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## R evolution: Improving perturbative QCD

André H. Hoang,<sup>1</sup> Ambar Jain,<sup>2</sup> Ignazio Scimemi,<sup>3</sup> and Iain W. Stewart<sup>2</sup>

<sup>1</sup>Max-Planck-Institut für Physik (Werner-Heisenberg-Institut), Föhringer Ring 6, 80805 München, Germany

<sup>2</sup>Center for Theoretical Physics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139, USA

<sup>3</sup>Departamento de Física Teórica II, Universidad Complutense de Madrid, 28040 Madrid, Spain

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Perturbative QCD results in the  $\overline{\text{MS}}$  scheme can be dramatically improved by switching to a scheme that accounts for the dominant power law dependence on the factorization scale in the operator product expansion. We introduce the “MSR scheme” which achieves this in a Lorentz and gauge invariant way and has a very simple relation to  $\overline{\text{MS}}$ . Results in MSR depend on a cutoff parameter  $R$ , in addition to the  $\mu$  of  $\overline{\text{MS}}$ .  $R$  variations can be used to independently estimate (i.) the size of power corrections, and (ii.) higher-order perturbative corrections (much like  $\mu$  in  $\overline{\text{MS}}$ ). We give two examples at three-loop order, the ratio of mass splittings in the  $B^*-B$  and  $D^*-D$  systems, and the Ellis-Jaffe sum rule as a function of momentum transfer  $Q$  in deep inelastic scattering. Comparing to data, the perturbative MSR results work well even for  $Q \sim 1$  GeV, and power corrections are reduced compared to  $\overline{\text{MS}}$ .

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### I. INTRODUCTION AND FORMALISM

The operator product expansion (OPE) is an important tool for QCD. In hard scattering processes two important scales are  $Q$ , a large momentum transfer or mass, and  $\Lambda_{\text{QCD}}$ , the scale of nonperturbative matrix elements. The Wilsonian OPE introduces a factorization scale  $\Lambda^f$ , where  $\Lambda_{\text{QCD}} < \Lambda^f < Q$ , and expands in  $\Lambda_{\text{QCD}}/Q$ . Consider a dimensionless observable  $\sigma$  whose OPE is

$$\sigma = C_0^W(Q, \Lambda^f)\theta_0^W(\Lambda^f) + C_1^W(Q, \Lambda^f)\frac{\theta_1^W(\Lambda^f)}{Q^p} + \dots \quad (1)$$

The  $C_{0,1}^W$  are dimensionless Wilson coefficients containing contributions from momenta  $k > \Lambda^f$  with perturbative expansions in  $\alpha_s$ , and  $\theta_{0,1}^W = \langle \mathcal{O}_{0,1} \rangle_W$  are nonperturbative matrix elements with mass dimensions 0 and  $p$ , containing contributions from  $k < \Lambda^f$ . If  $C_{0,1}^W(Q, \Lambda^f)$  are expanded they contain an infinite series of terms,  $(\Lambda^f/Q)^n$ , modulo  $\ln^m(\Lambda^f/Q)$  terms, and this reflects the fact that  $C_{0,1}^W$  only include contributions from momenta  $k > \Lambda^f$ . The Wilsonian OPE provides a clean separation of momentum scales, but can be technically challenging to implement. In particular, it is difficult to define  $\Lambda^f$  and retain gauge symmetry and Lorentz invariance, and perturbative computations beyond one-loop are atrocious.

A popular alternative is the OPE with dimensional regularization and the  $\overline{\text{MS}}$  scheme, which preserves the symmetries of QCD and provides powerful techniques for multiloop computations. In this case, Eq. (1) becomes

$$\sigma = \bar{C}_0(Q, \mu)\bar{\theta}_0(\mu) + \bar{C}_1(Q, \mu)\frac{\bar{\theta}_1(\mu)}{Q^p} + \dots, \quad (2)$$

where  $\mu$  is the renormalization scale and bars are used for  $\overline{\text{MS}}$  quantities. In  $\overline{\text{MS}}$ , the  $\bar{C}_i$  are simple series in  $\alpha_s$ ,

$$\bar{C}_i(Q, \mu) = 1 + \sum_{n=1}^{\infty} b_n\left(\frac{\mu}{Q}\right)\left[\frac{\alpha_s(\mu)}{(4\pi)}\right]^n, \quad (3)$$

