Problem Set 6

Part I - Some practice with production functions (30 points)

For each of the following production functions :

1) $Y = AK^{\alpha}N^{\beta}$; A, α and β are strictly positive constants

2) $Y = A[\alpha K^{-\rho} + (1-\alpha)N^{-\rho}]^{-1/\rho}$; A, α and ρ are strictly positive constants, and $0 < \alpha < 1$

3) $Y = K^{\alpha}(AN)^{1-\alpha}$, and $A = BK^{\phi}$; B, α and ϕ are strictly positive constants, and $0 < \alpha < 1$

do the following :

(a) (12 points) determine whether the production function exhibits diminishing marginal returns to capital

(b) (12 points) determine whether the production function is CRS, DRS or IRS

(c) (6 points) if possible, express Y/N as a function of K/N

In each of these questions, **show the math explicitly** and **provide extra conditions on the constants**, if necessary, to make a determination on the nature of the production function.

(1)

(a) $F_K = \alpha AK^{\alpha-1}N^{\beta}$

$F_{KK} = \alpha(\alpha-1)AK^{\alpha-2}N^{\beta}$, so for $\alpha < 1$, the production function has diminishing returns

(b) $F(\lambda K, \lambda N) = \lambda^{\alpha+\beta}AK^{\alpha}N^{\beta} = \lambda^{\alpha+\beta}F(K,N)$ for any $\lambda > 0$

so for $\alpha + \beta = 1$, the production function is CRS

$\alpha + \beta > 1$, the production function is IRS

$\alpha + \beta < 1$, the production function is DRS

(c) It is possible to represent y ($=Y/N$) as a function of k ($=K/N$) in the CRS case : $y = Ak^{\alpha}$

(2)

(a) $F_K = \alpha A[\alpha + (1-\alpha)K^{-\rho}N^{-\rho}]^{\frac{-\rho-1}{\rho}}$

$F_{KK} = \alpha A[\alpha + (1-\alpha)K^{-\rho}N^{-\rho}]^{\frac{-\rho-1}{\rho}-1}.(\frac{-\rho-1}{\rho}).(1-\alpha).\rho K^{\rho-1}N^{-\rho} < 0$, so this production function always has diminishing returns

(b) Clearly $F(\lambda K, \lambda N) = \lambda F(K,N)$ for any $\lambda > 0$, so the production function is CRS

(c) When the production function is CRS, it can always be representated in the form $y = f(k)$

In this case

$$y = A[\alpha k^{-\rho} + 1 - \alpha]^{-1/\rho}$$

(3) First substitute for A into the production function, so $Y = K^{\alpha+\phi(1-\alpha)}B^{1-\alpha}N^{1-\alpha}$

(a) $F_K = [\alpha + \phi(1-\alpha)]K^{\alpha+\phi(1-\alpha)-1}B^{1-\alpha}N^{1-\alpha}$

$F_{KK} = [\alpha + \phi(1-\alpha)][\alpha + \phi(1-\alpha) - 1]K^{\alpha+\phi(1-\alpha)-2}B^{1-\alpha}N^{1-\alpha}$

For $\alpha + \phi(1-\alpha) < 1$, the production function has diminishing returns.

(b) $F(\lambda K, \lambda N) = \lambda^{\alpha+\phi(1-\alpha)+1-\alpha} K^{\alpha+\phi(1-\alpha)} B^{1-\alpha} N^{1-\alpha}$

The production function is CRS when $\alpha + \phi(1-\alpha) + 1 - \alpha = 1$

i.e., when $\phi(1-\alpha) = 0$

i.e., when $\phi = 0$

Since the question asks you to assume that ϕ is a strictly positive constant and $0 < \alpha < 1$, this condition is never met, and so the production function is IRS.

(c) Since the production function is IRS, it is not possible to represent it as $y = f(k)$

Part II - A continuous time growth model (70 points)

So far, in the lectures and in the textbook, you have dealt with discrete time (ie, the subscript t takes discrete integer values 0,1,2,3 etc.) and the dynamic equation for capital accumulation (equation 11.3 in the textbook) is a difference equation in K/N. This excercise takes you through the algebra of a growth model where time is continuous, so that you work with differential instead of difference equations. Thus, if the savings rate is s, the production function is $F(K,N)$, and the depreciation rate is δ, the fundamental dynamic equation for capital in discrete time would be

$$K_{t+1} - K_t = sF(K,N) - \delta K$$

The continuous time analogue is

$$\dot{K} = sF(K,N) - \delta K$$

where the "dot" over the K on the left hand side indicates the time derivative (the rate of change of K or dK/dt).

Now consider the production function in Part I (1) with $\alpha + \beta = 1$. Express Y/N (call this y) as a function of K/N (call this k). Denote this function by $y = f(k)$ (Note: you get no additional points for doing this, since you are expected to do this in Part I itself).

Assume that employment grows at the constant rate n, ie, $N_t = N_0 e^{nt}$, and for the sake of simplicity, assume that $N_0 = 1$ (note that e is the exponential function)

(a) (5 points) Write down the dynamic equation for capital. Number this equation (I)

$$\dot{K} = sAK^{\alpha}N^{1-\alpha} - \delta K \qquad \text{(I)}$$

(b) (5 points) Divide the equation derived in (a) by N on both the LHS and RHS. Number this equation (II). On the left hand side you should have $\dot{K}/N$. Find an expression for $\dot{K}/N$ in terms of $\dot{k}$, n and k (Hint : start by calculating $\dot{k}$ and rearrange terms. Note that $\dot{K}/N$ which is $(dK/dt)/N$ is different from $\dot{k}$, which is $d(K/N)/dt$).

