15.093J/2.098J Optimization Methods
Assignment 5 Solutions

Exercise 5.1 BT, Exercise 10.2.
The decision variables are x_i, $i = \overline{1,20}$. $x_i = 1$ if the player p_i is selected; otherwise, $p_i = 0$. We have:
The total number of players in the team is 12: $\sum_{i=1}^{20} x_i = 12$
The team has at least 3 play makers: $\sum_{i=1}^{5} x_i \geq 3$
The team has at least 4 shooting guards: $\sum_{i=4}^{11} x_i \geq 4$
The team has at least 4 forwards: $\sum_{i=9}^{16} x_i \geq 4$
The team has at least 3 centers: $\sum_{i=16}^{20} x_i \geq 3$
The team has at least 2 NCAA players: $x_4 + x_8 + x_{15} + x_{20} \geq 2$
Average rebounding statistics constraint: $\sum_{i=1}^{20} r_i x_i \geq 12r$
Average assists statistics constraint: $\sum_{i=1}^{20} a_i x_i \geq 12a$
Average scoring statistics constraint: $\sum_{i=1}^{20} s_i x_i \geq 12s$
Average height statistics constraint: $\sum_{i=1}^{20} h_i x_i \geq 12h$
Average defense ability statistics constraint: $\sum_{i=1}^{20} d_i x_i \geq 12d$
Player p_5 is not in the team if the player p_9 is in the team: $x_5 \leq 1 - x_9$
Player p_2 and p_{19} can only be selected together: $x_2 = x_{19}$
At most 3 players from the same team $(p_1, p_7, p_{12}, p_{16})$ are selected: $x_1 + x_7 + x_{12} + x_{16} \leq 3$
With these constraints, the problem for the coach is to maximize the scoring average or total the score $\sum_{i=1}^{20} s_i x_i$

Exercise 5.2 BT, Exercise 10.4.
Consider x_{ij} is the number of module i we need to purchase in the year j, where $i = \overline{1,5}$ representing module A, B, C, D, and complete engine respectively and $j = \overline{1,3}$. We have: $x_{ij} \in Z_+$ for all i and j. We also need to know how many complete engines that will be broken into modules each year, let denote this quantity as x_{6j}, we have, $x_{6j} \in Z_+$ and $x_{6j} \leq x_{5j}$.
Demand constraint for a module i ($i = \overline{1,4}$) in year j: $\sum_{k=1}^{j}(x_{ik} + x_{6k}) \geq \sum_{k=1}^{j} d_{ik}$, where d_{ij} is the demand of the module i in year j.
Demand constraint for the complete engine in year j: $\sum_{k=1}^{j}(x_{5k} - x_{6k}) \geq \sum_{k=1}^{j} d_{5k}$, where d_{5j} is the demand of the complete engine in year j.
There is no inventory costs, no engine breaking costs, thus the total cost will be $\sum_{i=1}^{5}\sum_{j=1}^{3} c_{ij}x_{ij}$, where c_{ij} is the cost per unit of the module i, ($i = \overline{1,5}$), in year j. The objective is then to minimize this total cost.

Exercise 5.3 BT, Exercise 10.6.

(a) Define x_{ijk} to be the quantity of product k produced at plant i and shipped to market zone j, $x_{ijk} \geq 0$. z_{ik} is the quantity of product k produced at plant i, $z_{ik} \geq 0$. y_{ik} indicates whether product k is produced at plant i ($y_{ik} = 1$) or not ($y_{ik} = 0$). We can formulate the constraints as follows:
Shipping limit: $\sum_{j=1}^{J} x_{ijk} \leq z_{ik}$ for all i, k
Production limits: $m_{ik}y_{ik} \leq z_{ik} \leq M_{ik}y_{ik}$ for all i, k
Capacity constraint: $\sum_{k=1}^{K} q_{ik}z_{ik} \leq Q_i$ for all i
Demand constraint: $\sum_{i=1}^{I} x_{ijk} \geq d_{jk}$ for all j, k
The total cost includes the fixed cost, production variable cost, and transportation cost $\sum_{i=1}^{I}\sum_{k=1}^{K}(f_{ik}y_{ik} + v_{ik}z_{ik}) + \sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{k=1}^{K} c_{ijk}x_{ijk}$. The objective is to minimize this total cost.

