14.123 Microeconomic Theory III
Spring 2009

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Lecture 10: Auction Games

1. Symmetric IPV Model: Equilibria of First- and Second-Price Auctions and the English Auction
2. Vickrey's Efficiency Principle
3. General Symmetric Affiliated Values Model (Milgrom & Weber (1982))

Read: Vickrey (1961), Milgrom & Weber (1982)

Why Study Auctions?

- Learn general ideas (e.g., Vickrey's efficiency principle) as well as useful techniques (e.g., comparative statics proofs).
- Auctions are simple market games with incomplete information; clean environments in which interesting effects can be exhibited and studied in isolation.
- Auction games (in particular, double auctions) provide the theoretical foundations for competitive markets.
- Auction theory can be relatively cheaply tested in field experiments on EBay.
- Auction theory and mechanism design have been used quite successfully to allocate resources (FCC auctions, etc.).

Example of an Auction Game

- Vickrey (1961) introduced and analyzed the first Bayesian game, even before Bayesian games were invented by Harsanyi.
- Example: Two bidders in a First Price Auction.
- Model: Each bidder has a valuation, $v_i \sim$ iid uniform [0,1].
  This fact is "commonly known", but v_i is privately known by i.
  Submit bids $b_1, b_2 \in [0,1]$; highest bid wins and is paid to seller.
  Payoff of bidder i: $u_i(v_i,b_i,b_j) = \mathbf{1}_{\{b_i \geq b_j\}} (v_i - b_i)$.
- Result: A bayesian Nash equilibrium in pure strategies is that bidder i with valuation v_i submits bid $v_i/2$.

■ Submitting $b_i \leq ½$ yields payoff $\Pr(b_i \geq v_j/2)(v_i - b_i) = 2b_i(v_i - b_i)$.
This is maximized in b_i at $b_i = v_i/2$, as claimed. ■

1. Symmetric IPV Model

- Fixed number of potential buyers (n); each draws a valuation v_i independently from [0,1] according to the same cdf F.
- Valuations are private (bidder i knows his valuation, does not care about the signals others get) and are privately known.
- Suppose that bidders have vNM utility function u.
  Assume $u(0)=0$, $0 < u' < \infty$, $u'' \leq 0$.
- THM: Equilibrium in First-Price Auction is given by diff. eqn.

$$b'(x) = (n-1)f(x)/F(x) \cdot u(x - b(x))/u'(x - b(x)); \quad b(0) = 0.$$

■ If all other bidders use $b(.)$, then i's profit from bidding $b_i = b(v_i')$ with valuation v_i is $F(v_i')^{n-1}\, u(v_i - b(v_i'))$, which should attain its maximum in v_i' at $v_i' = v_i$, hence the differential equation. The boundary condition is from $u(0-b(0)) = 0$, no arbitrage.■

Comparative Statics

- Lemma. Let $g, h : [0,\infty)\rightarrow\mathbb{R}$ continuous, differentiable, $g(0) \geq h(0)$; $\forall x > 0$, $\{g(x) < h(x)\} \Rightarrow \{g'(x) \geq h'(x)\}$. Then $g(x) \geq h(x)$, $\forall x \geq 0$.

■ See Milgrom & Weber (1982), p.1108. Idea: If h ever overtakes g then it must "cross from below", which it cannot by assumption.■

- THM: If u undergoes concave transformation (keeping $u(0) = 0$), then the equilibrium bid in the FPA increases for every valuation.

■ For simplicity, compare equilibrium $b(.)$ under strictly concave u (see p. 4) with equilibrium bid $\beta(.)$ under risk neutrality given by

$$\beta'(x) = (n-1)f(x)/F(x)\cdot(x-\beta(x)) \quad \text{with } \beta(0) = 0.$$

If $\beta(x) \geq b(x)$, then $b'(x) > \beta'(x)$ as $u(x-b(x))/u'(x-b(x)) > (x-\beta(x))$ by the strict concavity of u. By the Lemma, $\beta(x) \leq b(x)$, $\forall x \geq 0$. ■

