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14.123 Microeconomic Theory III
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Lecture 9: Signaling Games

1. Intuitive Criterion and Divinity D1
2. Spencian Signaling Games
3. Cheap Talk

Read: FT 11.1-3; Crawford & Sobel: Strategic Information Transmission, *Econometrica*, 60 (1982), 1431-1450.

Previously in 14.123…

- In extensive-form, imperfect- or incomplete-information games, require sequential rationality (best reply to some beliefs) from all players. Beliefs are computed using Bayes' rule when possible.
- Sequential equilibrium needs some sequence $(\sigma^{\varepsilon},\mu^{\varepsilon})$ to converge to (σ,μ), such that each σ^{ε} is fully mixed and μ^{ε} is consistent with it.
- Trembling-hand perfection: $\exists\sigma^{\varepsilon}$ ε-constrained eqm, $\sigma^{\varepsilon} \to \sigma$ as $\varepsilon \to 0$. (Agent-normal form. Enough to find one such sequence.)
- An equilibrium is stable if it is trembling-hand perfect for any sequence of $\varepsilon(s_i)$ weight-constraints that converge to zero. Issue of existence resolved by defining stable sets of equilibria.

1. Signaling Games

- DEF: Signaling game. A two-player Bayesian game such that:
  1. Nature selects P1's type, $\theta \in \Theta$ with probs $\pi(\theta) > 0$, Θ finite.
  2. P1 (Sender) chooses action $m \in M$
     M is either finite or compact, $M \subset \mathbb{R}$.
  3. P2 (Receiver) observes m but not θ, picks $y \in Y$
     Y is either finite or compact, $Y \subset \mathbb{R}$.

  Payoffs are $u_1(\theta,m,y)$, $u_2(\theta,m,y)$.
- Commonly used in social sciences. Examples:
  - 'Beer-quiche'; 'Quants and Poets get MBAs' games.
  - Spence's labor market signaling (FT pp. 456-460).
  - Pure communication games.

PBE in Signaling Games

- Denote $\mu(m) \in \Delta(\Theta)$ P2's <u>belief</u> about P1's type after seeing m. $\mu_\theta(m)$ is the weight P2 puts on type $\theta \in \Theta$ after seeing m.
- Denote $y^*(\mu,m)$ P2's (mixed) <u>best response(s)</u> to m given beliefs μ.
- Denote $y^*(T,m)$ the set of (possibly mixed) best responses if P2 believes $\theta \in T$, i.e., all $y^*(\mu,m)$ such that $\mu(m)$ has support T.
- <u>DEF</u>: <u>Perfect Bayesian equilibrium</u>. Assessment (m,y,μ), with $m: \Theta \to \Delta(M)$, $y: M \to \Delta(Y)$, $\mu: M \to \Delta(\Theta)$ is PBE if
  - $m(\theta) \in \text{argmax}_{m'} \{u_1(\theta,m',y(m'))\}$; (P1 best responds)
  - $y \in y^*(\mu,m')$ for all $m' \in M$; (P2 best responds given beliefs)
  - μ satisfies Bayes rule (B) for all $m' \in m(\Theta)$.
- Introduced by Fudenberg and Tirole (1991).

Example: What PBE Rules Out

	m_1	m_2: y_1	m_2: y_2	π
θ_1	1,2	0,0	0,1	π_1
θ_2	1,2	-1,0	2,1	π_2

- Nature picks row (P1's type θ_i w/ prob π_i), P1 matrix, P2 column.
- P1 pooling on m_1, P2 replying y_1 to m_2 is (Bayesian) Nash eqm. Not PBE, because y_1 is not a best reply to any belief μ.
- <u>THM</u>: In signaling games, PBE $\Leftrightarrow$ sequential equilibrium.
  - ■ Beliefs of P2 generated by P1's trembles. FT p. 346.■
- PBE can involve P2 believing that P1 uses a weakly dominated strategy, or that P1 uses an equilibrium-dominated strategy.

