15.093J/2.098J Optimization Methods
Assignment 3 Solutions

Exercise 3.1 BT, Exercise 4.27.
Prove (a) $\Rightarrow$ (b)
Consider the problem

$$\begin{array}{rc} \min & -x_1 \\ \text{s.t.} & \mathbf{Ax} \geq \mathbf{0} \\ & \mathbf{x} \geq \mathbf{0} \end{array}$$

For every feasible solution of this problem, $x_1 = 0$. Thus, the problem has a finite optimum and the optimal objective is 0. The dual problem

$$\begin{array}{rc} \max & \mathbf{p}'\mathbf{0} \\ \text{s.t.} & \mathbf{p}'\mathbf{A}_1 \leq -1 \\ & \mathbf{p}'\mathbf{A} \leq \mathbf{0}' \\ & \mathbf{p} \geq \mathbf{0} \end{array}$$

also has a zero finite optimum. It means that the dual problem is feasible.
Thus there exists a vector $\mathbf{p} \geq \mathbf{0}$ such that $\mathbf{p}'\mathbf{A} \leq \mathbf{0}'$ and $\mathbf{p}'\mathbf{A}_1 \leq -1$ or we can say $\mathbf{p}'\mathbf{A}_1 < 0$.
Prove (b) $\Rightarrow$ (a)
Let p_0 be a vector that satisfies all conditions in (b). Consider the problem

$$\begin{array}{rc} \min & (\mathbf{p}_0'\mathbf{A}_1)x_1 \\ \text{s.t.} & \mathbf{Ax} \geq \mathbf{0} \\ & \mathbf{x} \geq \mathbf{0} \end{array}$$

Its dual problem is then

$$\begin{array}{rc} \max & \mathbf{p}'\mathbf{0} \\ \text{s.t.} & \mathbf{p}'\mathbf{A}_1 \leq \mathbf{p}_0'\mathbf{A}_1 \\ & \mathbf{p}'\mathbf{A} \leq \mathbf{0}' \\ & \mathbf{p} \geq \mathbf{0} \end{array}$$

The dual problem is feasible, $\mathbf{p} = \mathbf{p}_0$ is a feasible solution. It has the finite optimum of zero. Thus the primal problem is also feasible and has the optimal solution of zero. So, for all feasible solution $\mathbf{x}$, $\mathbf{Ax} \geq \mathbf{0}$, $\mathbf{x} \geq \mathbf{0}$, we have: $(\mathbf{p}_0'\mathbf{A}_1)x_1 \geq 0$. $\mathbf{p}_0'\mathbf{A}_1 < 0$ and $\mathbf{x} \geq \mathbf{0}$, thus we need to have $x_1 = 0$.
So we have (a) $\Leftrightarrow$ (b).
Exercise 3.2 BT, Exercise 4.29.
Consider the problem

$$\begin{array}{rc} \min & \mathbf{0}'\mathbf{x} \\ \text{s.t.} & \mathbf{Ax} \geq \mathbf{b} \end{array}$$

The dual problem is then

$$\begin{array}{rc} \min & \mathbf{b}'\mathbf{p} \\ \text{s.t.} & \mathbf{p}'\mathbf{A} = \mathbf{0}' \\ & \mathbf{p} \geq \mathbf{0} \end{array}$$

This problem is feasibe, $\mathbf{p} = \mathbf{0}$ is a feasible solution. The primal is infeasible, the the dual must be unbounded. Assume that all constraints are linearly independent; otherwise, we can remove the dependent constraints and work with a system with less constraints.
The dual polyhedron is in the fist orthant of $\mathbf{R}^m$, thus it does not contain a line, which means the polyhedron has at least an extreme point. Staring the simplex method with this extreme point, the simplex method will stop at a basis $\mathbf{B}$ at which we can find a non-basic index j such that the corresponding feasible direction calculated from the simplex method allows the objective to go to $-\infty$. This direction can be simply calculated with the basis $\mathbf{B}$ and the column $\mathbf{A}_j$.
Consider the modified dual problem with only these $n+1$ columns, we have: the problem is still unbounded. Thus the corresponding modified primal problem is infeasible. The modified primal problem is the problem that consists of $n+1$ constraints corresponding to the $n+1$ columns in the dual problem.

So, with this argument, we have proven that we can find $n+1$ inequalities such that the system is still infeasible. (If the constraints of the dual problem are not linearly independent, we can find less than $n+1$ inequalities that makes the system infeasible, which is a stronger statement.)

Exercise 3.3 BT, Exercise 4.43.

