14.123 Microeconomic Theory III
Spring 2009

MIT 14.123 (2009) by Peter Eso
Lectures 1-2: Expected Utility

1. Outline
2. Refresher on Preference Representations
3. Lotteries and Expected Utility
4. Positive and Normative Interpretations

Read: MWG 3.A-C, 6.A-6.B

Solve: 6.B.3, 6.B.4, 6.B.6, 6.B.7

Administrative Matters

- Instructor: Peter Eso.
- Office hours: Feel free to drop an email, propose two possible meeting times. I'll choose one, then you come by.
- Prerequisites: Fall semester Graduate Micro or Waiver.
- Grade: Weekly Problem Sets, One Exam (Midterm).
- Texts: Mas-Colell, Whinston & Green; Fudenberg & Tirole.
  Further readings: Rubinstein, Osborne & Rubinstein.
  See the Syllabus for details & precise references.
- Any questions? Ask now or write email.

Course Overview

- Decision Theory and Game Theory, 6 + 7 lectures.
- Decision Theory:

  Preferences over Lotteries; Expected Utility Theory;
  Measuring Risk and Risk Aversion; Applications;
  Beyond Expected Utility (Other Theories).

- Game Theory (Advanced Topics):

  Rationalizability; Advanced Equilibrium Notions;
  Applications: Signaling games, Auctions, Global games;
  Dynamic and Repeated Games.

Introduction

- Economics is about explaining and predicting choice.
- It is assumed that economic agents choose their most desirable alternative among the set of feasible ones.
  - Interpret it "as if", not necessarily "deliberate".
  - "This morning I took the shuttle to MIT because this was the best possible way to come in." Discuss.
- Desirability is represented by preferences and/or utility.
  - Attitudes may be expressed over outcomes never experienced (Would you prefer to be Superman or Spiderman?).

Preferences

- Set of alternatives: X. For all $x,y \in X$, answer the following quiz. Choose one:
  - ❑ I strictly prefer x to y.
  - ❑ I strictly prefer y to x.
  - ❑ I am indifferent between x and y.
- "Illegal" answers (see also Rubinstein (2007), p.2):
  - ❑ I don't know.
  - ❑ x and y are incomparable.
  - ❑ It depends (on circumstances, how you ask).
  - ❑ I strictly prefer x to y and y to x.
  - ❑ I don't just "prefer" x to y, I "love" x compared to y.

Preferences

- The answers induce a "strong" preference relation $\succ$ and a "weak" preference relation $\succsim$ exactly as one would expect:
  - $x \succ y$ if the answer is "I strictly prefer x to y";
  - $x \succsim y$ if "I strictly prefer x to y" or "I am indifferent".
- DEF. $\succsim$ is complete on X if $\forall x,y \in X$, either $x \succsim y$ or $y \succsim x$.
- DEF. $\succsim$ is transitive on X if $\forall x,y,z \in X$, $\{x \succsim y$ and $y \succsim z\} \Rightarrow x \succsim z$.

  [Note: Complete and transitive is called rational in MWG.]
- Transitivity is strong. Violations may arise when evaluating complex bundles (aggregation), or comparing similar bundles. Lack of it may be frustrating (e.g., for social planner or parent).

Utility Representation

- DEF. Utility fcn $u : X \to \mathbb{R}$ represents $\succsim$ if $u(x) \geq u(y) \Leftrightarrow x \succsim y$.
- THM: If u represents $\succsim$, then $\succsim$ is complete and transitive.
  - ■ Follows from the same properties of $\geq$ on real numbers. ■
- THM: If X is finite and $\succsim$ is complete and transitive, then there exists a utility function that represents $\succsim$.
  - ■ $u(x) = |\{y \in X : x \succsim y\}|$: # of alternatives that x beats weakly.■
- THM: If X is countable and $\succsim$ is complete and transitive, then there is a utility function with a bounded range that represents $\succsim$.
  - ■ $X \equiv \{x_1, x_2, \ldots\}$. Let $u(x_{-1})=0$, $u(x_0)=1$. For all $n=1,2,\ldots$, set $u(x_n) = [\max\{u(x_k) | x_n \succsim x_k, n>k\} + \min\{u(x_k) | x_k \succsim x_n, n>k\}]/2$.■

