14.123 Microeconomic Theory III
Spring 2009

Proofs for Lecture 5, 14.123 (Peter Eso)

On page 6 of the slides, there are two theorems regarding quasi-supermodularity. We provide their proofs here.

Recall, f on lattice X is quasi-spm if

$$\forall x, y \in X : \{f(x) - f(x \wedge y) \geq 0\} \Rightarrow \{f(x \vee y) - f(y) \geq 0\};$$

and if the first inequality is strict, the second one is strict too.

THM: Supermodularity or log-supermodularity of f implies quasi-supermodularity.

Proof: Supermodularity of f is

$$\forall x, y \in X : f(x \wedge y) + f(x \vee y) \geq f(x) + f(y),$$

which is equivalent to

$$\forall x, y \in X : f(x \vee y) - f(y) \geq f(x) - f(x \wedge y).$$

If the right-hand side is non-negative, then so is the left-hand side, implying that f is q-spm.

By definition, f is log-spm if

$$\forall x, y \in X : f(x \wedge y) f(x \vee y) \geq f(x) f(y),$$

or equivalently,

$$\forall x, y \in X : f(x \wedge y)/f(y) \geq f(x)/f(x \vee y).$$

If the LHS exceeds 1, so does the RHS, implying f is q-spm.$\blacksquare$

Recall that a function $h : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfies single-crossing differences if for all reals $z' > z$ and t, $g(t) \equiv h(z', t) - h(z, t)$ is single crossing, i.e., $\forall t' \geq t$, $\{g(t) \geq (>) 0\} \Rightarrow \{g(t') \geq (>) 0\}$.

THM: If X is a sublattice of $\mathbb{R}^n$, then f satisfies single-crossing differences in evey pair of coordinates provided it is q-spm.

Proof: Let $t' > t$ and $z' > z$ be four real numbers, and define

$$x = (z', t, x_{-ij}) \, , \, y = (z, t', x_{-ij})$$

for any x_{-ij} values of the coordinates other than i and j.

Since $x \vee y = (z', t', x_{-ij})$ and $x \wedge y = (z, t, x_{-ij})$; f being q-spm implies

$$\{f(z', t, x_{-ij}) - f(z, t, x_{-ij}) \geq 0\} \Rightarrow \{f(z', t', x_{-ij}) - f(z, t', x_{-ij}) \geq 0\} \, .$$

Fixing x_{-ij} and letting $g(t) \equiv f(z', t, x_{-ij}) - f(z, t, x_{-ij})$, this condition is equivalent to $g(t) \geq 0 \Rightarrow g(t') \geq 0$ for $z' > z$, single-crossing differences in coordinates i and j.■

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Recall that for sets $B, B' \subseteq X$ (subsets of a lattice) we say that $B' \geq B$ if, for all $b \in B$ and $b' \in B'$, we have $b \vee b' \in B'$ and $b \wedge b' \in B$.

On page 7 of the slides, we state Milgrom and Shannon's theorem:

THM: Suppose $(X, \geq)$ is a sublattice of $\mathbb{R}^n$, and $T \subseteq \mathbb{R}$. If B is a sublattice of X and $f : X \times T \to \mathbb{R}$ is q-spm, then $x^*(t, B) \equiv \arg\max \{f(x, t) | x \in B\}$ is increasing in t and B.

Proof: Let $B' \geq B$, $t' \geq t$, and let $x \in x^*(t, B)$ while $x' \in x^*(t', B')$. We want to show that (i) $x \vee x' \in x^*(t', B')$, and (ii) $x \wedge x' \in x^*(t, B)$.

Note: $x \wedge x' \in B$ and $x \vee x' \in B'$ because $x \in B$, $x' \in B'$ and $B' \geq B$.

(i) Since x maximizes f (given t) over B, and $x \wedge x' \in B$, we have

$$f(x, t) \geq f(x \wedge x', t).$$

Since f is q-spm, this inequality implies

$$f(x \vee x', t) \geq f(x', t).$$

Noting (again) that f is q-spm, this implies that for $t' \geq t$,

$$f(x \vee x', t') \geq f(x', t').$$

Thus, since $x' \in x^*(t', B')$ and $x \vee x' \in B'$, we must have $x \vee x' \in x^*(t', B')$.

(ii) Since x' maximizes f (given t') over B', and $x \vee x' \in B'$, we have

$$f(x', t') \geq f(x \vee x', t').$$

f is q-spm, hence for all $t \leq t'$,

$$f(x', t) \geq f(x \vee x', t).$$

Applied again,

$$f(x \wedge x', t) \geq f(x, t),$$

wihch implies $x \wedge x' \in x^*(t, B)$.■