15.093J

Midterm Exam

October, 20, 2004

1. This is an 1.5 hour exam.
2. You are required to submit both the questions and your answers.
3. You can use the textbook for the class, the notes from the lectures, your homeworks and homeworks solutions.
4. Good luck!

Problem 1 (40 points)

Please answer true or false. A correct answer is worth 1.5 points, a correct explanation 1.5 points.

1. A problem of maximizing a convex, piecewise-linear function over a polyhedron can be formulated as a linear programming problem.
2. The dual of the problem min x_1 subject to $x_1 = 0$, $x_1, x_2 \geq 0$ has a nondegenerate optimal solution.
3. If there is a nondegenerate optimal basis, then there is a unique optimal basis.
4. An extreme point $\mathbf{x}^*$ is degenerate. There is a unique basis corresponding to $\mathbf{x}^*$.
5. During the course of the simplex algorithm, the optimality conditions corresponding to a given basic feasible $\mathbf{x}^*$ solution are not satisfied. Then $\mathbf{x}^*$ is not optimal.
6. During Phase I of the simplex algorithm, unboundness may be detected.
7. An iteration of the simplex method may move the feasible solution by a positive distance while leaving the cost unchanged.
8. Consider a problem in standard form involving m equality constraints. If $\mathbf{x}$ is an optimal solution, no more than m of its components can be positive, where m is the number of equality constraints.
9. Consider a linear programming problem over a polyhedron P involving four variables, for which we know $x_1, x_2 \geq 0$ and $x_3, x_4 \leq 10$. P has an extreme point.
10. Consider the polyhedron $P = \{\mathbf{x} \in \mathcal{R}^n \mid 0 \leq x_i \leq 1, \ i = 1, \ldots, n\}$. P is a nondegenerate polyhderon with 2^n vertices.

Problem 2 (30 points) A company wants to decide the amounts x_1, x_2, x_3 of three products to produce. Its production problem can be formulated as a linear optimization problem:

$$\begin{aligned} \max \quad & 3x_1 + 4x_2 + 5x_3 \\ \text{s.t.} \quad & 2x_1 + 3x_2 + 4x_3 \leq 5 \\ & 2x_1 + x_2 + x_3 \leq 4 \\ & x_1 - x_2 \leq 0 \\ & x_1, x_2, x_3 \geq 0. \end{aligned}$$

Note that there are 5 units of Resource 1 available and 4 units of Resource 2. The optimal solution to the primal and the dual problems together with sensitivity information is given in Tables 1 and 2 below.

	Optimal Value	**Reduced Cost**	**Objective Coefficient**	**Allowable Increase**	**Allowable Decrease**
x_1	1	0	3	∞	0.333
x_2	1	0	4	0.5	0.75
x_3	0	-0.6	5	0.6	∞

Table 1: The optimal primal solution and its sensitivity with respect to changes in coefficients of the objective function. The last two columns describe the allowed changes in these coefficients for which the same solution remains optimal.

(a) What is the optimal solution and what is the optimal profit? Is the solution degenerate?

(b) You have a proposal to buy an extra unit of Resource 1 for \$ 1. Should you buy the extra unit?

(c) You have a proposal to buy two extra units of Resource 1 for \$ 1.5 for both units. Should you buy the extra two units?

(d) Under the current plan, we are not producing Product 3. How much should the profit of Product 3 increase so that you can produce it?

(e) Suppose that Resource 1 decreases by 3 units. What is the new solution $(x_1, x_2, x_3)'$?

(f) Suppose that the profit of the second product decreases by \$ 0.75. Is the current solution still optimal? Is there an alternative optimal solution $(x_1, x_2, x_3)'$?

(g) Is the current optimal solution unique?

	Slack Value	Dual Variable	Constr. RHS	Allowable Increase	Allowable Decrease
Resource 1	5	1.4	5	1.667	5
Resource 2	3	0	4	∞	1
Resource 3	0	0.2	0	1.25	1.667

Table 2: The optimal dual solution and its sensitivity. The column labeled "slack value" gives us the optimal values of the slack variables associated with each of the primal constraints. The third column simply repeats the right-hand side vector, while the last two columns describe the allowed changes in the components of the right-hand side vector for which the optimal dual solution remains the same.

(h) Suppose that Resource 1 increases by 3 units. Provide upper and lower bounds on the value of the optimal profit.

(i) What is the interpretation of the dual variable of 0.2 in Resource 3?

Problem 3 (30 points)

While solving a standard form linear programming problem

$$\begin{aligned} \min \quad & c_1x_1 + c_2x_2 \\ \text{s.t.} \quad & a_{11}x_1 + a_{12}x_2 + x_3 = b_1 \\ & a_{21}x_1 + a_{22}x_2 + x_4 = b_2 \\ & x_1 \geq 0, \ i = 1, \ldots, 4. \end{aligned}$$

using the simplex method, we arrive at the following tableau:

-10	0	0	2	γ
3	1	0	-2	-3
β	0	1	0	α

(a) Suppose that $\beta > 0$. Find a necessary and sufficient condition for the current solution to be optimal.

(b) Suppose that $\beta = 0$. Find a necessary and sufficient condition for the current solution to be optimal.

(c) If the feasible set is known to be bounded, what does this imply about α?

(d) Suppose that $\beta = 1$ and that the current solution is optimal. Find necessary and sufficient conditions for the existence of other optimal solutions.

(e) Suppose that $\beta = -1$. Find a sufficient condition for the problem to be infeasible.

(f) Suppose that $\beta = -1$, $\gamma = 3$ and the problem is feasible. Give an iteration of the dual simplex method.

(g) Suppose the current basis is optimal. What is the optimal cost? Is the solution degenerate? What is an optimal basis $\mathbf{B}$? What is $\mathbf{B}^{-1}$?

(h) Suppose the current basis is optimal. What are the dual variables?

(i) Assume that the basis associated with this tableau is optimal. Suppose also that b_1 in the original problem is replaced by $b_1 + \epsilon$. Give upper and lower bounds on ϵ so that this basis remains optimal.

(k) Assume that the basis associated with this tableau is optimal. Suppose also that c_1 in the original problem is replaced by $c_1 + \epsilon$. Give upper and lower bounds on ϵ so that this basis remains optimal.