15.093 Optimization Methods

Lecture 16: Dynamic Programming

1 Outline

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1. The knapsack problem
2. The traveling salesman problem
3. The general DP framework
4. Bellman equation
5. Optimal inventory control
6. Optimal trading
7. Multiplying matrices

2 The Knapsack problem

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$$\begin{aligned} \text{maximize} \quad & \sum_{j=1}^{n} c_j x_j \\ \text{subject to} \quad & \sum_{j=1}^{n} w_j x_j \le K \\ & x_j \in \{0,1\} \end{aligned}$$

Define

$$\begin{aligned} C_i(w) = \text{maximize} \quad & \sum_{j=1}^{i} c_j x_j \\ \text{subject to} \quad & \sum_{j=1}^{i} w_j x_j \le w \\ & x_j \in \{0,1\} \end{aligned}$$

2.1 A DP Algorithm

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- $C_i(w)$: the maximum value that can be accumulated using some of the first i items subject to the constraint that the total accumulated weight is equal to w
- Recursion

$$C_{i+1}(w) = \max\left\{C_i(w),\ C_i(w - w_{i+1}) + c_{i+1}\right\}$$

- By considering all states of the form (i, w) with $w \le K$, algorithm has complexity $O(nK)$

3 The TSP

SLIDE 4

- $G = (V, A)$ directed graph with n nodes
- c_{ij} cost of arc (i, j)
- Approach: choice of a tour as a sequence of choices
- We start at node 1; then, at each stage, we choose which node to visit next.
- After a number of stages, we have visited a subset S of V and we are at a current node $k \in S$

3.1 A DP algorithm

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- $C(S, k)$ be the minimum cost over all paths that start at node 1, visit all nodes in the set S exactly once, and end up at node k
- (S, k) a state; this state can be reached from any state of the form $\big(S \setminus \{k\}, m\big)$, with $m \in S \setminus \{k\}$, at a transition cost of c_{mk}
- Recursion

$$C(S, k) = \min_{m \in S \setminus \{k\}} \Big(C\big(S \setminus \{k\}, m\big) + c_{mk} \Big), \qquad k \in S$$

$$C\big(\{1\}, 1\big) = 0.$$

- Length of an optimal tour is

$$\min_k \Big(C\big(\{1, \ldots, n\}, k\big) + c_{k1} \Big)$$

- Complexity: $O\big(n^2 2^n\big)$ operations

4 Guidelines for constructing DP Algorithms

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- View the choice of a feasible solution as a sequence of decisions occurring in stages, and so that the total cost is the sum of the costs of individual decisions.
- Define the state as a summary of all relevant past decisions.
- Determine which state transitions are possible. Let the cost of each state transition be the cost of the corresponding decision.
- Write a recursion on the optimal cost from the origin state to a destination state.

The most crucial step is usually the definition of a suitable state.

5 The general DP framework

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- Discrete time dynamic system described by state x_k, k indexes time.
- u_k control to be selected at time k. $u_k \in U_k(x_k)$.
- w_k randomness at time k
- N time horizon
- Dynamics:

$$x_{k+1} = f_k(x_k, u_k, w_k)$$

- Cost function: additive over time

$$E\left(g_N(x_N) + \sum_{k=0}^{N-1} g_k(x_k, u_k, w_k)\right)$$

5.1 Inventory Control

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- x_k stock available at the beginning of the kth period
- u_k stock ordered at the beginning of the kth period
- w_k demand duirng the kth period with given probability distribution. Excess demand is backloged and filled as soon as additional inventory is available.
- Dynamics

$$x_{k+1} = x_k + u_k - w_k$$

- Cost

$$E\left(R(x_N) + \sum_{k=0}^{N-1} (r(x_k) + cu_k)\right)$$

6 The DP Algorithm

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- Define $J_k(x_k)$ to be the expected optimal cost starting from stage k at state x_k.
- Bellman's principle of optimality

$$J_N(x_N) = g_N(x_N)$$

$$J_k(x_k) =$$

$$\min_{u_k \in U_k(x_k)} E_{w_k}\left\{g_k(x_k, u_k, w_k) + J_{k+1}(f_k(x_k, u_k, w_k))\right\}$$

- Optimal expected cost for the overall problem: $J_0(x_0)$.

7 Inventory Control

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- If $r(x_k) = ax_k^2$, $w_k \sim N(\mu_k, \sigma_k^2)$, then

$$u_k^* = c_k x_k + d_k, \qquad J_k(x_k) = b_k x_k^2 + f_k x_k + e_k$$

- If $r(x_k) = p\max(0, -x_k) + h\max(0, x_k)$, then there exist S_k:

$$u_k^* = \begin{cases} S_k - x_k & \text{if } x_k < S_k \\ 0 & \text{if } x_k \geq S_k \end{cases}$$

8 Optimal trading

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- $\overline{S}$ shares of a stock to be bought within a horizon T.
- $t = 1, 2, \ldots, T$ discrete trading periods.
- Control: S_t number of shares acquired in period t at price P_t, $t = 1, 2, \ldots, T$
- Objective: $\min E\left[\sum_{t=1}^{T} P_t S_t\right]$

$$\text{s.t.} \quad \sum_{t=1}^{T} S_t = \overline{S}$$

- Dynamics:

$$P_t = P_{t-1} + \alpha S_t + \epsilon_t$$

where $\alpha > 0$, $\epsilon_t \sim N(0, 1)$

8.1 DP ingredients

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- State: (P_{t-1}, W_t)

