14.123 Microeconomic Theory III
Spring 2009

Additional Problems related to Background Risk
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Problem 1: Practice log-spm comparative statics.

There are two individuals (agents), with risk-averse vNM utility functions u_1 and u_2, respectively, on final wealth. Agent 1 is more risk averse than agent 2, i.e.,

$$r_A^1(w, u_1) \equiv -\frac{u_1''(w)}{u_1'(w)} > -\frac{u_2''(w)}{u_2'(w)} \equiv r_A^2(w, u_2),$$

where $r_A^i(w)$ agent i's degree of absolute risk aversion at final wealth w. Moreover, both agents' preferences exhibit decreasing absolute risk aversion (u_1 and u_2 are DARA), i.e., $r_A^i(w, u_i)$ is decreasing in w.

The agents have the same initial, uninsurable, random wealth $w_0 + \tilde{w}$ where w_0 is a real number and $\tilde{w}$ is a random variable. The agents can invest in an asset whose net return is the random variable $\tilde{x}$, which is affiliated with $\tilde{w}$, that is, the joint pdf of $\tilde{w}$ and $\tilde{x}$, $f(w, x)$, is log-supermodular. If agent i invests α_i dollars in the risky asset then his expected utility is $E[u_i(w_0 + \tilde{w} + \alpha_i \tilde{x})]$ where the expectation is taken over $\tilde{w}$ and $\tilde{x}$.

Finally, suppose that agent 2's expected utility maximizing (optimal) demand for the risky asset is $\alpha_2 = 1$.

Prove that agent 1's optimal demand is $\alpha_1 < 1$.

[In the proof, you can use theorems seen in class, but you cannot say "The problem is solved in FamousAuthor (2000)" or something like that.]

Problem 2: An alternative sufficient condition for a welfare-decreasing background risk to increase the risk aversion of a DARA agent.

There is a single decision maker with DARA utility function u and random initial wealth who faces an independent risk. The question is whether the "background risk" in his or her wealth makes him or her more risk averse.

Recall the definition of the precautionary premium seen in class (also slide 14 of Week 2 in the most recently revised version). The precautionary premium of risk $\tilde{z}$ at initial wealth level w is $\psi(w, u, \tilde{z})$ that solves

$$u'(w + E[\tilde{z}] - \psi(w, u, \tilde{z})) \equiv E\left[u'(w + \tilde{z})\right].$$

Also recall the definition of the coefficient of absolute prudence,

$$p_A(w, u) = -\frac{u'''(w)}{u''(w)}.$$

As a warm-up, show that if $p_A(w, u)$ is decreasing in w, then $\psi(w, u, \tilde{z})$ is decreasing in w for any risk $\tilde{z}$. (That is, show that decreasing prudence implies that the precautionary premium is decreasing in wealth.) You can invoke any theorem seen in class, or any analogy that makes the proof short.

Now comes the real question. Assume that u exhibits decreasing absolute risk aversion and decreasing absolute prudence, i.e., both $r_A(w, u) \equiv -u''(w)/u'(w)$ and $p_A(w, u)$ defined above are decreasing in w.

Prove that a non-positive expectation, uninsurable background risk, i.e., initial wealth $w_0 + \tilde{w}$ with $E[\tilde{w}] \leq 0$, makes the agent more risk averse when facing another independent risk, $\tilde{x}$. That is, prove that if $E[\tilde{z}] \leq 0$, then $v(w) \equiv E\left[u(w + \tilde{z})\right]$ is more risk averse than u. (You do not need to refer to w_0, $\tilde{w}$ and $\tilde{x}$ in the proof. Those are just the motivation for the problem.)

Hint: Follow these steps. First show that

$$-\frac{E\left[u''(w + \tilde{z})\right]}{E\left[u'(w + \tilde{z})\right]} = \left(1 - \frac{\partial}{\partial w}\psi(w, u, \tilde{z})\right) r_A(w + E[\tilde{z}] - \psi(w, u, \tilde{z})).$$

Then argue that $E[\tilde{z}] \leq 0$ implies $-E[u''(w + \tilde{z})]/E\left[u'(w + \tilde{z})\right] \geq r_A(w)$. Conclude that this is indeed the result we wanted to establish.