15.093 Optimization Methods

Lecture 2: The Geometry of LO

1 Outline

SLIDE 1

- Polyhedra
- Standard form
- Algebraic and geometric definitions of corners
- Equivalence of definitions
- Existence of corners
- Optimality of corners
- Conceptual algorithm

2 Central Problem

SLIDE 2

$$\begin{array}{lll}
\text{minimize} & \boldsymbol{c}'\boldsymbol{x} & \\
\text{subject to} & \boldsymbol{a_i}'\boldsymbol{x} = b_i & i \in M_1 \\
& \boldsymbol{a_i}'\boldsymbol{x} \leq b_i & i \in M_2 \\
& \boldsymbol{a_i}'\boldsymbol{x} \geq b_i & i \in M_3 \\
& x_j \geq 0 & j \in N_1 \\
& x_j \gtrless 0 & j \in N_2
\end{array}$$

2.1 Standard Form

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$$\begin{array}{ll}
\text{minimize} & \boldsymbol{c}'\boldsymbol{x} \\
\text{subject to} & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \\
& \boldsymbol{x} \geq \boldsymbol{0}
\end{array}$$

Characteristics

- Minimization problem
- Equality constraints
- Non-negative variables

2.2 Transformations

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$$\begin{array}{lcl}
\max \boldsymbol{c}'\boldsymbol{x} & & -\min(-\boldsymbol{c}'\boldsymbol{x}) \\
\boldsymbol{a_i}'\boldsymbol{x} \leq b_i & & \boldsymbol{a_i}'\boldsymbol{x} + s_i = b_i, \quad s_i \geq 0 \\
& \Leftrightarrow & \\
\boldsymbol{a_i}'\boldsymbol{x} \geq b_i & & \boldsymbol{a_i}'\boldsymbol{x} - s_i = b_i, \quad s_i \geq 0 \\
x_j \gtrless 0 & & x_j = x_j^+ - x_j^- \\
& & x_j^+ \geq 0, \; x_j^- \geq 0
\end{array}$$

2.3 Example

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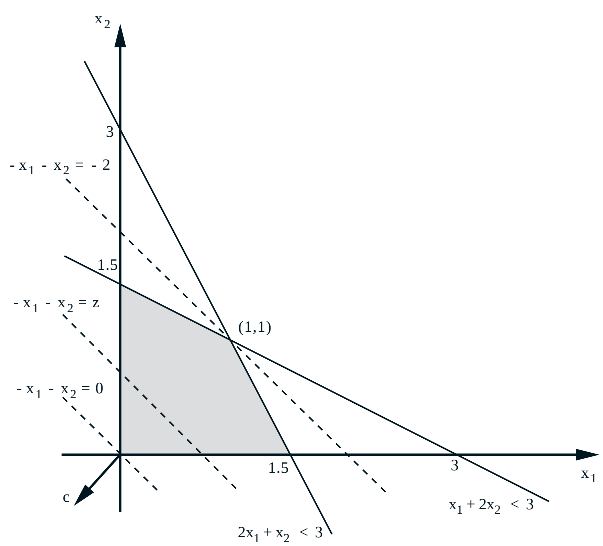
$$\begin{array}{ll} \text{maximize} & x_1 - x_2 \\ \text{subject to} & x_1 + x_2 \leq 1 \\ & x_1 + 2x_2 \geq 1 \\ & x_1 \gtrless 0, x_2 \geq 0 \\ & \Downarrow \\ -\text{minimize} & -x_1^+ + x_1^- + x_2 \\ \text{subject to} & x_1^+ - x_1^- + x_2 + s_1 \qquad = 1 \\ & x_1^+ - x_1^- + 2x_2 \qquad - s_2 = 1 \\ & x_1^+, x_1^-, x_2, s_1, s_2 \geq 0 \end{array}$$

3 Preliminary Insights

SLIDE 6

$$\begin{array}{rl} \text{minimize} & -x_1 - x_2 \\ \text{subject to} & x_1 + 2x_2 \leq 3 \\ & 2x_1 + x_2 \leq 3 \\ & x_1, x_2 \geq 0 \end{array}$$