(b) No plant may produce more than K_1 products: $\sum_{k=1}^{K} y_{ik} \leq K_1$ for all i

(c) Every products can be produced at most I_1 plants: $\sum_{i=1}^{I} y_{ik} \leq I_1$ for all k

(d) Plant 3 must produce product k_0 if neither plant 1 nor plant 2 produces it: $y_{3k_0} \geq 1 - y_{1k_0} - y_{2k_0}$

(e) Define u_{ij} to be the indicator whether market j is supplied by plant i ($u_{ij} = 1$) or not ($u_{ij} = 1$). The constraints are then can be formulated: $\sum_{i=1}^{I} u_{ij} = 1$ for all j and $x_{ijk} \leq d_{jk}u_{ij}$ for all i, j, k.

Exercise 5.4 BT, Exercise 10.8.
We can assume that the graph is complete; otherwise, just set the transportation cost to be infinity between two unconnected nodes. Consider the binary variable x_{ijk} which is used to indicate whether vehicle k on its route will travel on the arc from node i to node j ($x_{ijk} = 1$) or not ($x_{ijk} = 0$). Of course $x_{iik} = 0$ for all $i \in N$ and

$k = \overline{1,m}$. The constraints can be formulated as follows:
At node 1, there must be at most m vehicle leaving, $\sum_{j \in N, j \neq 1} x_{1jk} \leq 1$ for all k and these vehicles must return to node 1, $\sum_{i \in N, i \neq 1} x_{i1k} \leq 1$.
Each customer is visited once by only one vehicle $\sum_{k=1}^{m} \sum_{i \in N} x_{ijk} = 1$ (only one vehicle enters the node j, $j \in N, j \neq 1$) and $\sum_{k=1}^{m} \sum_{j \in N} x_{ijk} = 1$ (only one vehicle leaves the node i, $i \in N, i \neq 1$). This also means that for there will be at most one vehicle that travels from i to j.
Each vehicle cannot exceed its capacity $\sum_{i \in N, i \neq 1} \left(b_i \sum_{j \in N} x_{ijk} \right) \leq Q$ for all k.
The total cost is $\sum_{i \in N} \sum_{j \in N} \sum_{k=1}^{m} d_{\{i,j\}} x_{ijk}$. The objective is to minimize this transportation cost.
Exercise 5.5 BT, Exercise 11.1.

(a) $Z_{LP} = 7$ and $Z_{IP} = 6$.

(b) The convex hull of X is a the polyhedron with extreme points $(0,0), (2,0), (3,1), (2,2)$, and $(0,1)$.

(c) Convert the problem into standard form, we can apply the Gomory cut for the equality $x_2 + \frac{1}{6}s_1 + \frac{1}{6}s_2 = \frac{5}{2}$. The Gomory cut is then $x_2 + \lfloor \frac{1}{6} \rfloor s_1 + \lfloor \frac{1}{6} \rfloor s_2 = \lfloor \frac{5}{2} \rfloor$ or $x_2 \leq 2$.

(d) The LP optimal solution is $(2, 2.5)$. Two subproblem is created with additional constraint $x_2 \geq 3$, which is infeasible, or $x_2 \leq 2$, which results in optimal solution $(\frac{7}{3}, 2)$. We continue to branch from this subproblem. With the additional constraint $x_1 \geq 3$, we have integer solution $(1,3)$ and with additional constraint $x_1 \leq 2$, we also get an integer solution $(2,2)$ with higher cost. Thus the branch and bound can be stopped and the optimal integer solution is $(2,2)$.

(e) For part (e) and (f), we find the convex hull of X, construct the intersection with the relaxed constraint and then maximize the objective function over this polyhedron to obtain the value Z_D. It turns out that in this case, $Z_D = Z_{LP} = 7$.

(f) Similarly, we will obtain $Z_D = Z_{IP} = 6$ in this case.

Exercise 5.6 BT, Exercise 11.2.

(a) If the linear relaxation problem has the cost of $-\infty$, then there exists a nonegative feasible direction which improves the objective cost. In order to find that direction, one possible way is to convert the problem into standard form, applying the simplex method and the method will be terminated with an index j such that $c_j < 0$ and $d = -B^{-1} A_j \geq 0$. The direction we need is the direction d.
From the construction of the direction, if A and b are integers, then the direction d is a rational vector (B^{-1} can be constructed using Cramer's rule that deals with only integral determinants). Thus there exists an positive integer k such that kd is a nonnegative feasible integer direction that improves the objective cost. The integer problem is feasible, thus we conclude that the integer problem also has $-\infty$ cost.

(b) The answer is no. We can construct an example in which $Z_L P = 0$ and $Z_{IP} > 0$. In this case $Z_{IP} > a Z_{LP}$ for all $a > 0$. A simple example is $\min x_1$ subject to $0 \leq x_1 \leq 1, 0 \leq x_2 \leq 0.5, x_1 + 2x_2 \geq 1, x_1, x_2 \in Z$