Comparison of Auctions

- Consider <u>iid private values</u>, compare FPA with English auction or second-price auction (SPA); allow risk aversion.
- Recall that in SPA and English auctions, winner pays second-highest valuation (irrespective of risk preferences).
  - Under private values, bidding v_i in the SPA / keeping bidding while price < v_i in the English auction is dominant strategy. ■
- <u>THM</u>: In FPA with risk neutrality, $\beta(v_i) = \mathrm{E}[\max_{j\neq i}\{v_j\} \mid \forall j: v_j \leq v_i]$.
  - Differentiate $\beta(v_i) = \int_0^{v_i} x\,(n\text{-}1)F^{n-2}(x)f(x)\mathrm{d}x / F^{n-1}(v_i)$ in v_i:

$$\beta'(v_i) = v_i\,(n\text{-}1)F^{n-2}(v_i)f(v_i) \,/\, F^{n-1}(v_i)$$

$$- \int_0^{v_i} x\,(n\text{-}1)F^{n-2}(x)f(x)\mathrm{d}x\,(n\text{-}1)F^{n-2}(v_i)f(v_i) \,/\, F^{n-1}(v_i)$$

$$= v_i\,(n\text{-}1)f(v_i)/F(v_i) - \beta(v_i)\,(n\text{-}1)f(v_i)/F(v_i).$$

$\beta(v_i)$ indeed satisfies the differential equation on page 5. ■

Comparison of Auctions

- THM (Vickrey): Under iid private values and risk neutrality, the expected revenue of the FPA, SPA, and the English auction is the same: the expected value of the second-highest valuation.

  Expected revenues are equal, but the variances differ: the FPA is less risky for the seller than either the SPA or the English auction.

- THM: Under iid private values and risk aversion, the expected revenue of the first-price auction exceeds that of the second-price auction and/or the English auction.
  - Under risk neutrality, expected revenue equivalence.
    SPA and English auction equilibria same with risk aversion.
    FPA equilibrium bids increase if bidders are risk averse. ■

2. Vickrey and Efficiency

- William Vickrey was particularly interested in designing mechanisms that induce efficient use of economic resources.
- Vickrey suggested congestion pricing for toll roads and public transportation. (Transportation economics considers him its founding father.) Trivia: Vickrey invented a subway turnstile that automatically adjusted the access price as a function of traffic.
- Vickrey's Idea: An efficient mechanism (auction, etc.) should make participants pay their external effects on all affected parties.
- The winner of an auction "crowds out" the second-highest bid, hence the winner should pay the second-highest bid ($\Rightarrow$ SPA).
- K-units: Each bidder submits K bids; highest K bids win. If i wins k_i units then he pays the k_i highest losing bids submitted by others.

3. General Symmetric Model

- Milgrom and Weber (ECMA, 1982):
  General, symmetric model with affiliated values, risk neutrality.
- Information structure: Bidder i=1,…,n privately observes signal $X_i \in \mathbb{R}$; random variables $S = (S_1,\ldots,S_m)$ represent other risk.
- Buyer i's valuation is $V_i = \gamma(X_i, \{X_j\}_{j\neq i}, S)$, where γ is continuous, strictly increasing in its first argument, weakly in the rest.
  Note that i's valuation is symmetric in the signals of all $j \neq i$.
- Assume that f, the joint pdf of $(X_1,\ldots,X_n,S_1,\ldots,S_m)$, is symmetric in its first n arguments and that the expectation of V_i is finite.
- Affiliation: For all $z,z' \in \mathbb{R}^{n+m}$, $f(z\wedge z')\, f(z\vee z') \geq f(z)\, f(z')$.
  ($z\wedge z'$ is coordinate-wise min, $z\vee z'$ is coordinate-wise max.)