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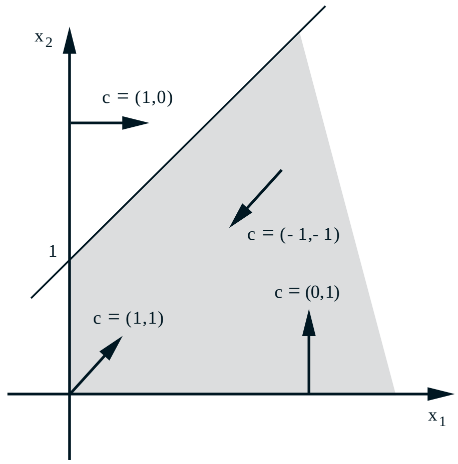
$$\begin{array}{rcl} -x_1 + x_2 & \leq & 1 \\ x_1 & \geq & 0 \\ x_2 & \geq & 0 \end{array}$$

SLIDE 8

- There exists a unique optimal solution.
- There exist multiple optimal solutions; in this case, the set of optimal solutions can be either bounded or unbounded.



- The optimal cost is $-\infty$, and no feasible solution is optimal.
- The feasible set is empty.

4 Polyhedra

4.1 Definitions

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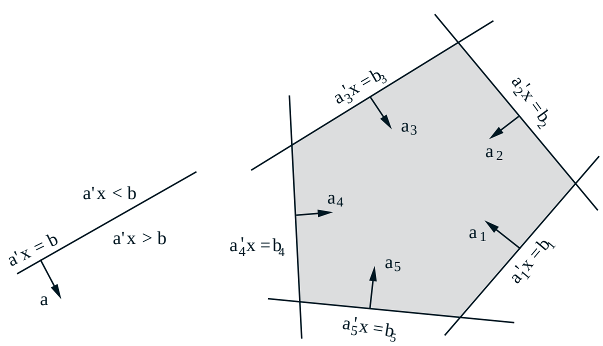
- The set $\{\boldsymbol{x} \mid \boldsymbol{a}'\boldsymbol{x} = b\}$ is called a **hyperplane**.
- The set $\{\boldsymbol{x} \mid \boldsymbol{a}'\boldsymbol{x} \geq b\}$ is called a **halfspace**.
- The intersection of many halfspaces is called a **polyhedron**.



(a) (b)

5 Corners

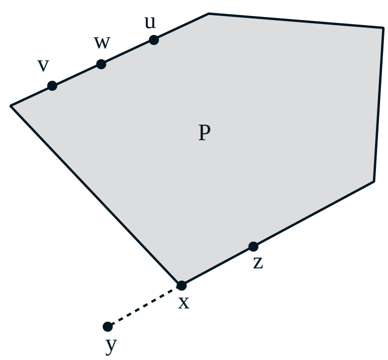
5.1 Extreme Points

SLIDE 10

- Polyhedron $P = \{\boldsymbol{x} \mid \boldsymbol{A}\boldsymbol{x} \geq \boldsymbol{b}\}$
- $\boldsymbol{x} \in P$ is **an extreme point** of P

  if $\nexists\ \boldsymbol{y}, \boldsymbol{z} \in P$ $(\boldsymbol{y} \neq \boldsymbol{x}, \boldsymbol{z} \neq \boldsymbol{x})$:

  $\boldsymbol{x} = \lambda \boldsymbol{y} + (1-\lambda)\boldsymbol{z}, 0 < \lambda < 1$



5.2 Vertex

SLIDE 11

- $\boldsymbol{x} \in P$ is **a vertex** of P if $\exists\ \boldsymbol{c}$:

  $\boldsymbol{x}$ is the unique optimum

$$\begin{array}{ll} \text{minimize} & \boldsymbol{c}'\boldsymbol{y} \\ \text{subject to} & \boldsymbol{y} \in P \end{array}$$