Utility Representation

- What can go wrong if X is a continuum? Lexicographic prefs.
- DEF: $\succsim$ is continuous on X, a set with a topology (e.g., $X \subseteq \mathbb{R}^n$): If $x \succ y$ (i.e., $x \succsim y$ and not $y \succsim x$), then for all x' near x and all y' near y, we have $x' \succ y'$. ("near •" ⇔ "in an open ball around •".)
- THM (Debreu): If $\succsim$ is complete, transitive and continuous on a connected set $X \subseteq \mathbb{R}^n$, then there exists a (continuous) utility function that represents $\succsim$.
  - Let Z be a countable, dense subset of X. (Such Z exists because X is assumed to be connected, hence separable.) By the last THM, there is a bounded u representing $\succsim$ on Z. $\forall x \in X$, let $u(x) = \sup\{u(z) \mid x \succ z, z \in Z\}$, or 0 if $\sup$ is empty. Works bc/ if $x \succ y$, then $\exists z, z' \in Z$ such that $x \succ z \succsim z' \succ y$. ■

Take Away on Preferences

- An economic agent's attitudes towards alternatives is expressed by a preference relation or a utility function maximized by his choice.
- This is a model of behavior; neither preferences nor utilities can be observed directly (e.g., in the brain). As such, they do not "exist".
- When can preferences be represented via a utility function?
  - Countable X: If $\succsim$ is complete and transitive.
  - $X \subseteq \mathbb{R}^n$, connected: If $\succsim$ is complete, transitive and continuous.
- Absolute utility levels are meaningless (only relative scale matters): THM: If u represents $\succsim$ and $f : \mathbb{R} \rightarrow \mathbb{R}$ is strictly increasing, then $v = f(u)$ also represents $\succsim$.

Model of Choice under Risk

- Up until now, we did not distinguish actions (choices) and outcomes, assuming choices have deterministic consequences.
- Framework for stochastic decision problems:
  Fix X, a finite set of prizes (outcome such as final wealth level or consumption bundle), and let actions correspond to lotteries (distributions) over X. Study preferences over objective lotteries.
- DEF: Set of lotteries over X is $\Delta(X) \equiv \{p \in [0,1]^{|X|} \mid \sum_i p_i = 1\}$.
  Denote $p(x)$ the probability of $x \in X$ according to lottery p.
- Many decision problems do not fit this framework – e.g., if the probabilities of outcomes are not objectively defined. Other frameworks apply without objective probs or even a state space.

Compound Lotteries

- Enrich the set of actions to include compound lotteries.
- DEF: Given lotteries $p^1,\ldots,p^K \in \Delta(X)$ and weights $\alpha_1,\ldots,\alpha_K \geq 0$ with $\sum_k \alpha_k = 1$, the corresponding compound lottery, $\oplus_k \alpha_k p^k$, is an action where Nature picks each lottery p^k with probability α_k, and the prize in X is picked according to the lottery chosen by Nature.
- For $K=2$, we can also write $\alpha_1 p^1 \oplus (1-\alpha_1) p^2$.
- Note that $\oplus_k \alpha_k p^k \notin \Delta(X)$ while $\sum_k \alpha_k p^k \in \Delta(X)$.
- DEF: A preference relation $\succsim$ for (compound) lotteries on X satisfies Reduction of Compound Lotteries if $\oplus_k \alpha_k p^k \sim \sum_k \alpha_k p^k$. (Here "$A\sim B$" denotes "$A\succsim B$ and $B\succsim A$".)
- From now on, we represent compound lotteries as reduced ones.

Preferences over Lotteries

- Suppose that the preferences $\succsim$ over $\Delta(X)$ are continuous, complete, transitive. Then there is a utility function v that represents them: $\forall p,p' \in \Delta(X)$, $v(p) \geq v(p') \Leftrightarrow p \succsim p'$. (Follows from Debreu's Thm.)
- Denote the sure outcome $x \in X$ by $\delta_x \in \Delta(X)$.
- $\succsim$ over $\Delta(X)$ induces complete, transitive preferences over X, which can be represented by utility u: $\forall x,x' \in X$, $u(x) \geq u(x') \Leftrightarrow \delta_x \succsim \delta_{x'}$.
- Questions for this week and next week:

  1) What additional assumptions on $\succsim$ result in a "nice" utility function v; in particular, $v(p) = \sum_{x \in X} p(x)u(x)$?