General Symmetric Model

- Recall that affiliation of f is equivalent to f being log-spm.
- In general, affiliation of (Y,Z) is stronger than $\text{Cov}(Y,Z) \geq 0$, and stronger than the non-negative covariance of all monotone transformations of Y and Z, and even positive regression dependence, $\Pr(Y>y|Z=z) \uparrow$ in z.
- Independence is a special case.
- Example:
  Suppose S is an "underlying common value" and X_i is i's "random sample" with conditional pdf $g(x_i|s)$ satisfying the Monotone Likelihood Ratio property: $g(x_i|s)/g(x_i|s')$ increasing in x_i for all $s > s'$. Then (X_i,S) are affiliated.

Preliminary Results

- Analyze behavior of bidder i=1 (wlog by symmetry), denote $Y_1,\ldots,Y_{n\text{-}1}$ the largest, …, smallest of $X_2,\ldots,X_n$.
- If $(X_1,\ldots,X_n,S)$ are affiliated then so are $(X_1,Y_1,\ldots,Y_{n\text{-}1},S)$.
- $V_1 = \gamma(X_1, Y_1,\ldots,Y_{n\text{-}1}, S)$.
- Theorem 5 of Milgrom-Weber 1982: Let $Z_1,\ldots,Z_k$ be affiliated random variables and H: $\mathbb{R}^k \rightarrow \mathbb{R}$ a weakly increasing function. Then, for all $a_1 \leq b_1, \ldots, a_k \leq b_k$,

  $h(a_1,b_1, \ldots, a_k,b_k) = \mathrm{E}[\, H(Z_1,\ldots,Z_k) \mid a_1 \leq Z_1 \leq b_1, \ldots, a_k \leq Z_k \leq b_k \,]$

  is weakly increasing in all of its arguments.
- Note: $[a_1, b_1], \ldots, [a_k, b_k]$ define a sublattice in $\mathbb{R}^k$. Theorem 5 says: If Z is an affiliated k-dim random variable, then its expected value conditional on a sublattice increases with the sublattice.

Equilibrium of the SPA

- Let $v(x,y) = \mathrm{E}[\ V_1 \mid X_1=x,\ Y_1=y\ ]$: Buyer 1's valuation conditional on his own signal and the highest of the other buyers' signals.
- <u>THM</u>: A symmetric eqm of the SPA is that all buyers bid $B^*(x) = v(x,x)$, their expected valuation conditional on winning in a tie.
- <u>Proof.</u> By Theorem 5, $B^*(x) = v(x,x)$ is strictly ↑ in x. Hence if the other bidders use B^* then Bidder 1 pays $B^*(Y_1)$ when he wins.

  Suppose Buyer 1 bids b with signal $X_1 = x$. His payoff is,

  $$\mathrm{E}[(V_1 - B^*(Y_1))\,\mathbf{1}_{\{B^*(Y_1)\le b\}} \mid X_1=x\ ]$$

  $$= \mathrm{E}[(v(X_1,Y_1) - v(Y_1,Y_1))\,\mathbf{1}_{\{B^*(Y_1)\le b\}} \mid X_1=x\ ]$$

  $$= \int_{-\infty}^{B^{*-1}(b)} [v(x,\eta) - v(\eta,\eta)]\, f_{Y_1}(\eta|x)\, \mathrm{d}\eta.$$

  The integrand is positive iff $\eta<x$, hence the integral is maximized by setting $B^{*-1}(b) = x$, i.e., by bidding $b=B^*(x)$. ■

Public Signal Disclosure in SPA

- Should the seller commit to publicly disclose S before the auction?
- Define $w(x,y,z) = \mathrm{E}[\ V_1 \mid X_1{=}x,\ Y_1{=}y,\ S{=}z\ ]$.
- If the Seller commits to disclose z (the realization of S) then an equilibrium of the SPA is for all buyers to bid $B^{**}(x) = w(x,x,z)$.
- <u>THM</u>: Commitment to disclosing S weakly increases revenue:

  $R_N = \mathrm{E}[v(Y_1,Y_1) \mid \{X_1{>}Y_1\}] \ \le\ R_I = \mathrm{E}[w(Y_1,Y_1,S) \mid \{X_1{>}Y_1\}]$.

■ Note, $v(x,y) = \mathrm{E}[v(X_1,Y_1) | X_1{=}x, Y_1{=}y] = \mathrm{E}[w(X_1,Y_1,S) | X_1{=}x, Y_1{=}y]$.