with coefficients  $b_n(\mu/Q) = \sum_{k=0} b_{nk} \ln^k(\mu/Q)$  containing only powers of  $\ln(\mu/Q)$ . We will always rescale  $\sigma$  and the matrix elements  $\bar{\theta}_i$  such that  $\bar{C}_i = 1$  at tree level. In  $\overline{\text{MS}}$ , all power law dependence on  $Q$  is manifest and unique in each term of Eq. (2). Also simple renormalization group equations (RGEs) in  $\mu$ , like

$$\frac{d \ln \bar{C}_0(Q, \mu)}{d \ln \mu} = \bar{\gamma}[\alpha_s(\mu)], \quad (4)$$

can be used to sum large logs in Eq. (2) if  $Q \gg \Lambda_{\text{QCD}}$ .

$C_i^W(Q, \Lambda^f)$  and  $\bar{C}_j(Q, \mu)$  are perturbatively related to each other, so Eqs. (1) and (2) are just the same OPE in two different schemes. The renormalization scale  $\mu$  in  $\overline{\text{MS}}$  plays the role of  $\Lambda^f$ . This is precisely true for logarithmic contributions,  $\ln \mu \leftrightarrow \ln \Lambda^f$ , and here the Wilsonian picture of scale separation in  $C_i^W$  and  $\theta_i^W$  carries over to  $\bar{C}_i$  and  $\bar{\theta}_i$  in  $\overline{\text{MS}}$ . The same is not true for power law dependences on  $\Lambda^f$ .  $\overline{\text{MS}}$  integrations are carried out over all momenta, so the  $\bar{C}_i$  contain some contributions from arbitrary small momenta, and the  $\bar{\theta}_i$  have contributions from arbitrary large momenta. For the power law terms there is no explicit scale separation in  $\overline{\text{MS}}$ , and correspondingly no powers of  $\mu$  appear in Eq. (3). While this simplifies higher-order computations, it leads to factorial growth in the perturbative coefficients. For the dominant terms in  $\bar{C}_0$ , one has  $b_{n+1}(\mu/Q) \simeq (\mu/Q)^p n! [2\beta_0/p]^n Z$  at large  $n$  [1], with constant  $Z$ . In practice this sometimes leads to poor convergence already at one- or two-loop order in QCD. This poor behavior is canceled by corresponding instabilities in  $\bar{\theta}_1$ , and is referred to as an order- $p$  infrared renormalon in  $\bar{C}_0$  canceling against an ultraviolet renormalon in  $\bar{\theta}_1$  [2–4]. The cancellation reflects the fact that the OPE in the  $\overline{\text{MS}}$  scheme does not strictly separate momentum scales. In Ref. [5] a convenient model to parametrize  $\bar{\theta}_1$  was provided based on the assumption that it is entirely related to the low energy behavior of the strong coupling.

The OPE can be converted to a scheme that removes this poor behavior, but still retains the simple computational features of  $\overline{\text{MS}}$ . Consider defining a new “ $R$  scheme” for  $C_0$  by subtracting a perturbative series

$$C_0(Q, R, \mu) = \bar{C}_0(Q, \mu) - \delta C_0(Q, R, \mu), \quad (5)$$

$$\delta C_0(Q, R, \mu) = \left(\frac{R}{Q}\right)^p \sum_{n=1}^{\infty} d_n \left(\frac{\mu}{R}\right) \left[\frac{\alpha_s(\mu)}{4\pi}\right]^n,$$

with

$$d_n(\mu/R) = \sum_{k=0}^n d_{nk} \ln^k(\mu/R). \quad (6)$$

If for large  $n$  the coefficients  $d_n$  are chosen to have the same behavior as  $b_n$ , so  $d_{n+1}(\mu/R) \simeq (\mu/R)^p n! [2\beta_0/p]^n Z$ , then the factorial growth in  $\bar{C}_0(Q, \mu)$  and  $\delta C_0(Q, R, \mu)$  cancel,

$$C_0(Q, R, \mu) \sim \left[\frac{\mu^p}{Q^p} - \frac{R^p}{Q^p} \frac{\mu^p}{R^p}\right] \sum_n n! \left[\frac{2\beta_0}{p}\right]^n Z. \quad (7)$$