  2) How are properties of u (utility fcn on X) and $\succsim$ related?

Independence and Continuity

- Independence Axiom: For any $p,q,r \in \Delta(X)$ and any $\alpha \in (0,1)$,
$$p \succsim q \Leftrightarrow \alpha p + (1-\alpha)r \succsim \alpha q + (1-\alpha)r \ .$$
- The "irrelevant, third lottery" that enters both compound lotteries with the same weight does not reverse the agent's preferences.
- Continuity (formal definition for preferences over lotteries):

  For any $p,q,r \in \Delta(X)$, the sets $\{\alpha\in[0,1] : \alpha p + (1-\alpha)q \succsim r\}$ and $\{\alpha\in[0,1] : r \succsim \alpha p + (1-\alpha)q\}$ are closed.
- This is a "topological" definition; an alternative definition can be given with "distances": If $p \succ q$, then all lotteries sufficiently close to p also dominate all lotteries sufficiently close to q.

Expected Utility

- THM (von Neumann & Morgenstern): If $\succsim$ is continuous, complete and transitive on $\Delta(X)$ (with X finite), and satisfies Independence, then there exists a collection of utility indices $u(x) \in \mathbb{R}$, $\forall x \in X$, such that $\succsim$ is represented by $v(p) \equiv \sum_{x \in X} p(x)u(x)$ for all $p \in \Delta(X)$.
- We say that in this case the utility index u over the sure alternatives represents the agent's preferences over all lotteries, because

$$\forall p,q \in \Delta(X): p \succsim q \Leftrightarrow \sum_{x \in X} p(x)u(x) \geq \sum_{x \in X} q(x)u(x).$$

- Preferences that satisfy the hypothesis of the Theorem are called "Expected Utility" preferences (for obvious reasons).
- The utility index u is sometimes called "Bernoulli utility index" or "von Neumann-Morgenstern [vNM] utility function".

Proof of the EU Theorem

- Lemma: Suppose $\succsim$ on $\Delta(X)$ satisfies the Independence Axiom. Let $x,y \in X$ be such that $\delta_x \succ \delta_y$. Then, for any $1 \geq \alpha > \beta \geq 0$,

$$\alpha\delta_x + (1\text{-}\alpha)\delta_y \succ \beta\delta_x + (1\text{-}\beta)\delta_y .$$

  - By Independence, $\alpha\delta_x + (1\text{-}\alpha)\delta_y \succ \delta_y$. Using it again, $\alpha\delta_x + (1\text{-}\alpha)\delta_y \succ \beta/\alpha[\alpha\delta_x+(1\text{-}\alpha)\delta_y] + (1\text{-}\beta/\alpha)\delta_y = \beta\delta_x + (1\text{-}\beta)\delta_y$.■

- Lemma: If $\succsim$ is continuous, then for any $x,y,z \in X$ with $\delta_x \succ \delta_y \succ \delta_z$, there exists $\alpha \in (0,1)$ such that $\alpha\delta_x + (1\text{-}\alpha)\delta_z \sim \delta_y$.
  - Archimedean Axiom, often used as an alternative to continuity. Let $\alpha = \inf\{\beta\in[0,1] : \beta\delta_x + (1\text{-}\beta)\delta_z \succ \delta_y\}$. Note $\alpha\in(0,1)$. It is easy to see that $\alpha\delta_x + (1\text{-}\alpha)\delta_z \succ \delta_y$ and $\delta_y \succ \alpha\delta_x + (1\text{-}\alpha)\delta_z$ both contradict continuity, hence $\alpha\delta_x + (1\text{-}\alpha)\delta_z \sim \delta_y$. ■

