14.123 Microeconomic Theory III
Spring 2009

MIT 14.123 (2009) by Peter Eso
Lecture 5: Background Risk

1. Calibrating Risk Aversion
2. Refresher on (Log-)Supermodularity
3. Background Risk & DARA

Solve: Problem set handed out in class.

Calibrating Risk Aversion

- Suppose u is CRRA(ρ) = $x^{1-\rho}/(1-\rho)$, and the agent's initial wealth is w = \$100,000. Consider a gamble ± \$$X$ with 50-50% chance.
  - X=30,000; ρ = 40: Risk premium is about \$28,700 – too high.
  - X=30,000; ρ = 2: Risk premium is about \$9,000 – OK?
  - X=500; ρ = 2: Risk premium is about \$2.5 – too low?
- It may be difficult to come up with reasonable parameters that match introspection and real-life evidence.

  Luckily, the Equity Premium Puzzle fizzled in 2008!

- Today: Background risk in real life (not present in bare-bones examples) may cause some of the apparent puzzles. Decision-making with risky initial wealth is non-trivial & interesting.

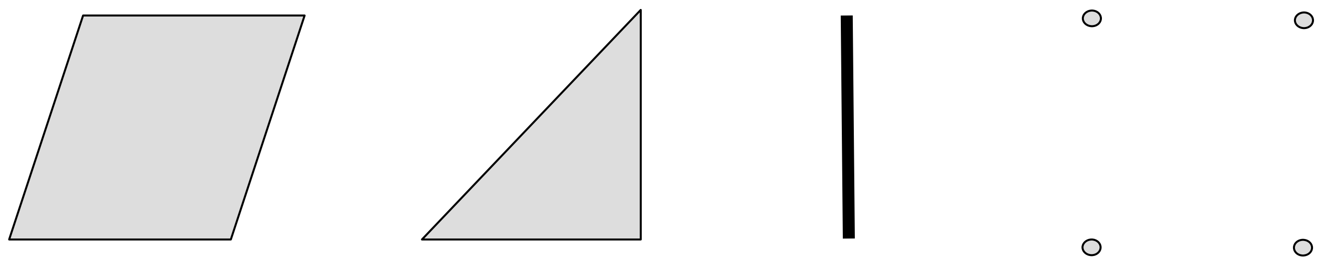
Lattices

- DEF: For any partially ordered set $(X, \geq)$ and all $x, y \in X$ define
  - The join $x \vee y = \inf\{z \in X : z \geq x, z \geq y\}$;
  - The meet $x \wedge y = \sup\{z \in X : x \geq z, y \geq z\}$.
- DEF: $(X,\geq)$ is a lattice if $\forall x, y \in X$: $x \vee y \in X, x \wedge y \in X$.
- DEF: Given $(X, \geq)$, for $S, Z \subseteq X$, let $S \geq Z$ ("S weakly exceeds Z in the strong set order") if $\{x \in S, y \in Z\} \Rightarrow \{x \vee y \in S, x \wedge y \in Z\}$.
- THM: $(X, \geq)$ is a lattice iff $X \geq X$. (trivial)
- DEF: $(X, \geq)$ is a complete lattice if $\forall S \subseteq X$, $\inf S \in X$, $\sup S \in X$.
- DEF: L is a sublattice of a partially ordered set $(X, \geq)$ if L is a subset of X and it is a lattice.

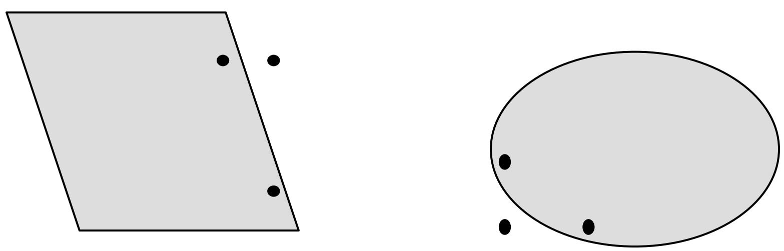
Sublattices of $\mathbb{R}^n$

- Example: $L = \mathbb{R}^n$, $\geq$ is the usual (coordinate-wise) order on vectors; $x \vee y$ is coordinate-wise maximum, $x \wedge y$ coordinate-wise minimum.

- Sublattices of $\mathbb{R}^2$:



- Not sublattices of $\mathbb{R}^2$:



(Log-)Supermodularity

- DEF: A function f: $X \to \mathbb{R}$ is supermodular if for all $x, y \in X$,

$$f(x \vee y) + f(x \wedge y) \geq f(x) + f(y).$$

- DEF: A function f: $X \to \mathbb{R}_+$ is log-supermodular if for all $x, y \in X$,

$$f(x \vee y) \cdot f(x \wedge y) \geq f(x) \cdot f(y).$$

  That is, h is log-spm if $\log(f)$ is supermodular.