What PBE Does Not Rule Out

	m_1	m_2: y_1	m_2: y_2	π
θ_1	1,2	0,2	0,1	π_1
θ_2	1,2	-1,0	2,1	π_2

- P1 pooling on m_1, P2 replying y_1 to m_2 is perfect Bayesian eqm supported by P2's beliefs $\mu(\theta_1|m_2) \geq 1/2$.
- This equilibrium is also sequential and trembling-hand perfect.
- But the only agent of P1 that can improve upon his equilibrium payoff by playing m_2 is θ_2 .
- Forward induction: P2 may try to figure out what P1 "wants to say" by deviating – P2 does not think that it is a tremble.

Equilibrium Dominance

- DEF: Action m' is equilibrium dominated for type θ of P1 in PBE (m,y,μ), if $u_1(\theta,m(\theta),y(m)) > u_1(\theta,m',y)$ for all $y \in y^*(\Theta,m')$.

  Weaker than dominance: m against $y(m)$ beats m' against $y^*(\Theta,m')$.

- Basic requirement of forward induction: Off-eqm beliefs should not put weight on nodes reached by equilibrium dominated actions.

  Beer-quiche: In 'pooling on quiche' eqm, 'beer' is equilibrium dominated for W, hence the equilibrium fails forward induction.

- Intuition? Cho & Kreps (QJE, 1987) propose a "speech":

  "By deviating to 'beer', you must believe that I am S as type W would not gain from this offer compared to his eqm payoff."

- Stiglitz Critique: P2 should infer W from lack of deviation to 'beer' and reply 'duel' to 'quiche'. Thus even W would prefer to deviate.

Intuitive Criterion

- DEF: For PBE assessment (m,y,μ) and off-eqm message m' define

  $$J(m') = \{ \theta \in \Theta : u_S(\theta,m(\theta),y(m)) > \max_{y' \in y^*(\Theta,m')} u_S(\theta,m',y') \}.$$

  The equilibrium fails the Intuitive Criterion at m' if $J(m') \neq \Theta$ and

  $$\{ \theta \in \Theta : u_S(\theta,m(\theta),y(m)) < \min_{y' \in y^*(\Theta \backslash J(m'),m')} u_S(\theta,m',y') \} \neq \emptyset.$$

- Due to Cho & Kreps (QJE, 1987).
- $J(m')$ is set of types for which m' is equilibrium-dominated.
  A PBE fails the Intuitive Criterion if P2's best response with any beliefs on $\Theta \backslash J(m')$ induces a deviation.
- Eliminates 'Pooling on quiche' in beer-quiche game.
  Generally weaker than stability in signaling games.

Divinity D1

			y_1	y_2	π
θ_1	2,2		1,0	4,3	2/3
θ_2	2,2		1,5	3,3	1/3
	m_1		m_2		

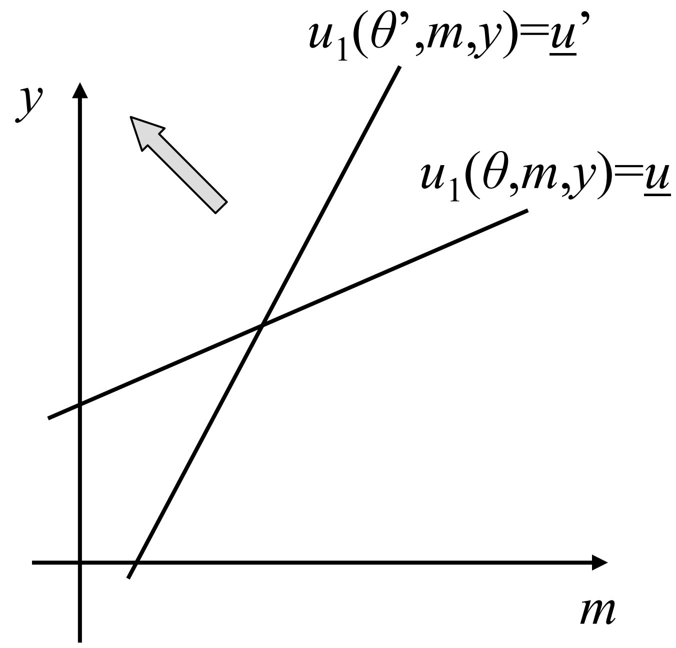
- Pooling on m_1 (with $m_2 \mapsto y_1$, $\mu_{\theta_1}(m_2) \leq 2/5$) is PBE with IC.
- But: θ_1 gains from deviation if $\sigma_2(y_2) \geq 1/3$, while θ_2 gains from deviation if $\sigma_2(y_2) \geq 1/2$. So, θ_1 gains for more responses to m_2.
- <u>DEF</u>: An equilibrium satisfies <u>Divinity "D1" Criterion</u> if P2's off-equilibrium beliefs only put weights on types of P1 that gain from the deviation for the most (mixed) best responses.
- D1 is generally weaker than stability. Similar, but stronger concepts (still weaker than stability) are discussed in FT Ch. 11.2.