(a) The general problem is

$$\begin{array}{ll} \min & c_1x_1 + c_2x_2 \\ \text{s.t.} & -x_1 + 2x_2 \geq -2 \\ & x_1 - x_2 \geq -3 \\ & x_1, x_2 \geq 0 \end{array}$$

Draw the polyhedron graphically, we can see that it has three extreme points $(0,0)$, $(2,0)$, and $(0,3)$. In addition to these three extreme points, we have two extreme rays $(1,1)$ and $(2,1)$. A feasible point in the polyhedron can be expressed as

$$(x_1, x_2) = \mu_1(2,0) + \mu_2(0,3) + \lambda_1(1,1) + \lambda_2(2,1)$$

where $0 \leq \mu_i \leq 1$, $\mu_1 + \mu_2 \leq 1$, and $\lambda_i \geq 0$.
So we have: $c_1x_1 + c_2x_2 = 2\mu_1c_1 + 3\mu_2c_2 + \lambda_1(c_1 + c_2) + \lambda_2(2c_1 + c_2)$.
$0 \leq \mu_i \leq 1$, thus $2\mu_1c_1 + 3\mu_2c_2$ is bounded. In order to have the finite optimum, the cost cannot go to $-\infty$. We have: $c_1x_1 + c_2x_2$, thus

$$c_1x_1 + c_2x_2 \not\to -\infty \Leftrightarrow \begin{cases} c_1 + c_2 \geq 0 \\ 2c_2 + c_2 \geq 0 \end{cases}$$

Thus the necessary and sufficient conditions are $c_1 + c_2 \geq 0$ and $2c_1 + c_2 \geq 0$.

(b) Consider the polyhedron associated with a feasible linear programming problem. Every linear programming problem can be transformed to the standard form with the same optimal objective value. Therefore, we will only consider the polyhedron in the standard form. The standard form polyhedron contains no line, thus we can always find the set of all extreme points $\mathbf{p^i}$ and extreme rays $\mathbf{w^j}$ for it. A feasible point within this polyhedron can be expressed as $\mathbf{x} = \sum_i \mu_i \mathbf{p}^i + \sum_j \lambda_j \mathbf{w}^j$, where $0 \leq \mu_i \leq 1$, $\sum_i \mu_i = 1$, and $\lambda_j \geq 0$.
We have, the objective value can be calculated as $\mathbf{c'x} = \sum_i \mu_i \mathbf{c'p}^i + \sum_j \lambda_j \mathbf{c'w}^j$.
Using the same arguments as in part (a), we have $\sum_i \mu_i \mathbf{c'p}^i$ is bounded because $0 \leq \mu_i \leq 1$. $\lambda_j \geq 0$, thus $\mathbf{c'x} \not\to -\infty \Leftrightarrow \mathbf{c'w}^j \geq 0 \quad \forall j$.
Thus the set of all vectors $\mathbf{c}$ such that a finite optimum will be obtained is the set $C = \{\mathbf{c} \in \mathbf{R}^n : \mathbf{c'w}^j \geq 0 \forall j\}$. Clearly, this is a polyhedron.

Exercise 3.4 BT, Exercise 5.5.

(a) The tableau is

	x_1	x_2	x_3	x_4	x_5
	0	0	$\bar{c}_3$	0	$\bar{c}_5$
$x_2 = 1$	0	1	-1	0	β
$x_4 = 2$	0	0	2	1	γ
$x_1 = 3$	1	0	4	0	δ

The necessary and sufficient conditions are $\bar{c}_3 \geq 0$ and $\bar{c}_5 \geq 0$.

(b) Continuing the simplex method with x_3 as the entering variable, we can see that x_1 will leave the basis. In the new tableau, a new optimal basic solution is obtained:

	x_1	x_2	x_3	x_4	x_5
	0	0	0	0	$\bar{c}_5$
$x_2 = 7/4$	$1/4$	1	0	0	$\beta + \frac{\delta}{4}$
$x_4 = 1/2$	$-1/2$	0	0	1	$\gamma - \frac{\delta}{4}$
$x_3 = 3/4$	$1/4$	0	1	0	$\frac{\delta}{4}$

(c) If $\bar{c}_3 \geq 0$ and $\bar{c}_5 \geq 0$, then the current solution is optimal.
Now consider the case when $\bar{c}_3 < 0$ or $\bar{c}_5 < 0$. We will construct a dual simplex tableau using the min-ratio

test between $\frac{\overline{c}_3}{2}$ and $\frac{\overline{c}_5}{\gamma}$ ($\gamma > 0$). This can be done with any values of $\overline{c}_3$ and $\overline{c}_5$ in this case. Thus, we always have a feasible dual simplex tableau, which means the dual problem is feasible. In addition, we have the primal is feasible. Thus both of them must have a finite optimum.
So we have for any values of $\overline{c}_3$ and $\overline{c}_5$, we can find an optimal basis solution.

