15.093J Optimization Methods
Assinment 6 (100 points)
Due December, 3

One of the exercises in this homework has a computational component. You may use any machine and any computer language to execute them. We recommend, however, that you use MATLAB. Some good sources for help on Matlab are:

http://www-math.bgsu.edu/˜gwade/matlabprimer/

The purpose of the computational exercises is to allow you to gain some hands-on computational experience with the methods you learn in this course. We recommend that you approach these exercises by implementing every "piece" of the algorithms as a separate subroutine, and then call them in appropriate combinations. For example, it is useful to have a bisection line search subroutine (in a separate file) and then call it at every iteration of the steepest descent algorithm. You might also want to give some thought to the question of what arguments to pass to your subroutines. For example, keep in mind that we will have to call the bisection search on a different function of one variable at each iteration of the steepest descent method, and you will have to try your steepest descent method on different functions and at different starting points etc.

Some of the specific details in the computation exercises are purposely left to your own discretion. For example, in cases where you must use a line-search routine, you must choose your own tolerance for the line-search. Also, you must choose your own stopping criteria for your computation. Please include your code or pseudo-code in your write-up.

Problem 1: (15 points)
Which of the following functions is convex, concave or neither? Why?
i) $f(x_1, x_2) = x_1^2 + 2x_1x_2 - 10x_1 + 5x_2$,
ii) $f(x_1, x_2) = -x_1^2 - 3x_2^2 - 2x_3^2 + 4x_1x_2 + 2x_1x_3 + 4x_2x_3$,
iii) $f(x_1, x_2) = x_1e^{-(x_1+x_2)}$.
iv) $f(x_1, x_2, x_3) = -\sum_{i=1}^{3} ln(x_i) + \frac{1}{x_1} + e^{x_2}$, for $x_i > 0$, $i = 1, 2, 3$.

Problem 2: (30 points)
(a) Program in MATLAB three functions that implement (i) the Steepest Descent Algorithm, (ii) the Conjugate Gradient Algorithm, and (iii) Newton's method to solve $\min f(\mathbf{x})$. The functions should be given as parameters.
Hints: You can use a subroutine function for the line search component.
Although we recommend MATLAB for this part of the homework, you may also use any programming language you prefer.
(b) For $i = 1, 2$ we let

$$f_i(\mathbf{x}) = \frac{1}{2}\mathbf{x}'\mathbf{Q}_i\mathbf{x} + \mathbf{c}_i'\mathbf{x} + 10;$$

$$g_i(\mathbf{x},\theta) = \frac{1}{2}\mathbf{x}'\mathbf{Q}_i\mathbf{x} + \mathbf{c}_i'\mathbf{x} + 10 - \theta(\mathbf{e}'\mathbf{log}(\mathbf{x}) + \log(1-\mathbf{e}'\mathbf{x})),$$

where $\mathbf{e}$ is the vector of ones, and $\mathbf{log}(\mathbf{x})$ is a vector with the operator log applied to each component.

Implement and run the three algorithms we learnt in class (steepest descent, Newton's method and conjugate gradient method) in MATLAB to solve the following four problems: $\min f_1(\mathbf{x})$, $\min f_2(\mathbf{x})$, $\min g_1(\mathbf{x},\theta)$ and $\min g_2(\mathbf{x},\theta)$ using

$$\mathbf{Q_1} = \begin{bmatrix} 1 & -5 \\ -5 & 100 \end{bmatrix}; \quad \mathbf{c_1} = \begin{bmatrix} -15 \\ 150 \end{bmatrix}$$

and starting from $\mathbf{x}_0 = (0.2, 0.2)'$.

$$\mathbf{Q_2} = \begin{bmatrix} 8.5 & -1.5 & 3 \\ -1.5 & 8.5 & 3 \\ 3 & 3 & 4 \end{bmatrix}; \quad \mathbf{c_2} = \begin{bmatrix} -10 \\ -10 \\ -10 \end{bmatrix}$$

and starting from $\mathbf{x}_0 = (0.3, 0.4, 0.1)'$ and $\theta = 1$.

(c) For each of the three algorithms plot $f(\mathbf{x}_k)$ vs. k and $\frac{f(\mathbf{x}_{k+1})-f(\mathbf{x}^*)}{f(\mathbf{x}_k)-f(\mathbf{x}^*)}$ vs. k, where $\mathbf{x}^*$ is the optimal solution.

(d) Using your answer to (c) discuss the rate of convergence for each of the three algorithms and compare it with the one predicted from theory.

(e) Illustrate numerically what happens to the solution of the problem $\min g_2(\mathbf{x},\theta)$ as θ varies in the interval $[0.01, 10]$.

Problem 3: (30 points)

Consider the following problem:

$$\min \quad f(x) = -x_1,$$

subject to the constraints

$$x_1^2 + x_2^2 \leq 1$$
$$(x_1 - 1)^3 - x_2 \leq 0$$

- Show that the KKT constraint qualification holds at point $\bar{x} = (1, 0)$.
- Show that point $\bar{x} = (1, 0)$ is a KKT point and also a global optimal solution.

Problem 4: (25 points)

(a) Use the KKT conditions to solve the following problem

$$\begin{aligned} \min \quad & f(\mathbf{x}) = \mathbf{x}'\mathbf{Q}\mathbf{x} + \mathbf{c}'\mathbf{x} \\ \text{s.t.} \quad & g(\mathbf{x}) = \mathbf{x}'\mathbf{R}\mathbf{x} \leq 1 \\ & \mathbf{e}'x = 1 \end{aligned}$$

where $\mathbf{Q}$ is an invertible matrix, although not necessarily positive definite and $\mathbf{R}$ is a positive definite matrix. Note that $\mathbf{e}$ is a vector of ones.
(b) Apply your solution to the case where

$$\mathbf{Q} = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}; \quad \mathbf{c} = \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix} \quad \mathbf{R} = \begin{bmatrix} 1 & -0.5 & -0.4 \\ -0.5 & 1 & 0 \\ -0.4 & 0 & 1 \end{bmatrix};$$