Proof of the EU Theorem

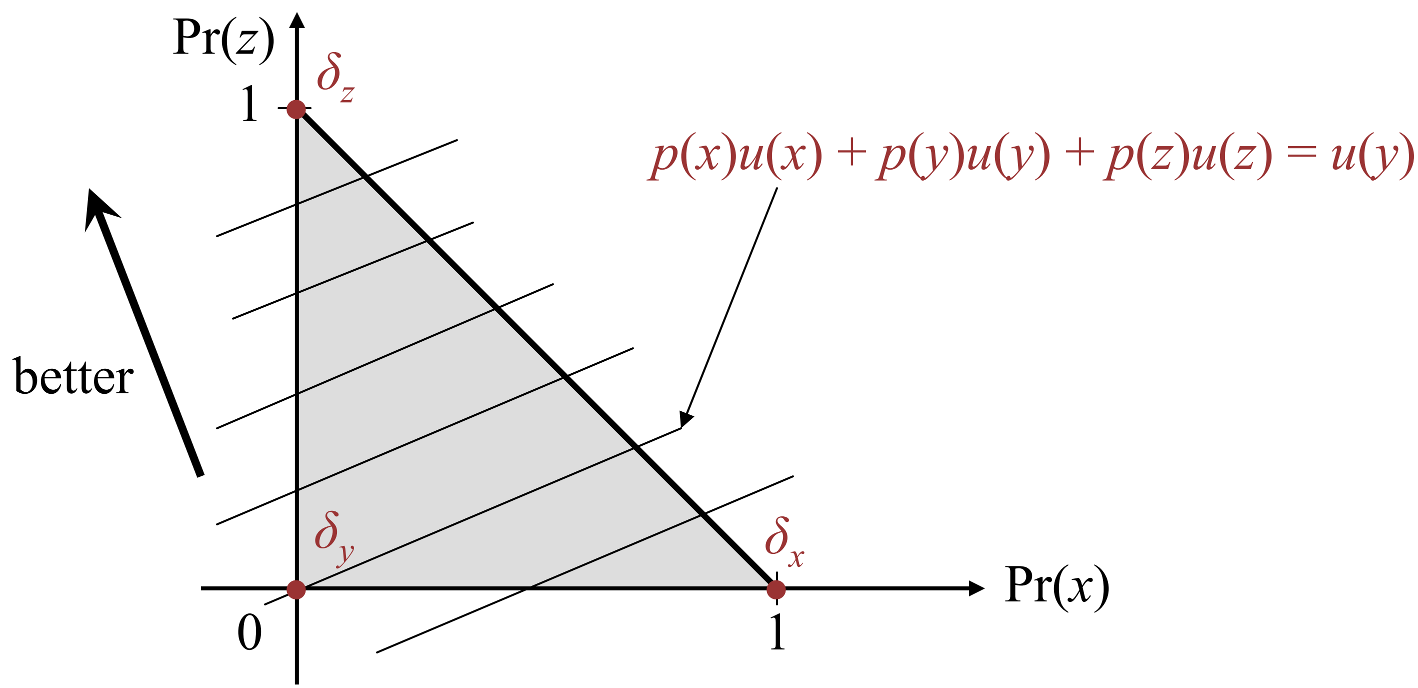
- Let M be a maximal and m a minimal element of X according to $\succsim$.
- By the two Lemmas, for all $x \in X$, there exists a unique $u(x) \in [0,1]$ such that $u(x)\delta_M + [1\text{-}u(x)]\delta_m \sim \delta_x$. Note $u(M) = 1$ and $u(m) = 0$.
- Clearly, for any $p \in \Delta(X)$, $p \equiv \sum_{x \in X} p(x)\delta_x$.
- Fix $p \in \Delta(X)$, and successively replace each $\delta_x \in X$ with the equivalent lottery $u(x)\delta_M + [1\text{-}u(x)]\delta_m$.
- By the Independence Axiom, for all $q \in \Delta(X)$, $p \succsim q \Leftrightarrow$ $p(x)\{u(x)\delta_M + [1\text{-}u(x)]\delta_m\} + [1\text{-}p(x)]\{\sum_{z \neq x} p(z)/[1\text{-}p(x)]\delta_z\} \succsim q$.
- Hence, $p \succsim q \Leftrightarrow \{\sum_{x \in X} p(x)u(x)\}\delta_M + \{\sum_{x \in X} p(x)(1\text{-}u(x))\}\delta_m \succsim q$.
- Therefore $v(p) \equiv \sum_{x \in X} p(x)u(x)$ indeed represents $\succsim$. ■

Take Away on EU (Basics)

- Model: Finite set of outcomes, X. Decision maker has preferences $\succsim$ over lotteries, $p \in \Delta(X)$. Compound lotteries are reduced.
- Assumptions: $\succsim$ is complete and transitive over $\Delta(X)$; moreover, it satisfies Archimedes' Axiom and the Independence Axiom.
- Main Result: $\succsim$ can be represented by a utility function $v: \Delta(X) \rightarrow \mathbb{R}$ of the form $v(p) = \sum_{x \in X} p(x)u(x)$, where $u(x) \equiv v(\delta_x)$.
- Interpretation: Under the assumptions, the decision maker has a utility function over the deterministic *outcomes*; the decision maker evaluates *lotteries* according to their *expected utility*.