  P_{t-1} price realized at the previous period

  W_t # of shares remaining to be purchased
- Control: S_t number of shares purchased at time t
- Randomness: ϵ_t
- Objective: $\min E\left[\sum_{t=1}^{T} P_t S_t\right]$
- Dynamics: $P_t = P_{t-1} + \alpha S_t + \epsilon_t$ $W_t = W_{t-1} - S_{t-1}$, $W_1 = \overline{S}$, $W_{T+1} = 0$

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Note that $W_{T+1} = 0$ is equivalent to the constraint that $\overline{S}$ must be executed by period T

8.2 The Bellman Equation

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$$J_t(P_{t-1}, W_t) \;=\; \min_{S_t} \; E_t \left[\; P_t S_t \;+\; J_{t+1}(P_t, W_{t+1}) \; \right]$$

$$J_T(P_{T-1}, W_T) \;=$$

$$\min_{S_T} \; E_T[P_T W_T] \;=\; (P_{T-1} + \alpha W_T) W_T$$

Since $W_{T+1} = 0 \;\;\Rightarrow\;\; S_T^* = W_T$

8.3 Solution

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$$J_{T-1}(P_{T-2}, W_{T-1}) =$$

$$\begin{aligned} &= \min_{S_{T-1}} \; E_{T-1} \left[\; P_{T-1} S_{T-1} \;+\; J_T(P_{T-1}, W_T) \; \right] \\ &= \min_{S_{T-1}} \; E_{T-1} \Big[\; (P_{T-2} + \alpha S_{T-1} + \epsilon_{T-1}) S_{T-1} \; + \\ &\qquad J_T \Big(P_{T-2} + \alpha S_{T-1} + \epsilon_{T-1}, W_{T-1} - S_{T-1} \Big) \; \Big] \end{aligned}$$

$$\begin{aligned} S_{T-1}^* &= \frac{W_{T-1}}{2} \\ J_{T-1}(P_{T-2}, W_{T-1}) &= W_{T-1}(P_{T-2} + \frac{3}{4}\alpha W_{T-1}), \end{aligned}$$

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Continuing in this fashion,

$$\begin{aligned} S_{T-k}^* &= \frac{W_{T-k}}{k+1} \\ J_{T-k}(P_{T-k-1}, W_{T-k}) &= W_{T-k}(P_{T-k-1} + \frac{k+2}{2(k+1)}\alpha W_{T-k}) \end{aligned}$$

$$\begin{aligned} S_1^* &= \frac{\overline{S}}{T} \\ J_1(P_0, W_1) &= P_0 \overline{S} \;+\; \frac{\alpha \overline{S}^2}{2} \left(1 + \frac{1}{T} \right) \end{aligned}$$

$$S_1^* \;=\; S_2^* \;=\; \cdots \;=\; S_T^* \;=\; \frac{\overline{S}}{T}$$

8.4 Different Dynamics

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$$
\begin{aligned}
P_t &= P_{t-1} + \alpha S_t + \gamma X_t + \epsilon_t \quad , \quad \alpha > 0 \\
X_t &= \rho X_{t-1} + \eta_t \quad , \quad X_1 = 1 \quad , \quad \rho \in (-1, 1)
\end{aligned}
$$

where $\epsilon_t \sim N(0, \sigma_\epsilon^2)$ and $\eta_t \sim N(0, \sigma_\eta^2)$

8.5 Solution

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$$
\begin{aligned}
S^*_{T-k} &= \frac{W_{T-k}}{k+1} + \frac{\rho b_{k-1}}{2a_{k-1}} X_{T-k} \\
J_{T-k}(P_{T-k-1}, X_{T-k}, W_{T-k}) &= P_{T-k-1} W_{T-k} + a_k W^2_{T-k} + \\
&\qquad b_k X_{T-k} W_{T-k} + c_k X^2_{T-k} + d_k
\end{aligned}
$$

for $k = 0, 1, \ldots, T-1$, where:

$$
\begin{aligned}
a_k &= \frac{\alpha}{2}\left(1 + \frac{1}{k+1}\right), \quad , \quad a_0 = \alpha \\
b_k &= \gamma + \frac{\alpha \rho b_{k-1}}{2a_{k-1}} \quad , \quad b_0 = \gamma \\
c_k &= \rho^2 c_{k-1} - \frac{\rho^2 b^2_{k-1}}{4a_{k-1}} \quad , \quad c_0 = 0 \\
d_k &= d_{k-1} + c_{k-1}\sigma_\eta^2 \quad , \quad d_0 = 0 \, .
\end{aligned}
$$

9 Matrix multiplication

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- Matrices: M_k: $n_k \times n_{k+1}$
- Objective: Find $M_1 \cdot M_2 \cdots M_N$
- Example: $M_1 \cdot M_2 \cdot M_3$; $M_1 : 1 \times 10$, $M_2 : 10 \times 1$, $M_3 : 1 \times 10$.

  $M_1(M_2 M_3)$ 200 multiplications;

  $(M_1 M_2) M_3$ 20 multiplictions.
- What is the optimal order for performing the multiplication?

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- $m(i,j)$ optimal number of scalar multiplications for multiplying $M_i \ldots M_j$.
- $m(i,i) = 0$
- For $i < j$:

$$m(i,j) = \min_{i \leq k < j} (m(i,k) + m(k+1,j) + n_i n_{k+1} n_{j+1})$$