \frac{\lambda_{\max}}{\lambda_{\min}}$ is the *condition number* of $\boldsymbol{Q}$
- $\kappa(\boldsymbol{Q}) \geq 1$
- $\kappa(\boldsymbol{Q})$ plays an extremely important role in analyzing computation involving $\boldsymbol{Q}$

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$$\frac{f(\boldsymbol{x}^{k+1}) - f(\boldsymbol{x}^*)}{f(\boldsymbol{x}^k) - f(\boldsymbol{x}^*)} \leq \left(\frac{\kappa(\boldsymbol{Q}) - 1}{\kappa(\boldsymbol{Q}) + 1} \right)^2$$

$\kappa(Q) = \frac{\lambda_{\max}}{\lambda_{\min}}$	Upper Bound on Convergence Constant δ	Number of Iterations to Reduce the Optimality Gap by 0.10
1.1	0.0023	1
3.0	0.25	2
10.0	0.67	6
100.0	0.96	58
200.0	0.98	116
400.0	0.99	231

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For $\kappa(Q) \sim O(1)$ converges fast.
For large $\kappa(Q)$

$$\left(\frac{\kappa(\boldsymbol{Q}) - 1}{\kappa(\boldsymbol{Q}) + 1} \right)^2 \sim (1 - \frac{1}{\kappa(\boldsymbol{Q})})^2 \sim 1 - \frac{2}{\kappa(\boldsymbol{Q})}$$

Therefore

$$(f(\boldsymbol{x}^k) - f(\boldsymbol{x}^*)) \leq (1 - \frac{2}{\kappa(\boldsymbol{Q})})^k (f(\boldsymbol{x}^0) - f(\boldsymbol{x}^*))$$

In $k \sim \frac{1}{2}\kappa(\boldsymbol{Q})(-ln\epsilon)$ iterations, finds $\boldsymbol{x^k}$:

$$(f(\boldsymbol{x}^k) - f(\boldsymbol{x}^*)) \leq \epsilon(f(\boldsymbol{x}^0) - f(\boldsymbol{x}^*))$$

13.2 Example 2

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$$f(\boldsymbol{x}) = \frac{1}{2}\boldsymbol{x}'\boldsymbol{Q}\boldsymbol{x} - \boldsymbol{c}'\boldsymbol{x} + 10$$

$$\boldsymbol{Q} = \begin{bmatrix} 20 & 5 \\ 5 & 1 \end{bmatrix} \qquad c = \begin{pmatrix} 14 \\ 6 \end{pmatrix}$$

$\kappa(\boldsymbol{Q}) = 30.234$

$\delta = \left(\frac{\kappa(\boldsymbol{Q})-1}{\kappa(\boldsymbol{Q})+1}\right)^2 = 0.8760$

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k	x_1^k	x_2^k	$\|\|\boldsymbol{d}^k\|\|_2$	λ^k	$f(\boldsymbol{x}^k)$	$\frac{f(\boldsymbol{x}^k)-f(\boldsymbol{x}^*)}{f(\boldsymbol{x}^{k-1})-f(\boldsymbol{x}^*)}$
1	40.000000	−100.000000	286.06293014	0.0506	6050.000000	
2	25.542693	−99.696700	77.69702948	0.4509	3981.695128	0.658079
3	26.277558	−64.668130	188.25191488	0.0506	2620.587793	0.658079
4	16.763512	−64.468535	51.13075844	0.4509	1724.872077	0.658079
5	17.247111	−41.416980	123.88457127	0.0506	1135.420663	0.658079
6	10.986120	−41.285630	33.64806192	0.4509	747.515255	0.658079
7	11.304366	−26.115894	81.52579489	0.0506	492.242977	0.658079
8	7.184142	−26.029455	22.14307211	0.4509	324.253734	0.658079
9	7.393573	−16.046575	53.65038732	0.0506	213.703595	0.658079
10	4.682141	−15.989692	14.57188362	0.4509	140.952906	0.658079

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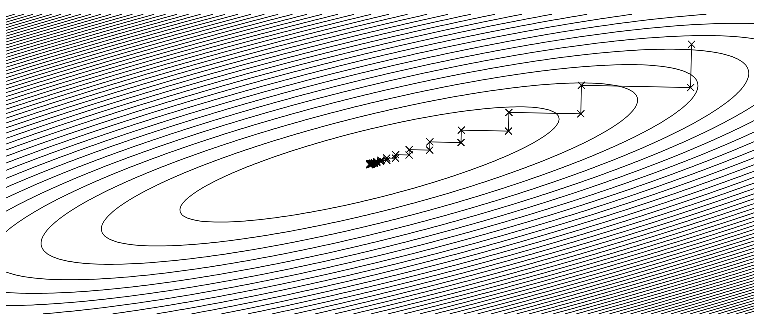
k	x_1^k	x_2^k	$\|\|\boldsymbol{d}^k\|\|_2$	λ^k	$f(\boldsymbol{x}^k)$	$\frac{f(\boldsymbol{x}^k)-f(\boldsymbol{x}^*)}{f(\boldsymbol{x}^{k-1})-f(\boldsymbol{x}^*)}$
20	0.460997	0.948466	1.79847660	0.4509	3.066216	0.658079
30	−0.059980	3.038991	0.22196980	0.4509	0.965823	0.658079
40	−0.124280	3.297005	0.02739574	0.4509	0.933828	0.658079
50	−0.132216	3.328850	0.00338121	0.4509	0.933341	0.658079
60	−0.133195	3.332780	0.00041731	0.4509	0.933333	0.658078
70	−0.133316	3.333265	0.00005151	0.4509	0.933333	0.658025
80	−0.133331	3.333325	0.00000636	0.4509	0.933333	0.654656
90	−0.133333	3.333332	0.00000078	0.0000	0.933333	0.000000