For $x \ge y$, $v(y,y) = \mathrm{E}[w(X_1,Y_1,S) | X_1{=}y, Y_1{=}y]$

$= \mathrm{E}[w(Y_1,Y_1,S) | X_1{=}y, Y_1{=}y] \le \mathrm{E}[w(Y_1,Y_1,S) | X_1{=}x, Y_1{=}y]$.

So, $R_N = \mathrm{E}[v(Y_1,Y_1) \mid \{X_1{>}Y_1\}] \le \mathrm{E}[\ \mathrm{E}[w(Y_1,Y_1,S) | X_1,Y_1] \mid \{X_1{>}Y_1\}\ ]$

$= \mathrm{E}[w(Y_1,Y_1,S) \mid \{X_1{>}Y_1\}] = R_I$. ■

Equilibrium of the English Auction

- "Button" auction: Continuous price clock, irreversible public exit.
- Strategy: Drop-out price given history of exits and own signal.
- Let $b_0(x) = E[\ V_1 \mid X_1{=}x,\ Y_1{=}x,\ \ldots,\ Y_{n-1}{=}x\ ]$, and for all $k{=}1,\ldots,n{-}1$ and prices $(p_1,\ldots,p_k)$, set $b_k(x,p_1,\ldots,p_k)$ recursively equal to $E[\ V_1 \mid X_1{=}Y_1{=}\ldots{=}Y_{n-k-1}{=}x,\ b_0(Y_{n-1}){=}p_1,\ \ldots,\ b_{k-1}(Y_{n-k},\ p_1,\ldots,p_{k-1}){=}p_k]$.
- THM: $(b_0,\ldots,\ b_{n-1})$ played by all bidders is an equilibrium.
- Proof. By Theorem 5, b_k is strictly increasing in x for all k. Bidders exit in increasing order of signals, losers' signals revealed. If Buyers $2,\ldots,n$ use $(b_0,\ldots,\ b_{n-1})$ then, if Buyer 1 wins, he pays $E[\ V_1 \mid X_1{=}y_1,\ Y_1{=}y_1,\ \ldots,\ Y_{n-1}{=}y_{n-1}]$, which is less than his valuation, $E[\ V_1 \mid X_1{=}x,\ Y_1{=}y_1,\ \ldots,\ Y_{n-1}{=}y_{n-1}]$, iff $x \geq y_1$. Using $(b_0,\ldots,b_{n-1})$ Buyer 1 wins iff $X_1 \geq Y_1$, exactly when his profit is non-negative.■

Comments

- Ex post equilibrium: $(b_0, \ldots, b_{n\text{-}1})$ is best response to all others playing $(b_0, \ldots, b_{n\text{-}1})$ even if the buyers know each others' signals (i.e., given $y_1, \ldots, y_{n\text{-}1}$). But: $(b_0, \ldots, b_{n\text{-}1})$ is not dominant strategy.
- Interpretation of equilibrium strategy: Bid expected valuation conditional on winning in a tie with all remaining participants. (In SPA equilibrium strategy was to condition on a two-way tie.)
- The seller's revenue from buyer 1 in the English Auction is the same as it is in the SPA with $Y_2=y_2, \ldots, Y_{n\text{-}1}=y_{n\text{-}1}$ publicly revealed.
- The seller gains from the public revelation of signals affiliated with the buyers' valuations, hence the expected revenue of EA exceeds that of SPA. This is called the Linkage Principle.

