15.093J/2.098J Optimization Methods
Assignment 2 Solutions

Exercise 2.1 BT, Exercise 3.17.
The initial tableau in Phase I is

-5	-1	-1	-3	-1	-2	0	0	0
$x_6 = 2$	1^*	3	0	4	1	1	0	0
$x_7 = 2$	1	2	0	-3	1	0	1	0
$x_8 = 1$	-1	-4	3	0	0	0	0	1

The final tableau in Phase I is

0	0	1	0	7	0	2	0	1
$x_1 = 2$	1	3	0	4	1	1	0	0
$x_7 = 0$	0	-1^*	0	-7	0	-1	1	0
$x_3 = 1$	0	1/3	1	4/3	1/3	1/3	0	1/3

Drive the artifical variable x_7 out of the basis

0	0	0	0	0	0	1	1	1
$x_1 = 2$	1	0	0	-17	1	-2	3	0
$x_2 = 0$	0	1	0	7	0	1	-1	0
$x_3 = 1$	0	0	1	11/3	1/3	2/3	$-1/3$	1/3

The initial tableau in Phase II is

-7	0	0	0	3	-5
$x_1 = 2$	1	0	0	-17	1^*
$x_2 = 0$	0	1	0	7	0
$x_3 = 1$	0	0	1	11/3	1/3

The final tableau in Phase II is

3	5	82/7	0	0	0
$x_5 = 2$	1	17/7	0	0	1
$x_4 = 0$	0	1/7	0	1	0
$x_3 = 1/3$	$-1/3$	$-4/3$	1	0	0

Exercise 2.2 BT, Exercise 3.19.

-10	δ	-2	0	0	0
4	1	η	1	0	0
1	α	-4	0	1	0
β	γ	3	0	0	1

(a) The current solution is optimal but the current basis is not. Thus the current solution is a degenerate optimal solution. So we have $\beta = 0$. Update the tableau using one simplex iteration

-10	$\delta + \frac{2\gamma}{3}$	0	0	0	2/3
$x_3 = 4$	$-1 - \frac{\gamma\eta}{3}$	0	1	0	$\frac{-\eta}{3}$
$x_4 = 1$	$\alpha + \frac{4\gamma}{3}$	0	0	1	4/3
$x_2 = 0$	$\frac{\gamma}{3}$	1	0	0	1/3

There are multiple optimal solutions, thus $\delta + \frac{2\gamma}{3} = 0$. In addition, we need to have a feasible direction, which requires $\frac{\gamma}{3} \leq 0$.
The conditions could be $\beta = 0$, $\delta + \frac{2\gamma}{3} = 0$, and $\gamma \leq 0$.

(b) The optimal cost is $-\infty$ when we have a feasible solution in the current tableau, a nonbasic variable x_i with $\bar{c}_i < 0$ and $\mathbf{u_i} = \mathbf{B^{-1}A_i} \leq \mathbf{0}$. We need $\beta \geq 0$ for problem feasiblity.

The variable x_2 cannot satisfy all the conditions for the $-\infty$ cost. For the variable x_1, the conditions then can be expressed as follows: $\alpha \le 0$, $\lambda \le 0$, and $\delta < 0$.

(c) The current solution is feasible if $\beta \ge 0$. If the solution is not degenerate then the current solution is definitely not optimal. Thus a condition could be simply $\beta > 0$.

Exercise 2.3 BT, Exercise 3.31.

(a) The set of all (b_1, b_2) is the convex hull of four points $A_1(2,1)$, $A_2(3,2)$, $A_3(1,1)$, and $A_4(1,3)$.

(b) There are more than one basis corresponding to a degenerate basic feasible solution. The set of all (b_1, b_2) is the union of six segments A_1A_2, A_1A_3, A_1A_4, A_2A_3, A_2A_4, and A_3A_4.

(c) If (b_1, b_2) is one of the four points A_i, then there is a basic feasible solution associated with it and this basic feasible solution has three bases.
If (b_1, b_2) is the intersection of A_1A_4 and A_2A_3 then there are two basic feasible solutions associated with it; each of them has two bases.
If (b_1, b_2) is not one of those points mentioned above, then there is one basic feasible solutions associated with it and this basic feasible solution has two bases.