2. Spencian Signaling Games

- DEF: Spencian signaling game: M, $Y \subset [0,\infty)$, $\Theta \subset \mathbb{R}$ finite;
  - For all $y' > y$, $u_1(\theta,m,y') > u_1(\theta,m,y)$. (Monotonicity)
  - For any belief $\mu \in \Delta(\Theta)$, unique best response $y^*(\mu,m) < \infty$, and y^* increases in μ (increases as μ increases in FSD sense).
  - $[\partial u_1(\theta,m,y)/\partial m]/[\partial u_1(\theta,m,y)/\partial y]$ is increasing in θ. (Spence-Mirrlees sorting condition.)
- Situations it intends to model:
  - Sender (P1) prefers Receiver (P2) to take higher actions;
  - Receiver wants his reply correlate with the Sender's type;
  - It is relatively less costly for higher Sender types to send higher messages.

Example

- Labor-market signaling:
  θ is productivity,
  m is schooling, y is wage.
- $u_1(\theta,m,y) = y - m/\theta$.
  Schooling is "cheaper" for higher type (talent?).
- $y^*(\theta,m) = \theta$.
  Firm(s) pay market wage for productivity.
- In this formulation schooling is unproductive; does not have to be.



Indifference curves, $\theta' < \theta$
(arrow indicates increasing prefs)

Riley Outcome

- Denote $\Theta = \{\theta_1, \ldots, \theta_k\}$ with $\theta_1 < \ldots < \theta_k$.
- DEF: The Riley outcome is a list of $(\underline{m}_i, \underline{y}_i)$, $i=1,\ldots,k$, such that

$$\underline{m}_1 = \text{argmax}_m\, u_S(\theta_1, m, y^*(\theta_1, m)) \text{ and } \underline{y}_1 = y^*(\theta_1, \underline{m}_1);$$

while for all $i = 2,\ldots,k$, $\underline{m}_i$ maximizes $u_S(\theta_i, m, y^*(\theta_i, m))$ subject to $u_S(\theta_j, m, y^*(\theta_i, m)) \leq u_S(\theta_j, \underline{m}_j, \underline{y}_j)$, $\forall j<i$, and $\underline{y}_i = y^*(\theta_i, \underline{m}_i)$.

- This is the minimal-cost separating equilibrium in the game.
- Riley outcome in the Labor Market Example:

$$\underline{m}_1 = 0,\ \underline{y}_1 = \theta_1; \quad \forall i > 1: \underline{m}_i = \underline{m}_{i\text{-}1} + \theta_{i\text{-}1}(\theta_i - \theta_{i\text{-}1}),\ \underline{y}_i = \theta_i.$$

  - Type $\theta_{i\text{-}1}$ prefers $\underline{m}_{i\text{-}1}$ to $\underline{m}_i$ iff $\theta_{i\text{-}1} - \underline{m}_{i\text{-}1}/\theta_{i\text{-}1} \geq \theta_i - \underline{m}_i/\theta_{i\text{-}1}$. In the minimal-cost separating equilibrium, set $\underline{m}_i$ as low as possible and still satisfy this incentive constraint. ■

