15.093 Optimization Methods

Final Examination

Instructions:

1. There are 5 problems each with 20 points for a maximum of 100 points.
2. You are allowed to use class notes, your homeworks, solutions to homework exercises, and the book by Bertsimas and Tsitsiklis. You are not allowed to use any other book.
3. You have 3 hours to work in the examination.
4. Please explain your work carefully.
5. Good luck!

Problem 1 (20 points)

Please answer true or false. No explanation is needed. A correct answer is worth 2 points, no answer 0 points, a wrong answer -1.

1. A problem of maximizing a convex, piecewise-linear function over a polyhedron can be formulated as a linear programming problem.
2. The dual of the problem min x_1 subject to $x_1 = 0$, $x_1, x_2 \geq 0$ has a nondegenerate optimal solution.
3. If there is a nondegenerate optimal basis, then there is a unique optimal basis.
4. An optimal basic feasible solution is strictly complementary.
5. The convergence of the primal-dual barrier interior point algorithm is affected by degeneracy.
6. Given a local optimum $\bar{x}$ for a nonlinear optimization problem it always satisfies the Kuhn-Tucker conditions when the gradients of the tight constraints and the gradients of the equality constraints at the point $\bar{x}$ are linearly independent.
7. In a linear optimization problem with multiple solutions, the primal-dual barrier algorithm always finds an optimal basic feasible solution.
8. In the minimum cost flow problem with integral data all basic feasible solutions have integral coordinates.
9. The convergence of the steepest descent method for quadratic problems $\min f(x) = x'Qx$ highly depends on the condition number of the matrix Q. The larger the condition number the slower the convergence.
10. The convergence of the steepest descent method highly depends on the starting point.

Problem 2 (20 points)

Let $f(\cdot)$ be a concave function. Consider the problem

$$\begin{aligned} \text{minimize} \quad & \sum_{j=1}^{n} f(x_j) \\ \text{subject to} \quad & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \geq \boldsymbol{0}. \end{aligned}$$

Let $P = \{\boldsymbol{x} \in \mathcal{R}^n \mid \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b},\ \boldsymbol{x} \geq \boldsymbol{0}\}$.

(a) Prove that if there exists an optimal solution, there exists an optimal solution which is an extreme point of P.

(b) Suppose we now add the constraints that $x_j \in \{0, 1\}$ for all j, i.e., the problem becomes a 0-1 nonlinear integer optimization problem. Show that the problem can be reformulated as a linear integer optimization problem.

Problem 3 (20 points) Let $\boldsymbol{Q}, \boldsymbol{\Sigma}$ be $n \times n$ matrices. The matrix $\boldsymbol{\Sigma}$ is positive semidefinite.

$$\begin{aligned} \text{minimize} \quad & \boldsymbol{c}'\boldsymbol{x} + \frac{1}{2}\boldsymbol{x}'\boldsymbol{Q}\boldsymbol{x} \\ \text{subject to} \quad & \boldsymbol{d}'\boldsymbol{x} + \tfrac{1}{2}\boldsymbol{x}'\boldsymbol{\Sigma}\boldsymbol{x} \leq a. \end{aligned}$$

(a) Write the Kuhn-Tucker conditions.

(b) Propose an algorithm for the problem based on the Kuhn-Tucker conditions.

(c) Suppose the matrix $\boldsymbol{Q}$ is also positive semidefinite. Reformulate the problem as a semidefinite optimization problem.

Problem 4 (20 points)

(a) You are given points $(\boldsymbol{x}_i, a_i)$, $i = 1, \ldots, m$ where $\boldsymbol{x}_i$ are vectors in $\Re^n$ and a_i are either 0 or 1. The interpretation here is that point $\boldsymbol{x}_i$ is of category 0 or 1. We would like to decide whether it is possible to separate the points $\boldsymbol{x}_i$ by a hyperplane $\boldsymbol{f}'\boldsymbol{x} = 1$ such that all points of category 0 satisfy $\boldsymbol{f}'\boldsymbol{x} \leq 1$ and all points of category 1 satisfy $\boldsymbol{f}'\boldsymbol{x} > 1$. Propose a linear optimization problem to find the vector $\boldsymbol{f}$.

(b) You are given points $(\boldsymbol{x}_i, y_i)$, $i = 1, \ldots, m$, where $\boldsymbol{x}_i$ are vectors in $\Re^n$ and $y_i \in \Re$ are response variables. We would like to find a hyperplane $\boldsymbol{a}'\boldsymbol{x} = 1$, such that for all points $\boldsymbol{a}'\boldsymbol{x}_i \leq 1$, $y_i \approx \boldsymbol{\beta_1}'\boldsymbol{x}_i$, while if $\boldsymbol{a}'\boldsymbol{x}_i > 1$, $y_i \approx \boldsymbol{\beta_2}'\boldsymbol{x}_i$. More formally, for all those points $\boldsymbol{x}_i$ with $\boldsymbol{a}'\boldsymbol{x}_i \leq 1$, we will choose $\boldsymbol{\beta_1}$ in order to minimize

$$\sum_{i:\ \boldsymbol{a}'\boldsymbol{x}_i \leq 1} |y_i - \boldsymbol{\beta_1}'\boldsymbol{x}_i|,$$

while for all those points $\boldsymbol{x}_i$ with $\boldsymbol{a}'\boldsymbol{x}_i > 1$, we will choose $\boldsymbol{\beta_2}$ in order to minimize

$$\sum_{i:\ \boldsymbol{a}'\boldsymbol{x}_i > 1} |y_i - \boldsymbol{\beta_2}'\boldsymbol{x}_i|.$$

Propose an integer programming problem to find the vectors $\boldsymbol{a}, \boldsymbol{\beta_1}, \boldsymbol{\beta_2}$.

Problem 5 (20 points) Consider the problem

$$\begin{aligned} Z^* = \min \quad & \boldsymbol{c}'\boldsymbol{x} \\ s.t. \quad & \boldsymbol{a_1}'\boldsymbol{x} \geq b_1 \\ & \boldsymbol{a_2}'\boldsymbol{x} \geq b_2 \\ & \boldsymbol{x} \in \{0,1\}^n \end{aligned}$$

with $\boldsymbol{a_1}, \boldsymbol{a_2}, \boldsymbol{c} \geq \boldsymbol{0}$. We denote by Z_{LP} the value of the LP relaxation.

(a) Consider the relaxation:

$$\begin{aligned} Z_1 = \max_{\lambda \geq 0} \min \quad & \boldsymbol{c}'\boldsymbol{x} - \lambda(b_2 - \boldsymbol{a_2}'\boldsymbol{x}) \\ s.t. \quad & \boldsymbol{a_1}'\boldsymbol{x} \geq b_1 \\ & \boldsymbol{x} \in \{0,1\}^n \end{aligned}$$

Indicate how you can compute the value of Z_1. What is the relation among Z^*, Z_1, Z_{LP}. Please justify your answer.

(b) Propose a dynamic programming algorithm for computing Z^*.