- THM (Topkis): A twice-differentiable f: $\mathbb{R}^n \to \mathbb{R}$ is supermodular iff for all $i,j = 1,\ldots,n$, $i \neq j$, and $x \in \mathbb{R}^n$, $\partial^2 f(x)/\partial x_i \partial x_j \geq 0$.

- Examples: If $X = \mathbb{R}$, then f is supermodular, as well as log-spm. If $X = \mathbb{R}^n$ and $f(x) \equiv f(\sum x_n)$, then f is log-spm iff log-convex.

- (Log-)supermodularity captures complementarity.

Single Crossing

- DEF: Given lattice $(X, \geq)$, function $f: X \rightarrow \mathbb{R}$, is quasi-supermodular if $\forall x,y \in X$, $f(x) - f(x \wedge y) \geq (>)\ 0$ implies $f(x \vee y) - f(y) \geq (>)\ 0$.
- THM: If a function is supermodular or log-spm then it is quasi-spm.
- DEF: $g: \mathbb{R} \rightarrow \mathbb{R}$ is single-crossing if $\forall t' \geq t$: $g(t) \geq (>)\ 0 \Rightarrow g(t') \geq (>)\ 0$.
- DEF: $f: \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ satisfies single-crossing differences if $\forall z' > z$, $g(t) \equiv f(z',t) - f(z,t)$ is single-crossing.
- THM: If $(X, \geq)$ is a sublattice of $\mathbb{R}^n$, then quasi-supermodularity $\Rightarrow$ single-crossing differences in every pair of coordinates.
  - ■ Prove both Theorems in Recitation. ■
- Single-crossing conditions are used in a variety of settings.

Monotonic Comparative Statics

- DEF: Let $B,B' \subseteq X$. $B' \geq B$ if $\forall b \in B$, $b' \in B'$: $b \wedge b' \in B$ and $b \vee b' \in B'$.
- THM (Topkis): Let $(X, \geq)$ be a partially ordered set, $f: X \times \mathbb{R} \to \mathbb{R}$ a supermodular function, B a sublattice of $(X, \geq)$, and $t' \geq t$. Then,

$$x^*(t,B) \equiv \text{argmax } \{ f(x,t) \mid x \in B \}$$

  is sublattice of $(X, \geq)$ that is increasing ("isotone") in t and B.
- THM (Milgrom & Shannon): Let $(X, \geq)$ be a sublattice of $\mathbb{R}^n$ and $T \subseteq \mathbb{R}$. If B is a sublattice of X and $f: X \times T \to \mathbb{R}$ is q-spm, then $x^*(t,B) \equiv \text{argmax } \{ f(x,t) \mid x \in B \}$ is increasing in B and t.

  ■ Prove the latter Theorem in Recitation. ■

Instances of Log-Supermodularity

- In mathematical statistics: Total Positivity of Order 2 (Karlin). (Re-)discovered and first applied in economics by Ian Jewitt, Paul Milgrom, and Xavier Vives (separately) in the 80's.

- The price-elasticity of demand, $P \cdot D_P(P,t)/D(P,t)$, is weakly increasing in t iff the demand function, $D(P,t)$, is log-spm.

  - $\partial \ln(D(P,t))/\partial P = D_P(P,t)/D(P,t)$. By Topkis' Thm: $D(P,t)$, is log-spm iff $D_P(P,t)/D(P,t) \uparrow$ in t. ■

- A vector of random variables is affiliated (a notion of "positively correlated" used in auction theory) iff their joint pdf is log-spm.

  - Definition of affiliated pdf f : $f(z \wedge z') \, f(z \vee z') \geq f(z) \, f(z')$. Non-negative correlation conditional on any outcome-pair.■

Instances of Log-Supermodularity

- A parametrized family of payoff-distributions $F(x,t)$ is increasing in t in the MPR sense iff F is log-spm.
  - $F(x,1)$ MPR-dominates $F(x,0)$ iff $F(x,1)/F(x,0)$ ↑ in x. ■
- A parametrized family of payoff-distributions $F(x,t)$ is increasing in t in the MLR sense iff F' is log-spm.
  - $F(x,1)$ MLR-dominates $F(x,0)$ iff $F'(x,1)/F'(x,0)$ ↑ in x. ■
- A Bernoulli-vNM utility index u is DARA iff $u'(w+z)$ is log-spm in wealth (w) and the realization of the prize (z).
  - u' is log-spm iff log-convex; $\partial \ln(u'(x))/\partial x = u''(x)/u'(x)$. ■
- Agent 1 is more risk averse than 2 if $u_i'(w)$ is log-spm in (w,i).
  - log-spm: $\partial \ln(u_i'(w))/\partial w = u_i''(w)/u_i'(w)$ is increasing in i. ■