Recall D1 Criterion

- Fix a PBE (m,y,μ) where $m: \Theta \rightarrow M$, $y: M \rightarrow Y$, $\mu: M \rightarrow \Delta(\Theta)$; define $u_S^*(\theta) = u_S(\theta,m(\theta),y(m(\theta)))$, equilibrium payoff of type θ.
- Informal definition: (m,y,μ) satisfies the D1 Criterion if P2's off-equilibrium beliefs only put weights on types of P1 that gain from the deviation for the most best responses.
- DEF: (m,y,u) fails the D1 Criterion if there exists $m' \in M \backslash m(\Theta)$ and $\theta, \theta' \in \Theta$, such that $\mu(\theta|m') > 0$, and

$$\{ y' \in y^*(\Theta,m') \mid u_S^*(\theta) \leq u_S(\theta,m',y') \} \subsetneq \{ y' \in y^*(\Theta,m') \mid u_S^*(\theta') \leq u_S(\theta',m',y')\}.$$

D1 Selects Riley

- THM: PBE outcome satisfying D1 $\Leftrightarrow$ Riley outcome.

  For simplicity, work with $u_1(\theta,m,y) = y - m/\theta$ and $y^*(\theta,m) = \theta$.

- Lemma: If θ plays m with positive probability in a PBE, then D1 implies that for all off-equilibrium $m' > m$ and $\theta' < \theta$: $\mu(\theta'|m') = 0$.

■ For all θ' and $m' > m$, define $\hat{y}(\theta',m') = u_1^*(\theta') + m'/\theta'$.
Type θ' gains from deviating to $m' > m$ iff reply is $y > \hat{y}(\theta',m')$.

Now, $\hat{y}(\theta',m') \leq \hat{y}(\theta,m')$ iff $u_1^*(\theta')+m'/\theta' \leq u_1^*(\theta) + m'/\theta$.
By equilibrium, $y(m) - m/\theta' \leq u_1^*(\theta')$ and $y(m) - m/\theta = u_1^*(\theta)$, so

$$y(m) - m/\theta' + m'/\theta' \leq y(m) - m/\theta + m'/\theta.$$

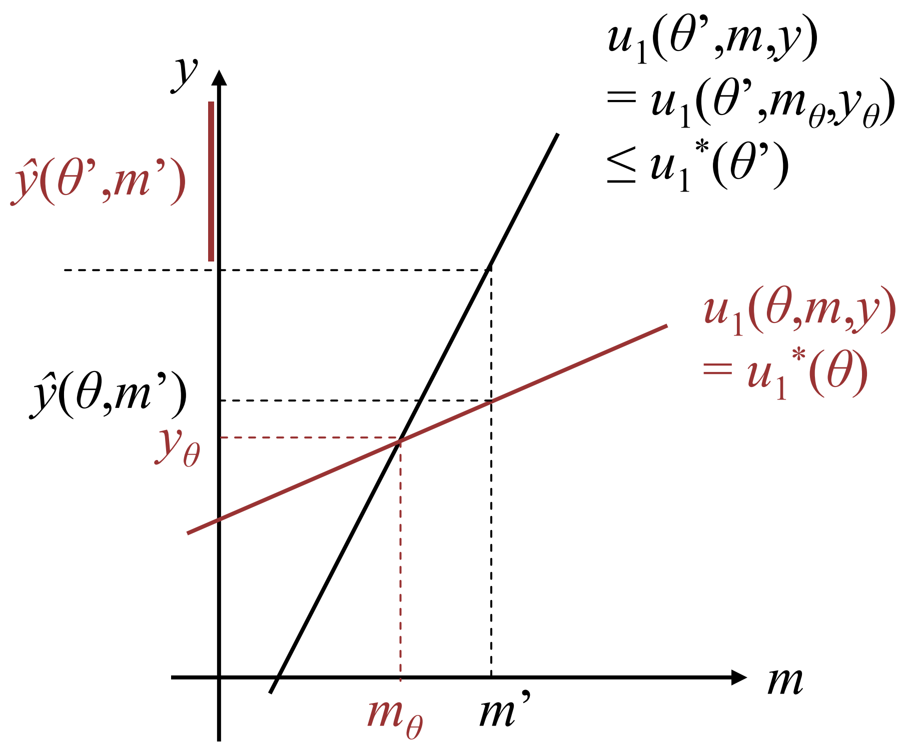
Therefore $\hat{y}(\theta',m') \leq \hat{y}(\theta,m')$ and $m' > m$ imply $\theta \leq \theta'$.