Equilibrium of the FPA

- THM: There exists a strictly increasing symmetric equilibrium in the FPA where each bidder i with signal value x_i submits $\beta(x_i)$.
- We characterize β as the solution to a differential equation. If the other bidders use β, then buyer 1 with signal x bidding b gets
  $\pi(b,x) = \mathrm{E}[\ (V_1-b)\ \mathbf{1}_{\{\beta(Y_1)\leq b\}} |\ X_1=x\ ] = \int_{\underline{x}}^{\beta^{-1}(b)} [v(x,\eta) - b]\, f_{Y_1}(\eta|x)\ \mathrm{d}\eta.$
- Maximization in b yields the FOC,
  $[v(x,\beta^{-1}(b)) - b]\, f_{Y_1}(\beta^{-1}(b)|x)\ /\ \beta'(\beta^{-1}(b)) - \int_{\underline{x}}^{\beta^{-1}(b)} f_{Y_1}(\eta|x)\ \mathrm{d}\eta = 0.$
- In equilibrium it is optimal to bid $b= \beta(x)$, hence
  $[v(x,x) - \beta(x)]\, f_{Y_1}(x|x)\ /\ \beta'(x) - F_{Y_1}(x|x) = 0$, or equivalently
  $\beta'(x) = [v(x,x) - \beta(x)]\, f_{Y_1}(x|x)/F_{Y_1}(x|x)$, which is positive.
- If the support of X_i is bounded, i.e. $\underline{x} > -\infty$, then the boundary condition for this differential equation becomes $\beta(\underline{x}) = v(\underline{x},\underline{x})$.

Comparison of Auctions

- We already established (linkage principle): Expected revenue of English Auction ≥ Expected revenue of SPA.
- <u>THM</u>: Expected revenue of SPA ≥ Expected revenue of FPA.

■ Let $W^M(x,z)$ denote the expected payment Buyer 1 makes in mechanism $M \in \{\text{SPA,FPA}\}$ conditional on $X_1 = z$, playing as if his signal realization were x, and winning.

- $W^{\text{FPA}}(x,z) = \beta(x)$
  $W^{\text{SPA}}(x,z) = \text{E}[\ v(Y_1,Y_1) \mid X_1 = z,\ Y_1 \le x\ ]$.
- Note: $\partial W^{\text{FPA}}(x,z)/\partial z = 0 \le \partial W^{\text{SPA}}(x,z)/\partial z$.
- Define $R(x,z) = \text{E}[\ V_1 \mathbf{1}_{\{Y_1 \le x\}} \mid X_1{=}z\ ]$, Buyer 1's expected valuation conditional on $X_1 = z$, pretending $X_1 = x$ and winning.

Proof, continued

- In mechanism $M \in$ {FPA, SPA}, Buyer 1 maximizes in x

$$R(x,z) - W^M(x,z)\, F_{Y_1}(x|z).$$

  In equilibrium, the maximum is attained at $x = z$.

- FOC: $\partial R(x,z)/\partial x - \partial W^M(x,z)/\partial x\, F_{Y_1}(x|z) = W^M(x,z)\, f_{Y_1}(x|z)$ at $x=z$.

- $W^{FPA}(z,z) > W^{SPA}(z,z) \Rightarrow \partial W^{FPA}(x,z)/\partial x < \partial W^{SPA}(x,z)/\partial x$ at $x=z$.

- Combined with $\partial W^{FPA}(x,z)/\partial z \leq \partial W^{SPA}(x,z)/\partial z$, this gives:

  If $W^{FPA}(z,z) > W^{SPA}(z,z)$ then $dW^{FPA}(z,z)/\mathrm{d}z < \partial W^{SPA}(z,z)/\mathrm{d}z$.

- Since $W^{FPA}(\underline{x},\underline{x}) = W^{SPA}(\underline{x},\underline{x})$, Lemma implies $W^{FPA}(z,z) \leq W^{SPA}(z,z)$ for all $z \geq \underline{x}$. The expected payment made by the winner is weakly greater in the SPA than it is in the FPA. ■

Summary

- Symmetric, iid private values, risk neutrality: Expected revenues of FPA and SPA are equal (Vickrey, 1961).

  Generally: "Revenue Equivalence Thm" in Mechanism Design.

- Risk aversion of the buyers (or the seller) favors FPA for seller.
- Affiliated valuations (positive, statistical correlation of information) favors SPA, and English auction is even better.
- Asymmetries would make revenue comparison inconclusive.
- Other interesting (solved) questions:
  - Bidders' preferences over auction forms.
  - Stochastic number of bidders; entry
  - Information acquisition in auctions.