5.3 Basic Feasible Solution

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$$P = \{(x_1, x_2, x_3) \mid x_1 + x_2 + x_3 = 1, \quad x_1, x_2, x_3 \geq 0\}$$

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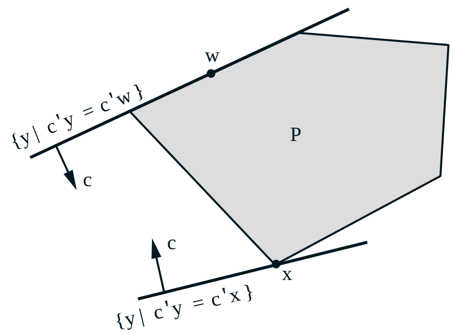
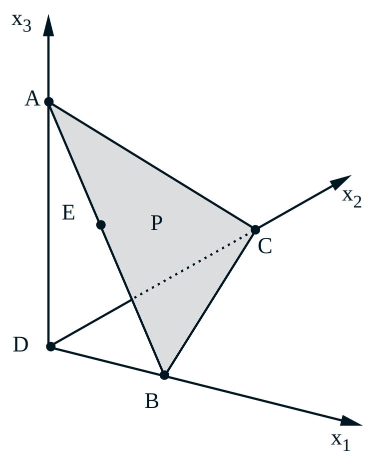
Points A,B,C : 3 constraints active

Point E: 2 constraints active

suppose we add $2x_1 + 2x_2 + 2x_3 = 2$.





Then 3 hyperplanes are tight, but constraints are not linearly independent.

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Intuition: a point at which n inequalities are tight and corresponding equations are linearly independent.
$P = \{\boldsymbol{x} \in \Re^n \mid \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}\}$

- $\boldsymbol{a_1}, \ldots, \boldsymbol{a_m}$ rows of $\boldsymbol{A}$
- $\boldsymbol{x} \in P$
- $I = \{i \mid \boldsymbol{a_i}'\boldsymbol{x} = b_i\}$

Definition $\boldsymbol{x}$ is **a basic feasible solution** if subspace spanned by $\{\boldsymbol{a_i}, i \in I\}$ is $\Re^n$.

5.3.1 Degeneracy

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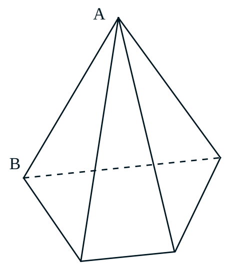
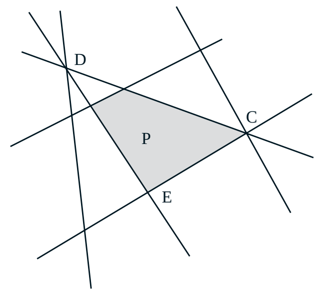
- If $|I| = n$, then $\boldsymbol{a_i},\ i \in I$ are linearly independent; $\boldsymbol{x}$ nondegenerate.

- If $|I| > n$, then there exist n linearly independent $\{\boldsymbol{a_i}, i \in I\}$; $\boldsymbol{x}$ degenerate.



(a)



(b)

6 Equivalence of definitions

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Theorem: $P = \{\boldsymbol{x} \mid \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}\}$. Let $\boldsymbol{x} \in P$.
$\boldsymbol{x}$ is a vertex $\Leftrightarrow$ $\boldsymbol{x}$ is an extreme point $\Leftrightarrow$ $\boldsymbol{x}$ is a BFS.