A Theorem from Statistics

- Let $X = X_1 \times \ldots \times X_n$ and $Z = Z_1 \times \ldots \times Z_m$ be sublattices of $\mathbb{R}^n$ and $\mathbb{R}^m$ with $X_i \subseteq \mathbb{R}$ and $Z_j \subseteq \mathbb{R}$ for all i and j. Let $T \subseteq \mathbb{R}$.
- Suppose u: $X \times Z \rightarrow \mathbb{R}_+$ is a bounded utility function and f: $Z \times T \rightarrow \mathbb{R}_+$ is a probability density function on Z for all $t \in T$. Define

$$U(x,t) = \int u(x,z)\, f(z,t) \mathrm{d}z.$$

- THM (Karlin): If u and f are log-spm, then U is log-spm.
- Remark: Products of log-spm functions are clearly log-spm, but arbitrary sums of log-spm functions are not log-spm.

MCS in Decision Theory

- THM: If u and f are log-spm, then $\forall t \in T$ and sublattice $B \subseteq X$,

$$x^*(t,B) \equiv \text{argmax } \{U(x,t) \mid x \in B\} \text{ is increasing in } t \text{ and } B.$$

That is, for all $t' \geq t$ and sublattices $B' \geq B$ (in strong set order), we have $x^*(t',B') \geq x^*(t,B)$.

  ■ Combine Karlin's Thm (previous slide) with Milgrom & Shannon's Thm (slide #6). ■

Problem with Background Risk

- Agent has vNM utility u for wealth, strictly increasing & concave.
- The agent is exposed to uninsurable risk: Her initial wealth is $w_0 + \tilde{w}$, where w_0 is a scalar, $\tilde{w}$ is a random variable.
- Can invest in asset with random net return $\tilde{x}$, independent of $\tilde{w}$.
- Problem: Invest α to maximize $\mathrm{E}[u(w_0+\tilde{w}+\alpha\tilde{x})]$.
- Define $v(z) = \mathrm{E}[u(z+\tilde{w})]$. Problem $\Leftrightarrow \max_\alpha \mathrm{E}[v(w_0+\alpha\tilde{x})]$.
- Are "good properties" of u inherited by v ?
  - Clearly, $v' > 0$, $v'' < 0$. (Differentiation goes through E.)
  - If u is DARA, is v DARA as well?
  - If u is DARA & $\mathrm{E}[\tilde{w}] \leq 0$, then is v more risk averse than u?

DARA with Background Risk

- THM: If $u: \mathbb{R}\to\mathbb{R}$ is a DARA utility and f a pdf on $Z \subseteq \mathbb{R}$, then

$$v(x) \equiv \int_Z u(x+z)\, f(z)\mathrm{d}z,\ \forall x \in \mathbb{R},$$

is a DARA utility function.

  ■ u is DARA $\Leftrightarrow$ $u'(x_1+x_2+z)$ is log-spm in (x_1,x_2,z).

  f is log-spm because Z is one-dimensional.

  Let $v'(x_1+x_2) \equiv \int_Z u'(x_1+x_2+z)\, f(z)\mathrm{d}z.$

  By Karlin's Thm, $v'(x_1+x_2)$ is log-spm in (x_1,x_2),
  hence v is DARA. ■

- Similar theorems are not true if u is not DARA.

DARA with Background Risk

- THM: Given utility u: $\mathbb{R}\to\mathbb{R}$ and pdf f with $\int zf(z)\mathrm{d}z \le 0$, if $r_A(x,u)$ is decreasing and convex in x, then $v(x) \equiv \int_Z u(x+z)\, f(z)\mathrm{d}z,\ \forall x \in \mathbb{R}$, is more risk averse than u.
  - To show: $-\int_Z u''(x+z)\, f(z)\mathrm{d}z \,/ \int_Z u'(x+z)\, f(z)\mathrm{d}z \ge r_A(x,u)$, that is, $\int_Z r_A(x+z,u)\, u'(x+z)\, f(z)\mathrm{d}z \ge r_A(x,u) \int_Z u'(x+z)\, f(z)\mathrm{d}z$.

    Left-hand side exceeds $\int_Z r_A(x+z,u)\, f(z)\mathrm{d}z \int_Z u'(x+z)\, f(z)\mathrm{d}z$ because both r_A and u' are decreasing in z. ($\mathrm{Cov}(r_A,u') \ge 0$.)

    $\int_Z r_A(x+z,u)\, f(z)\mathrm{d}z \ge r_A(x+\mathrm{E}[z],u)$ by convexity of r_A, and $r_A(x+\mathrm{E}[z],u) \ge r_A(x,u)$ because $\mathrm{E}[z] \le 0$ and DARA. ■
- Why assume $\mathrm{E}[z] \le 0$? Otherwise background risk could increase wealth, possibly reducing risk aversion (DARA).