15.093J/2.098J Optimization Methods
Assignment 6 Solutions

Exercise 6.1
These functions are twice differentiable; therefore, we can check the convexity using the Hessian matrices. A function $f(x)$ is convex if its Hessian matrix is positive semidefinite for all x. The definiteness of matrices can be determined by using the definition, eigenvalues or principal minors.

i) $f(x_1, x_2) = x_1^2 + 2x_1x_2 - 10x_1 + 5x_2$
$H(x_1, x_2) = \begin{pmatrix} 2 & 2 \\ 2 & 0 \end{pmatrix}$ is indefinite (using principal minors), thus f is neither concave nor convex.

ii) $f(x_1, x_2, x_3) = -x_1^2 - 3x_2^2 - 2x_3^2 + 4x_1x_2 + 2x_1x_3 + 4x_2x_3$
$H(x_1, x_2, x_3) = \begin{pmatrix} -2 & 4 & 2 \\ 4 & -6 & 4 \\ 2 & 4 & -4 \end{pmatrix}$ is indefinite (using principal minors again), thus f is neither convex nor concave.

iii) $f(x_1, x_2) = x_1 e^{-(x_1+x_2)}$
$H(x_1, x_2) = \begin{pmatrix} x_1 - 2 & x_1 - 1 \\ x_1 - 1 & x_1 \end{pmatrix} e^{-(x_1+x_2)}$ is not a definite matrix for all x, thus f is neither convex nor concave.

iv) $f(x_1, x_2, x_3) = -\sum_{i=1}^{3} \ln(x_i) + \frac{1}{x_1} + e^{x_2}$ for $x_i > 0$
We can prove this function to be convex by computing its Hessian matrix and showing that it is a positive definite matrix. The second way is to show that f is the sum of convex functions, namely $-\ln(x)$, $\frac{1}{x}$, and e^x.

Exercise 6.2
This a computational exercise. The main point is to calculate the direction in each iteration and the corresponding parameter α.$f_i(x)$ is unconstrained while $g_i(x)$ is constrained with the penalty function $p(x) = e' \log(x) + \log(1 - e'x)$. For unconstrained functions, bisection method seems to sufficient; however, the Armijo's rule yields better results for constrained problems.
For the final part, we can expect that when $\theta \to 0$, the solution tends to the optimal solution of optimizing $f_2(x)$ subject to $x > 0$ and $e'x < 1$. Conversely, if θ is large, the penalty function $p(x)$ will dominate the objective function. It means that the solution will tends to the optimal solution of optimizing the penalty function.

Exercise 6.3
Let $g_1(x) = x_1^2 + x_2^2 - 1$, $g_2(x) = (x_1 - 1)^3 - x_2$, we have:
Both constraints are tight at $\overline{x} = (1, 0)$. The gradients calculated at $\overline{x} = (1, 0)$ are $\nabla g_1(\overline{x}) = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$ and $\nabla g_2(\overline{x}) = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$. These two vectors are linearly independent, thus the constraint qualification condition holds at $\overline{x} = (1, 0)$.
$\nabla f(\overline{x}) = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$, the KKT condition is that there exists nonnegative a and b such that $\nabla f(\overline{x}) + a\nabla g_1(\overline{x}) + b\nabla g_2(\overline{x}) = 0$. Solving this system of two equations, we obtain $a = \frac{1}{2}$ and $b = 0$. Thus $\overline{x}$ is a KKT point.
$f(\overline{x}) = -1$, $x_1^2 + x_2^2 \leq 1$, thus $|x_1| \leq 1$. So we have: $f(x) = -x_1 \geq -1 = f(\overline{x})$. Thus $\overline{x}$ is also a global optimal solution.

Exercise 6.4
We assume here Q and R are symmetric matrices (common assumption) so that the gradient of $x'Qx$ can be calculated as $2Qx$ instead of the general form $(Q + Q')x$.

(a) First of all, we need to find conditions such that the given problem is feasible. These conditions can be found by solving the minimization problem $\min_{e'x=1} x'Rx$. R is a positive definite matrix, thus this problem is a convex problem. Applying the KKT conditions, we obtain the system of equation

$$\begin{cases} 2Rx + ue = 0 \\ e'x = 1 \end{cases}$$

The optimal solution is $\left(e'R^{-1}e\right)^{-1}$ with $x = \left(e'R^{-1}e\right)^{-1} R^{-1}e$. Thus the feasible condition is $e'R^{-1}e \geq 1$. With this feasible condition, the KKT condtions for the original problem are:

$$\begin{cases} 2Qx + c + 2uRx + ve = 0 \\ u[g(x) - 1] = 0 \\ u \geq 0 \\ g(x) \leq 1 \\ h(x) = 1 \end{cases}$$

We consider two cases as follows:
$u = 0$: Explicitly, the solution can be calculated using the formula

$$\overline{x} = \frac{1}{2}Q^{-1}\left(\frac{e'Q^{-1}c + 2}{e'Q^{-1}e}e - c\right)$$

if $e'Q^{-1}e \neq 0$. We also need to check the condition $g(\overline{x}) \leq 1$. Thus if $e'Q^{-1}e \neq 0$ and $g(\overline{x}) \leq 1$, then $x = \overline{x}$ is the solution.
Now consider the general case, if these two conditions do not hold and we consider $u \neq 0$. We need to solve the system of equations:

$$\begin{cases} 2Qx + c + 2uRx + ve = 0 \\ x'Rx = 1 \\ e'x = 1 \\ u \leq 0 \end{cases}$$

This system of nonlinear equations can be solved using the Newton method with the Jacobian matrix

$$J(x, u, v) = \begin{pmatrix} 2Q + 2uR & 2Rx & e \\ (2Rx)' & 0 & 0 \\ e' & 0 & 0 \end{pmatrix}$$

The Newton method can be implemented with addition nonnegative checking for u in each iteration (affecting the selection of the step size).
The process of finding a solution for the original problem is as follows:

1. $e'R^{-1}e < 1$, the problem is infeasible.
2. $e'R^{-1}e \geq 1$, $e'Q^{-1}e \neq 0$, and $g(\overline{x}) \leq 1$, then the solution is

$$x = \overline{x} = \frac{1}{2}Q^{-1}\left(\frac{e'Q^{-1}c + 2}{e'Q^{-1}e}e - c\right)$$

3. Otherwise, applying the Newton method to solve the system of equations

$$\begin{cases} 2Qx + c + 2uRx + ve = 0 \\ x'Rx = 1 \\ e'x = 1 \\ u \geq 0 \end{cases}$$

to find the numerical solution.

(b) This is again a computational exercise. The main point here is to select a good intial solution and to handle the case when the Jacobian matrix is ill-conditioned. We can select the solution $x = \left(e'R^{-1}e\right)^{-1} R^{-1}e$ as the intial solution and when the matrix is ill-conditioned, we can use the matrix $J(x)'J(x) + I$.
The final solution for this problem is $x = (0.687844, -0.389943, 0.702099)'$ and the optimal value is $f = -1.813610$.