7 BFS for standard form polyhedra

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- $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ and $\boldsymbol{x} \geq \boldsymbol{0}$
- $m \times n$ matrix $\boldsymbol{A}$ has linearly independent rows
- $\boldsymbol{x} \in \Re^n$ is a basic solution if and only if $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$, and there exist indices $B(1), \ldots, B(m)$ such that:
  - The columns $\boldsymbol{A}_{B(1)}, \ldots, \boldsymbol{A}_{B(m)}$ are linearly independent
  - If $i \neq B(1), \ldots, B(m)$, then $x_i = 0$

7.1 Construction of BFS

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Procedure for constructing basic solutions

1. Choose m linearly independent columns $\boldsymbol{A}_{B(1)}, \ldots, \boldsymbol{A}_{B(m)}$
2. Let $x_i = 0$ for all $i \neq B(1), \ldots, B(m)$
3. Solve $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ for $x_{B(1)}, \ldots, x_{B(m)}$

$$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \quad \rightarrow \quad \boldsymbol{B}\boldsymbol{x}_B + \boldsymbol{N}\boldsymbol{x}_N = \boldsymbol{b}$$
$$\boldsymbol{x}_N = 0, \quad \boldsymbol{x}_B = \boldsymbol{B}^{-1}\boldsymbol{b}$$

7.2 Example

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$$\begin{bmatrix} 1 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 1 & 6 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \boldsymbol{x} = \begin{bmatrix} 8 \\ 12 \\ 4 \\ 6 \end{bmatrix}$$

- $\boldsymbol{A}_4, \boldsymbol{A}_5, \boldsymbol{A}_6, \boldsymbol{A}_7$ basic columns
- Solution: $\boldsymbol{x} = (0, 0, 0, 8, 12, 4, 6)$, a BFS
- Another basis: $\boldsymbol{A}_3, \boldsymbol{A}_5, \boldsymbol{A}_6, \boldsymbol{A}_7$ basic columns.
- Solution: $\boldsymbol{x} = (0, 0, 4, 0, -12, 4, 6)$, not a BFS

7.3 Geometric intuition

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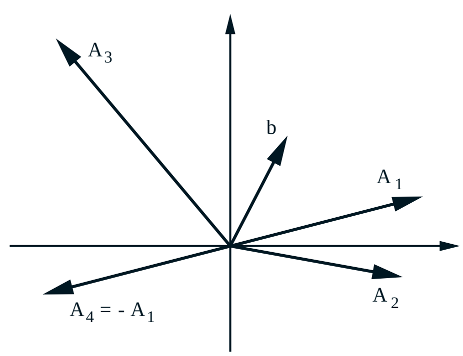

8 Existence of BFS

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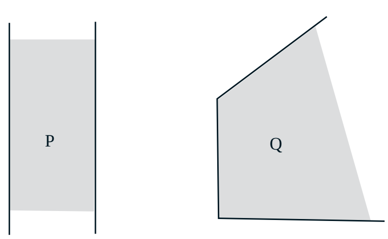

$$P = \{(x_1, x_2) : 0 \le x \le 1\}$$

$$Q = \{(x_1, x_2) : -x_1 + x_2 \le 2,\ x_1 \ge 0,\ x_2 \ge 0\}.$$

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Definition: P contains a line if $\exists \boldsymbol{x} \in P$; and $\boldsymbol{d} \in \Re^n$:

$$\boldsymbol{x} + \alpha \boldsymbol{d} \in P \quad \forall \alpha.$$

Theorem: $P = \{x \in \Re^n \mid Ax \ge b\} \ne \emptyset$.
P has a BFS $\Leftrightarrow$ P does not contain a line.
Implications

- Polyhedra in standard form $P = \{\boldsymbol{x} \mid \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}, \boldsymbol{x} \ge \boldsymbol{0}\}$ contain a BFS
- Bounded polyhedra have a BFS.

9 Optimality of BFS

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$$\begin{array}{rl} \min & \boldsymbol{c}'\boldsymbol{x} \\ \text{s.t.} & \boldsymbol{x} \in P = \{\boldsymbol{x} \mid \boldsymbol{A}\boldsymbol{x} \ge \boldsymbol{b}\} \end{array}$$

Theorem: Suppose P has at least one extreme point. Either optimal cost is $-\infty$ or there exists an extreme point which is optimal.

10 Conceptual algorithm

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- Start at a corner
- Visit a neighboring corner that improves objective.