$\theta' < \theta$ gains from $m' > m$ for fewer replies than θ does; apply D1.■

Illustration of the Lemma

- For all θ' and $m' > m_\theta$, $\hat{y}(\theta',m') = u_1^*(\theta')+m'/\theta'$. Type θ' gains from deviating to $m' > m$ iff reply is $y > \hat{y}(\theta',m')$.
- Graph shows:

  If $m' > m_\theta$ and $\theta' < \theta$, then $\hat{y}(\theta',m') > \hat{y}(\theta,m')$.



Proof that D1 Selects Riley:

- Recall lemma: If θ plays m in equilibrium, then D1 implies that for all $m' > m$ and $\theta' < \theta$, we have $\mu(\theta'|m') = 0$.
- If θ is the highest type that plays m in equilibrium with positive probability and $\theta' < \theta$ also plays m in equilibrium, then $y(m) < \theta$. By deviating to $m' = m+\varepsilon$, type θ guarantees a reply $y \geq \theta$ because by the lemma, $\forall\theta' < \theta$, $\mu(\theta'|m') = 0$. Profitable, hence no pooling.
- Non-minimal separation by type θ is ruled out by D1 because θ gains from deviating to $m(\theta)$-ε for more replies than any $\theta' < \theta$.■
- Issues with D1: (i) Not intuitive (called "divine" for a reason); (ii) Specifies beliefs too strictly: P2 must put zero weight on all types of P1 that do not gain for the most replies to a deviation – not needed for sustaining the Riley-outcome in equilibrium.

3. Cheap Talk

- Crawford & Sobel (1982): Bayesian sender-receiver game, where
  - Sender knows $\omega \in [0,1]$, pdf f; sends message $m \in M \supseteq [0,1]$. All messages are available in all states ("talk is cheap").
  - Receiver takes action $y \in \mathbb{R}$ given S's message. R cannot commit to a "reply rule" prior to S's message.
  - Message does not enter utility (second meaning of "cheap"). Single-peaked, concave utilities in y given ω. $u_i(\omega,y)$, i=S,R; $\forall\omega$, $\exists y$: $\partial u_i(\omega,y)/\partial y = 0$, $\partial^2 u_i(\omega,y)/\partial y^2 < 0$.
  - Sorting condition: $\partial^2 u_i(\omega,y)/\partial\omega\partial y > 0$. $\Rightarrow y_i(\omega) = \text{argmax}_y\, u_i(\omega,y)$ is <u>increasing</u> in ω.
    - ■ $\partial u_i(\omega,y_i(\omega))/\partial y \equiv 0$, use implicit fcn theorem. ■

Crawford-Sobel Cheap Talk

- Perfect Bayesian / sequential equilibrium (Q,P,y):
  - Sender's strategy: measure $Q(m|\omega)$, $\int_M dQ(m|\omega) = 1$, $\forall\omega$.
    Suppose Q has a density, $q(m|\omega)$ on M. PBE condition:
    If m is in the support of q, then $m \in \text{argmax}_{m'}\, u_1(\omega,y(m'))$.
  - Receiver's belief: measure $P(\omega|m)$, density $p(\omega|m)$ on Ω.
    R's strategy is y: $M \to \mathbb{R}$.

    PBE conditions: (i) $y(m) = \text{argmax}_{y'} \int_0^1 u_2(\omega,y')\, dP(\omega|m)$.
    (ii) Bayes rule: $p(\omega|m) = q(m|\omega)f(\omega) / \int_0^1 q(m|w)\, f(w)dw$.

<u>Note</u>: Since m does not have payoff consequences, we can assume without loss of generality that all messages are sent in equilibrium, so there is no off-equilibrium message m'.