(d) A feasible basis is optimal if the reduced cost vector is nonnegative. The reduced cost in this geometry setting can be calculated as the vertical distance from the dual plane to a point (A_j, c_j). Checking carefully with this optimality condition in mind, we can see that S_1 and S_4 are empty while S_2 is the convex hull of A_1, A_3, and A_4, S_3 is the convex hull of A_1, A_2, and A_4.

(e) The set of all (b_1, b_2) is the intersection of $S_2 \cup S_3$ with the set defined in part b). It is clear that this set is the union of five segments A_1A_2, A_1A_3, A_1A_4, A_2A_4, and A_3A_4.

(f) The first basis is not optimal (result from part d)). The entering variable is x_1. There are two options for leaving variables, x_2 or x_3. Both options lead to optimal bases (result from part d)); therefore, after one simplex iteration, we obtain the optimal solution and it is degenerate.

Exercise 2.4 BT, Exercise 4.2.

$$\begin{array}{ll} \min & \mathbf{c}'\mathbf{x} \\ \text{s.t.} & \mathbf{A}\mathbf{x} \ge \mathbf{b} \\ & \mathbf{x} \ge \mathbf{0} \end{array} \tag{1}$$

The dual problem is

$$\begin{array}{ll} \max & \mathbf{b}'\mathbf{p} \\ \text{s.t.} & \mathbf{p} \ge \mathbf{0} \\ & \mathbf{p}'\mathbf{A} \le \mathbf{c}' \end{array} \tag{2}$$

Convert the dual problem into the minimization format, we have:

$$\begin{array}{ll} -\min & (-\mathbf{b})'\mathbf{p} \\ \text{s.t.} & \mathbf{p} \ge \mathbf{0} \\ & (-\mathbf{A}')\mathbf{p} \ge -\mathbf{c} \end{array} \tag{3}$$

In order for these two problems are identical, the first thing is both primal and dual variable vector must have the same size. Thus $n = m$.
There must be a one-to-one mapping $d : \{1, \ldots, m\} \mapsto \{1, \ldots, m\}$ such that one constraint of the dual is mapped exactly with one constraint of the primal and vice versa. These two constraints must be exactly the same; therefore, we have: $a_{ij} = -ka_{jd(i)}$ and $b_i = -kc_{d(i)}$, $k > 0$. This means that the polyhedra of these two problems are exactly the same.
In addition, we need to have the cost vectors to be the same. It means that $-\mathbf{b} = \mathbf{c}$ or $-b_i = c_i$ for all i. With this condition, we will have $Z_P = -Z_D$; therefore, $Z_D = Z_P = 0$. The minus sign has no effect in this case and we can consider two problems are identical.
Thus the conditions are:

1. $m = n$
2. $\exists d : \{1, \ldots, m\} \mapsto \{1, \ldots, m\}, k > 0 : a_{ij} = -ka_{jd(i)}, b_i = -kc_{d(i)} \quad \forall i = \overline{1, m}$
3. $-b_i = c_i \quad \forall i = \overline{1, m}$

One possible mapping d is $d(i) = i$ for all i and we can select $k = 1$.

Exercise 2.5 BT, Exercise 4.2.

(a) True. If the optimal cost is equal to $\mathbf{c}'\mathbf{x}$ then there is an optimal basis associated with the given basic feasible solution. The corresponding dual basic solution is feasible and optimal.

(b) True. The primal auxiliary problem is always feasible, $(\mathbf{0}, \mathbf{b})$ is a feasible solution given that $\mathbf{b} \geq \mathbf{0}$. The objective cost is bounded from below (the objective cost is always nonnegative). Therefore, the primal problem has finite optimum, which means the dual problem is also feasible and has finite optimum.