Graphical Representation

- Three outcomes, $z \succ y \succ x$ with $p(x)+p(y)+p(z)=1$.
- The indifference curves are parallel, straight lines with slope $[u(y)-u(x)]/[u(z)-u(y)]$, preference $\succsim$ increases up- and leftward.



Why Parallel, Straight Lines?

- DEF: Preferences $\succsim$ on $\Delta(X)$ satisfy betweenness if

$$\forall p,q \in \Delta(X),\ \forall \lambda \in [0,1]:\ p \sim q \Rightarrow \lambda p+(1-\lambda)q \sim q.$$

- Betweenness follows from the Independence Axiom (set $p \sim q = r$). It implies that the indifference curves are straight lines.
- Why are the indifference lines parallel?
- Pick any two lotteries $p \sim q$ (i.e., on the same indifference curve). Mix in δ_z (the best, sure alternative) with the same weight λ. By the Independence Axiom, $\lambda p+(1-\lambda)\delta_z \sim \lambda q+(1-\lambda)\delta_z$. This defines a parallel indifference curve closer to δ_z .

Discussion: Basic Properties

- What other preferences over lotteries may be reasonable? (Examples taken from Rubinstein (2007), pp. 95-96.)

  #1 Preference for "less dispersion", $\sum_{x \in X} (p(x) - 1/|X|)^2$.

  #2 Preference for "more certainty", $\max_{x \in X} p(x)$.

  #3 Increase "prob. of good outcomes", $\sum_{x \in G} p(x)$, where $G \geq X \backslash G$.

  #4 Better "worst-case'', $\min_{x \in X} \{u(x) \mid p(x)>0\}$.

  #5 Better "most-likely prize", $\operatorname{argmax}_{x \in X} \{p(x)\}$.

- Only Expected Utility satisfies both Achimedean Continuity and Independence. (Note that #3 is a special case of EU).

  For example, #4 (often used in Computer Science to evaluate stochastic outcomes) fails Continuity; #2 fails Independence.

Discussion: Basic Properties

- Expected Utility is linear in probabilities:

  If v is an EU representation of $\succsim$, then $\forall p,q \in \Delta(X)$, $\forall \lambda \in [0,1]$:
  $v(\lambda p+(1-\lambda)q) = \lambda v(p) + (1-\lambda)v(q)$.

- THM (Uniqueness): If $\sum_{x\in X} p(x)u(x)$ and $\sum_{x\in X} p(x)w(x)$ both represent $\succsim$, then $\exists \alpha > 0$ and β such that $w(x) = \alpha u(x) + \beta$.

  - Let $\alpha > 0$ and β solve $w(M) = \alpha u(M) + \beta$ and $w(m) = \alpha u(m) + \beta$. For any $x \in X$, there exists p_x such that $\delta_x \sim p_x\delta_M + (1-p_x)\delta_m$, so
    $w(x) = p_x w(M) + (1-p_x)w(m) = p_x[\alpha u(M)+\beta] + (1-p_x)[\alpha u(m)+\beta]$
    $= \alpha[p_x u(M)+(1-p_x)u(m)] + \beta = \alpha u(x) + \beta$. ■

- Any positive monotone transformation of v also represents $\succsim$, but it is not an EU representation unless the transformation is affine.

Discussion: Normative Appeal

- Violators of EU can fall victim to "Dutch book" bets and die poor.
- Suppose (in violation of Independence) that there exist $p, q \in \Delta(X)$ and $\alpha \in (0,1)$ such that $q \succ p$ but $\alpha p \oplus (1\text{-}\alpha)q \succ q$. Compensate agent to accept q as the default lottery outcome.
  1. Offer agent to change the outcome to p with probability α; he is willing to pay for this bet as $\alpha p \oplus (1\text{-}\alpha)q \succ q$.
  2. If the probability α event occurs and p becomes the default, ask agent to pay to change it back to $q \succ p$. Repeat from Step 1.
- Harsanyi (JPE 1955) suggested normative EU approach to moral preferences. Result: Behind the veil of ignorance, *be utilitarian*. Precedes and contradicts Rawls' egalitarian moral philosophy.