Equilibrium Characterization

- DEF: Sender has upward bias if $\forall \omega \in [0,1]$, $y_S(\omega) > y_R(\omega)$.
- DEF: In equilibrium (Q,P,y), action y' is induced in state ω' if $\int_{M'} dQ(m'|\omega') > 0$, where $M' = \{m' : y(m') = y'\}$.
$Y' = \{y' : \exists \omega' \text{ s.t. } y' \text{ is induced in } \omega'\}$, the set of all induced actions.
- DEF: (Q,P,y) is an interval-partition equilibrium if for all induced actions $y' \in Y'$, the set of states in which y' is induced is an interval.
- THM: If the sender has upward bias, then all equilibria of the Crawford-Sobel cheap talk game are interval-partition equilibria with finitely many induced actions.
- Note: There can be infinitely many different messages (and even mixing over messages) in a finite interval-partition equilibrium, it does not matter: There are only finitely many induced actions.

Proof

- Suppose $x, z \in Y'$ with $x < z$. If x is induced in state ω_x and z in ω_z, then S weakly prefers x to z in state ω_x, opposite is true in state ω_z. By continuity of u_S, there is ω' such that $u_S(\omega',x) = u_S(\omega',z)$.
- By the strict concavity of u_S: $x < y_S(\omega') < z$.
- By the sorting condition: x is not induced by $\omega > \omega'$ and z is not induced by any $\omega < \omega'$. To see this: $\omega < \omega'$ iff $u_S(\omega,z) - u_S(\omega,x) = \int_x^z \partial u_S(\omega,y)/\partial y < \int_x^z \partial u_S(\omega',y)/\partial y = u_S(\omega',z) - u_S(\omega',x)$.
- R knows this, so (by the sorting condition), $x \le y_R(\omega') \le z$.
- Then, $z - x \ge y_S(\omega') - y_R(\omega') \ge \varepsilon > 0$ by upward bias assumption, which implies Y' is finite in any equilibrium.
- By sorting: ω' that induce same $y' \in Y'$ form an interval. Q.E.D.

Leading Example

- Uniform f, constant bias $b = y_S(\omega) - y_R(\omega)$, quadratic loss utility functions $u_i(\omega,y) = -(y-y_i(\omega))^2$ for i=S,R. Normalize $y_R(\omega) = \omega$.
- Interval-partition equilibrium with N induced outcomes: $\{ 0=a_0, \ldots, a_N=1 \}$ such that $(a_i+a_{i+1})/2 - (a_i+b) = a_i+b - (a_{i-1}+a_i)/2$.
- Eqm condition yields second-order difference equation,

$$a_{i+1} = 2a_i - a_{i-1} + 4b \quad \text{for } i=1,\ldots,N;\ a_0 = 0,\ a_N = 1.$$

- For initial value a_1, get $a_i = ia_1 + 2i(i-1)b$, $i = 1,\ldots,N$. Terminal condition $a_N = 1$ gives $a_1 = [1 - 2N(N-1)b]/N$. N-partition equilibrium exists iff $a_1 > 0$, i.e., $N(N-1) \leq 1/(2b)$.
- If there exists an equilibrium with N different induced actions, then there is also an equilibrium with $(N$-1) different induced actions. (This is true in general, not only in the example.)

Comments

- Positive message: talk is cheap, yet it may be informative.
  Negative message: continuum of states, finite # of induced actions.
- No information transmission (only "babbling") if bias is too high.
- Can use all feasible messages in any equilibrium.
  E.g, $M = [0,1]$: let S send m = uniform on $[a_{i-1},a_i]$ when $\omega \in [a_{i-1},a_i]$.
  Standard equilibrium refinements do not restrict the equilibrium set.
- *Chen, Kartik & Sobel (2008): Denote y_0 action induced in $\omega = 0$.

  *<u>THM</u>: If $u_S(0,y_R(0)) \geq u_S(0,y_0)$, then there is another equilibrium with more induced actions. Under certain conditions, $u_S(0,y_R(0)) \leq u_S(0,y_0)$ holds in the equilibrium with the most induced actions.