(c) False. We have to remove the variable p_i from the dual problem. The problem with $p_i = 0$ added is not the dual problem of the primal problem with the i^{th} equality constraint removed. If we take the dual of the dual problem, an additional free variable needs to be introduced in the primal problem.

(d) True. The unboundedness criterion in the primal simplex method is that the vector $\mathbf{B^{-1}A_j}$ is nonpositive for a negative reduced cost $\bar{c}_j$. In this case, the primal cost is $-\infty$. Thus the dual problem must be infeasible.

Exercise 2.6

(1) A. The objective funtion is $\sum_{i=1}^{4} \sum_{j=1}^{7} c_{ij} x_{ij}$
B. Machine hour constraints are $\sum_{i=1}^{4} a_{ij} x_{ij} \leq b_j$. For Ambrosi mill, the constraint is $0.4x_{21} + 0.375x_{31} + 0.25x_{41} \leq 2,500$.
C. Demand constraints are $\sum_{j=1}^{7} x_{ij} \geq d_i$. For the extra fine yarn, the constraint is $x_{12} + x_{13} + x_{15} + x_{16} + x_{17} \geq 25,000$.
Solving the problem with an optimization tool, we get the optimal solution. In this optimal solution, all mills use up to their capacity except for De Blasi, which still has $1,886$ machine hours available. Filatoir produces all types of yarn at a specific amount as compared to the previous results in Feburary. The total cost is $1,382,544$.

(2) This scenario indicates a change in RHS. We need to check feasibility for the new solution. It turns out we only need to check for x_{34}, which has a new value of $628 > 0$. The cost will be reduced by $1,270.59$ while we need to pay extra $1,500$. The change is not recommended.

(3) The previous analysis showed that a 600 machine hour increase only shifts some production of medium size yarn from De Blasi to Filatoir. So it will be the same if the machine hours are increased by 300 hours. If the production cost is the same, the total cost is reduced by only 635.29. However, the production cost decreases for extra medium size yarn produced by Filatoir, which means there is additional cost reduction. The additional cost reduction is $4,023.53$. The total cost reduction is then higher than the rental cost of $3,000$, thus it is recommended that Filatoir rents this machine.

(4) Again, a change in RHS. With this current plan (current optimal basis), the maximum extra medium size yarn production is $5388kg$ and the shadow price for this production is 12.3 per kg. If the customer asks for $6,000kg$, then Filatoir need to change its current plan. Using the dual simplex method, a new optimal solution can be found after few iterations. In this new solution, De Blasi uses all its capacity and Filatoir has to outsource to Brescani. The new shadow price is 13.76. Thus Filatoir should use flexible price scheme and it would ask the customer whether they want to pay higher price than the current price if their extra demand exceeds $5388kg$.

(5) We need to consider the worst case and the best case: all the cost goes up by 5% or goes down by 5%. In both cases, the optimality condition remains. Thus the plan does not need to be changed.

(6) Changing the capacity of De Blasi mill will affect the RHS vector. We need to check the feasibility. There are $2,600$ machine hours and we only use up to $1,886$. Thus, 20% change in the capacity (520 hours) will not affect the current plan.

(7) The maximum cost that Filatoir is willing to pay must be equal to the extra cost that Filatoir has to pay for another plan if the Ambrosi mill will not produce any fine-size yarn at all. Setting the corresponding varibles to zero and solving the problem again, we will get the extra cost per month is $2,368$ or $28,426$ for twelve month contract as a one-time setup cost.

(8) Consider the overtime capacity of Guliani is another mill with the same machine hours required for each type of yarn, same capacity, same transportation cost but not the production cost. The production cost of this new mill is calculated using the formula: $c_{i8} = 0.5c_{i7} + 0.5 \times 1.13c_{i7}$. Construct new variables and constraints into the current problem, then we can use the current solution and current basis to create the initial basis for the new problem. Solving the new problem, we get the plan in which Filatoir will outsource $2040kg$ medium size yarn to Guiliani instead of De Blasi. Looking at the new cost matrix, we can see that the cost for medium size yarn overtime production of Guiliani is smaller than that of De Blasi even with 13% increase in worker wages.