(d) The inverse matrix of $\mathbf{B}$ can be found in the last three columns of the tableau because the last three columns of matrix $\mathbf{A}$ form the identity matrix.
The primal feasibility needs to be checked. The condition is then $\mathbf{B}^{-1}\mathbf{b} + \epsilon\mathbf{B}_1^{-1} \geq \mathbf{0} \Leftrightarrow -3/4 \leq \epsilon \leq 1$.

(e) The cost c_1 is changed; however, in the current basis, x_1 is the third column. Thus, the third row of $\mathbf{B}^{-1}\mathbf{A}$ matrix needs to be considered.
The conditions are then $\overline{c}_3 - 4\epsilon \geq 0$ and $\overline{c}_5 - \delta\epsilon \geq 0$. $\overline{c}_3 \geq 0$ and $\overline{c}_5 \geq 0$, thus the conditions can be expressed as follows

$$\begin{cases} \epsilon \leq \frac{\overline{c}_3}{4} & \delta = 0 \\ \epsilon \leq \min\left\{\frac{\overline{c}_3}{4}, \frac{\overline{c}_5}{\delta}\right\} & \delta > 0 \\ \frac{\overline{c}_5}{\delta} \leq \epsilon \leq \frac{\overline{c}_3}{4} & \delta < 0 \end{cases}$$

Exercise 3.5 BT, Exercise 5.6.

(a) The formulation is

$$\begin{array}{ll} \min & 35\sum_{i=1}^{4} x_i + 50\sum_{i=1}^{4} y_i + 5\sum_{i=1}^{4} z_i \\ \text{s.t.} & x_i + y_i + z_{i-1} - z_i = d_i \\ & x_i \leq 160 \\ & x_i, y_i, z_i \in \mathbf{Z}_+, z_0 = 0 \end{array}$$

(b) Remove the integrality constraints, we can solve the problem with Excel solver or Matlab, or any other linear programming package.
The solution is that the company A will produce the maximum amount of lamps within each month. It then has to hold 10 lamps in January and February. In March and April, the company will buy 55 and 20 lamps respectively from company C to meet the demand of company B. The total cost is \$26,250. We can keep this as the solution of the original problem because all values in the solution are integers.

(c) Change in RHS. Analyzing three cases seperately, maintenance in January, February, or March, we can see that the optimal basis is still the same (primal feasibility is still satisfied). Using the dual prices, the maintenance cost will be \$45, \$70, and \$75 for January, February, and March respectively. Thus, maintenance in January is recommended.

(d) We need to buy 55 lamps from the company C in March with the price of \$50 each while no lamp purchase is needed in January and February. The offered price from the company D is \$45. Thus we can purchase 50 lamps from the company D in March, the savings will be \$250.

(e) Change in cost. In February, the company A does not buy any lamps from the company C, this is a non-basic variable with the reduced cost 5. Thus if the company C makes the price more than \$5 lower than the original price, the company A will start to buy lamps in February.

(f) Change in cost again. This time the inventory in February is a basic variable. The optimality conditions are checked and it turns out that the current basis is still optimal. The cost will be increased by \$3 × 10 = \$30.

(g) The allowable decrease is 55, which is smaller than the amount we want to decrease, 60. The objective function is a convex function in terms of this change in RHS. Thus the lower bound will be \$26,250 - \$40 × 60 = \$23,850. Of course, with the lower demand, the cost must be lower. Thus the upper bound will be \$26,250 - \$40 × 55 = \$24,050.

Exercise 3.6 BT, Exercise 5.8.

(a) Only are Currier and Bluetail service sets produced, 2 Currier sets and 5 Bluetail ones. The total profit is \$102 × 2 + \$89 × 5 = \$649.

(b) For clay, \$1.429 is the amount of money we can make (lose) if we have more (less) one pound of clay within the allowable range.
For enamel and dry room hour, we do not gain or lose any money if these resources vary within their allowable ranges.

For kiln, \$20.143 is the amount of money we can make (lose) if we have more (less) one kiln hour within the allowable range.
For Primose, \$11.429 is the amount of money we will make if the number of Primose sets made by the method 1 is one more than that of Primose sets made by the method 2.

(c) 20 is within the allowable range. The dual price is \$1.429, which is greater than \$1.1, thus the manufacture should buy these additional 20 lbs. of clay.

(d) 30 is outside the allowable decrease range. We have, if we produce one Bluetail less, the dry room constraint will not be violated any more (other constraints are of course still satisfied). The decrease in total profit is then bounded by the value \$89.

(e) The dual price for this constraint is positive; therefore, it is recommended to have a positive number of Primose service sets produced with the method 1. We can say that in the new optimal solution, the number of Primose sets made by the method 1 will be positive.