Discussion: Positive Appeal

- Introspection and observation of economic behavior often conform to the expected utility hypothesis (more on applications next week).
- Allais paradox. Choose A or B, then C or D.

  (A) Win $1 million for sure.

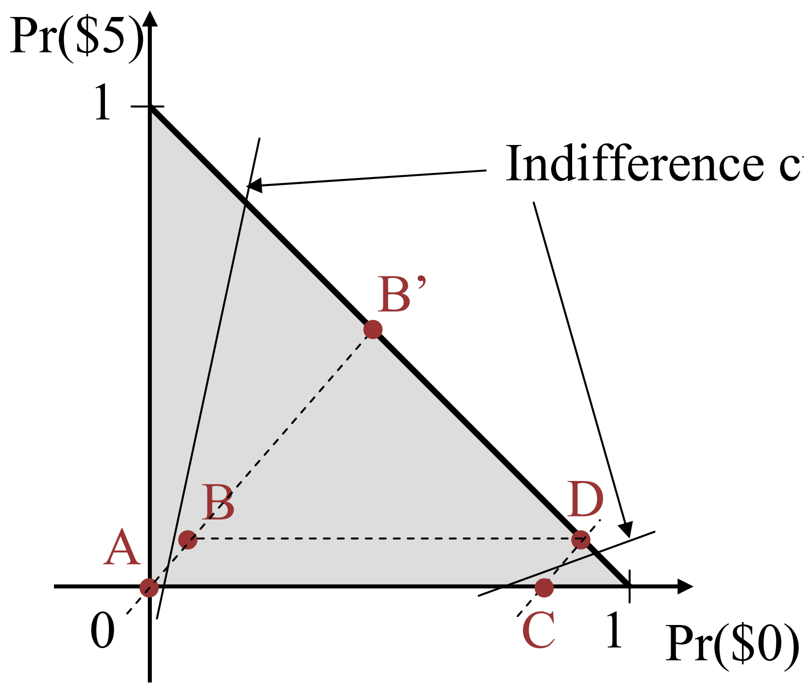
  (B) Win $5M with 10% chance, $1M with 89%, nothing with 1%.

  (C) Win $1M with 11% chance, nothing with 89%.

  (D) Win $5M with 10% chance, nothing with 90%.

Many subjects choose A and D in violation of expected utility: If $u(1) > .1*u(5) + .89*u(1)$, then adding $.89*u(0)-.89*u(1)$ to both sides yields $.11*u(1) + .89*u(0) > .1*u(5) + .9*u(0)$.

Allais Paradox, Graphically



"Common consequence" paradox: A $\succ$ B but D $\succ$ C.

"Common ratio" paradox: A $\succ$ B' but D $\succ$ C.

Resolutions

- A systematic violation of expected utility appears to be indifference curves that *fan out*. (Explains Allais and some other paradoxes.)
- Resolution: Discard Independence, require Betweenness. Weighted Expected Utility: $W(p) = \sum_{x \in X} \gamma(x)p(x)u(x)/[\sum_{x \in X} \gamma(x)p(x)]$.
- Other axiom-systems yield Rank-Dependent Expected Utility, $R(p) = \sum_{x \in X} \pi(p(x))\, x$. Expected value with distorted prob weights.
- Machina (1982) weakens Independence to get "local" expected utility – the indifference curves are curved, but differentiable.
- Big literature, very thoroughly researched, especially within the framework of "preferences over objective lotteries". But is the framework the right one? (Probs are given? Compound lotteries?)

Fundamental Challenges

- Tversky and Kahnemann (1981). "Outbreak of disease is about to kill 600 people. Choose treatment program A or B; then C or D."

  (A) 400 people die.

  (B) Nobody dies with 1/3 chance, 600 people die with 2/3 chance.

  (C) 200 people saved.

  (D) All saved with 1/3 chance, nobody saved with 2/3 chance.

  78% of subjects pick B, 28% of subjects (in different group) pick D. But A is equivalent to C, B is equivalent to D (apart from wording).

- Possible resolution: People infer probabilities from how a question is framed, not only from the direct meaning of the question. The role of *language in decision theory* is an open research area.