$$\frac{\dot{K}}{N} = sA\left(\frac{K}{N}\right)^{\alpha} - \delta\frac{K}{N} \qquad \text{(II)}$$

$$\dot{k} = \frac{d(K/N)}{dt} = \frac{dK}{dt}.\frac{1}{N} - \frac{K}{N^2}\frac{dN}{dt} = \frac{\dot{K}}{N} - \frac{K}{N}\frac{\dot{N}}{N} = \frac{\dot{K}}{N} - kn$$

So $\quad \frac{\dot{K}}{N} = \dot{k} + kn$

(c) (5 points) Put the expression for $\dot{K}/N$ you just derived in (b) into the LHS of (II). This gives you the continuous time analogue of equation 11.3 in the textbook. Number this equation (III). Interpret this equation in words (Hint : in interpreting this equation, it might help to consider what would happen if s were to be 0).

$$\dot{k} + kn = sAk^{\alpha} - \delta k$$

So $\quad \dot{k} = sAk^{\alpha} - (\delta + n)k \qquad$ (III)

In words, this equation says that the addition to the stock of capital per worker is the investment minus the capital per worker that is "lost" due to depreciation and growth in workers.

(d) (5 points) Find an expression for the steady state k. Call this k^*.

The steady state is given by $\dot{k} = 0$, so equating the RHS of (III) to 0, and rearranging, we get

$$k^* = \left(\frac{sA}{n+\delta}\right)^{\frac{1}{1-\alpha}}$$

Note that the steady state capital stock is an increasing function of s

(e) (5 points) Find expressions for steady state output per capita (call this y^*), and steady state consumption per capita (call this c^*).

$$y^* = f(k^*) = A\left(\frac{sA}{n+\delta}\right)^{\frac{\alpha}{1-\alpha}}$$
$$c^* = (1-s)f(k^*) = (1-s)A\left(\frac{sA}{n+\delta}\right)^{\frac{\alpha}{1-\alpha}}$$

Verify that y^* **is an increasing function of** s**, but that the sign of** $\frac{dc^*}{ds}$ **is ambiguous**

(f) (5 points) Consider a 2 dimensional graph. Along the x-axis plot k. Along the y-axis plot $f(k)$, $sf(k)$, and the straight line $(\delta + n)k$. Show on the diagram the value k^*, and the associated c^*

(I do not draw the graph here, but it should be clear that k^* is given by the point along the x-axis where the $sf(k)$ curve intersects the $(\delta + n)k$ .line, while c^* is given by the vertical distance between $f(k^*)$ and $(\delta + n)k^*$)

For the next two parts, consider a general CRS production function $Y = F(K,N)$ which can be expressed as $y = f(k)$. The savings rate is s, the depreciation rate δ and the growth rate of employment n

(g) (10 points) Show that steady state consumption c^* can be written as

$$c^*(s) = f(k^*(s)) - (n+\delta)k^*(s)$$

where $k^*(s)$ is the steady state capital per capita, and both c^* and k^* are functions of s (if you are not convinced that they should be, just look at the expressions you derived for k^* and c^* in parts (d) and (e)). Intrepret this equation.

$c^*(s) = (1-s)f(k^*(s))$ where $k^*(s)$ is defined by solving $sf(k^*) = (n+\delta)k^*$ for $k^* = k^*(s)$

So, $c^*(s) = f(k^*(s)) - sf(k^*(s)) = f(k^*(s)) - (n+\delta)k^*(s)$ since $sf(k^*(s)) = (n+\delta)k^*(s)$

This is merely an algebraic restatement of the graphical notion of c^* as described in (f). In the steady state, consumption is given by the gap between the concave production schedule which denotes output and the straight line $(\delta + n)k$, which denotes investment/savings.

(h) (10 points) The golden rule savings rate is the savings rate that will maximize steady state consumption per capita. Associated with this golden rule savings rate is a golden rule capital per capita and a golden rule consumption per capita. Using the equation from (g), derive an equation that **implicitly** defines the golden rule capital per capita. Call the golden rule capital per capita k_{gold}.

$$c^*(s) = f(k^*(s)) - (n+\delta)k^*(s)$$

In order to find the savings rate that will maximize $c^*(s)$, we differentiate this function with respect to s, and set its derivative to 0.

So, $\frac{dc^*}{ds} = \frac{df}{dk^*}\frac{dk^*}{ds} - (n+\delta)\frac{dk^*}{ds} = 0$

which gives us $\frac{df(k^*(s))}{dk^*} = (n+\delta)$ since $\frac{dk^*}{ds} > 0$

This equation implicitly defines k_{gold}, since k_{gold} must satisfy $\frac{df(k_{gold}(s))}{dk_{gold}} = (n + \delta)$

(i) (10 points) On a graph like the one you drew in (f), identify k_{gold} (Hint : to do this, you will need to use the equation you derived in (h)).

Find the point along f on your graph, where the slope of f is equal to $(\delta + n)$, draw a vertical line down from this point to the x-axis - where it meets the x-axis is k_{gold}

(j) (10 points) Identifying k_{gold} should help you to pin down both the golden rule consumption per capita c_{gold} and the golden rule savings rate s_{gold}. Show these on the diagram you drew in (i). (Notice that for any other savings rate, steady state consumption per capita would be lower). Explain in words what would happen to current consumption and future steady state consumption if the economy started at the savings rate s_{gold} and then increased or decreased its savings rate.

If the savings rate were increased, current and the new steady state consumption would both be lower than c_{gold}. Moreover at every date along the adjustment path, consumption would be lower than c_{gold}. Clearly, increasing the savings rate would take the economy to a dynamically inefficient savings rate. On the other hand, reducing the savings rate from s_{gold} would increase current consumption above c_{gold} but the new steady state consumption would be strictly lower than c_{gold}.