SLIDE 47



13.3 Example 3

SLIDE 48

$$f(\boldsymbol{x}) = \frac{1}{2}\boldsymbol{x}'\boldsymbol{Q}\boldsymbol{x} - \boldsymbol{c}'\boldsymbol{x} + 10$$

$$\boldsymbol{Q} = \begin{bmatrix} 20 & 5 \\ 5 & 16 \end{bmatrix} \qquad \boldsymbol{c} = \begin{pmatrix} 14 \\ 6 \end{pmatrix}$$

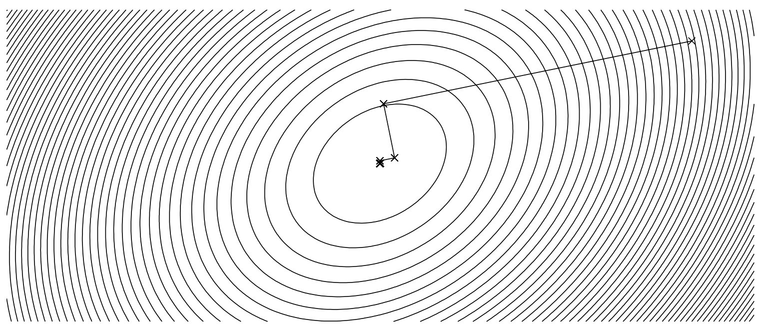
$\kappa(\boldsymbol{Q}) = 1.8541$

$$\delta = \left(\frac{\kappa(\boldsymbol{Q}) - 1}{\kappa(\boldsymbol{Q}) + 1}\right)^2 = 0.0896$$

SLIDE 49

k	x_1^k	x_2^k	$\|\|\boldsymbol{d}^k\|\|_2$	λ^k	$f(\boldsymbol{x}^k)$	$\frac{f(\boldsymbol{x}^k) - f(\boldsymbol{x}^*)}{f(\boldsymbol{x}^{k-1}) - f(\boldsymbol{x}^*)}$
1	40.000000	−100.000000	1434.79336491	0.0704	76050.000000	
2	19.867118	−1.025060	385.96252652	0.0459	3591.615327	0.047166
3	2.513241	−4.555081	67.67315150	0.0704	174.058930	0.047166
4	1.563658	0.113150	18.20422450	0.0459	12.867208	0.047166
5	0.745149	−0.053347	3.19185713	0.0704	5.264475	0.047166
6	0.700361	0.166834	0.85861649	0.0459	4.905886	0.047166
7	0.661755	0.158981	0.15054644	0.0704	4.888973	0.047166
8	0.659643	0.169366	0.04049732	0.0459	4.888175	0.047166
9	0.657822	0.168996	0.00710064	0.0704	4.888137	0.047166
10	0.657722	0.169486	0.00191009	0.0459	4.888136	0.047166
11	0.657636	0.169468	0.00033491	0.0704	4.888136	0.047166
12	0.657632	0.169491	0.00009009	0.0459	4.888136	0.047161
13	0.657628	0.169490	0.00001580	0.0704	4.888136	0.047068
14	0.657627	0.169492	0.00000425	0.0459	4.888136	0.045002
15	0.657627	0.169491	0.00000075	0.0000	4.888136	0.000000

SLIDE 50



13.4 Empirical behavior

SLIDE 51

- The convergence constant bound is not just theoretical. It is typically experienced in practice.

- Analysis is due to Leonid Kantorovich, who won the Nobel Memorial Prize in Economic Science in 1975 for his contributions to optimization and economic planning.

SLIDE 52

- What about non-quadratic functions?

  – Suppose $\boldsymbol{x}^* = \arg\min_{\boldsymbol{x}} f(\boldsymbol{x})$

- $\nabla^2 f(\boldsymbol{x}^*)$ is the Hessian of $f(\boldsymbol{x})$ at $\boldsymbol{x} = \boldsymbol{x}^*$

- Rate of convergence will depend on $\kappa(\nabla^2 f(\boldsymbol{x}^*))$

14 Summary

SLIDE 53

1. Optimality Conditions
2. The steepest descent algorithm - Convergence
3. Rate of convergence of Steepest Descent

15 Choices of step sizes

SLIDE 54

- $Min_\lambda f(x^k + \lambda d^k)$
- Limited Minimization: $Min_{\lambda \in [0,s]} f(x^k + \lambda d^k)$
- Constant stepsize $\lambda^k = s$ constant
- Diminishing stepsize: $\lambda^k \to 0, \ \sum_k \lambda^k = \infty$
- Armijo Rule