Thus the  $R$  scheme introduces power law dependence on the cutoff,  $(R/Q)^p$ , in  $C_0(Q, R, \mu)$ , which captures the dominant  $(\Lambda^f/Q)^p$  behavior of the Wilsonian  $C_0^W$ . In practice, this improves the convergence in  $C_0$  even at low orders in the  $\alpha_s$  series. The dominant effect is compensated by a scheme change to  $\bar{\theta}_1$ ,  $\bar{\theta}_1(\mu) = \theta_1(R, \mu) - [Q^p \delta C_0(Q, R, \mu)] \bar{\theta}_0(\mu)$ . This new  $\theta_1$  remains  $Q$  independent and will exhibit improved stability. In the  $R$  scheme the OPE becomes

$$\sigma = C_0(Q, R, \mu) \bar{\theta}_0(\mu) + \bar{C}_1(Q, \mu) \frac{\theta_1(R, \mu)}{Q^p} + \bar{C}'_1(Q, \mu) \frac{\theta'_1(R, \mu)}{Q^p} + \dots, \quad (8)$$

where  $\theta'_1 = [Q^p \delta C_0] \bar{\theta}_0$  and  $\bar{C}'_1 = 1 - \bar{C}_1 \sim \alpha_s$ . Both  $C_0$  and  $\theta_1$  are free of order- $p$  renormalons. The severity of an ambiguity can be quantified by the singularity structure in the Borel transform, and we will neglect  $\bar{C}'_1 \theta'_1$  which only contributes a subdominant cut.

To setup an appropriate  $R$  scheme it remains to define the  $d_n$ . In the renormalon literature such scheme changes are well known for heavy quark masses [6,7]. For OPE predictions a “renormalon subtraction” (RS) scheme has been implemented in Ref. [8]. In the RS scheme an approximate result for the residue of the leading Borel renormalon pole is used to define the  $d_n$ , which adds a source of uncertainty. The approach of Ref. [9] for event shape distributions is based on powerlike subtractions derived from the assumption that the power corrections are related to the low energy behavior of the strong coupling. This is a model for QCD power corrections. Since this setup lies outside the strict OPE framework, the uncertainties introduced by the model dependent assumptions are unclear and could have the same size as the subtracted pieces. The issue of large logarithms in the subtraction series is also not addressed. The  $R$  evolution that we propose in this work implements subtractions completely in the framework of

the OPE and also resums large logarithms in the subtractions.

For our analysis we define the “MSR” scheme for  $C_0$  by simply taking the coefficients of the subtraction to be exactly the  $\overline{\text{MS}}$  coefficients. In general, it is more convenient to use  $\ln \bar{C}_0$  rather than  $\bar{C}_0$ , since this simplifies renormalization group equations. Writing the series as

$$\ln \bar{C}_0(Q, \mu) = \sum_{n=1}^{\infty} a_n \left(\frac{\mu}{Q}\right) \left[\frac{\alpha_s(\mu)}{4\pi}\right]^n, \quad (9)$$

with  $a_n(\mu/Q) = \sum_{k=0}^n a_{nk} \ln^k(\mu/Q)$  we define the MSR scheme by the series

$$\ln C_0(Q, R, \mu) \equiv \sum_{n=1}^{\infty} \left\{ a_n \left(\frac{\mu}{Q}\right) - \frac{R^p}{Q^p} a_n \left(\frac{\mu}{R}\right) \right\} \frac{\alpha_s^n(\mu)}{(4\pi)^n}. \quad (10)$$

This definition still cancels the order- $p$  renormalon for large  $n$ , as in Eq. (7). It yields the very simple relation

$$C_0(Q, R, \mu) = \bar{C}_0(Q, \mu) [\bar{C}_0(R, \mu)]^{-(R/Q)^p}, \quad (11)$$

which must be expanded order-by-order in  $\alpha_s(\mu)$  to remove the renormalon. Thus, the coefficient  $C_0(Q, R, \mu)$  for the MSR scheme is obtained directly from the  $\overline{\text{MS}}$  result. Note  $C_0(Q, Q, \mu) = 1$  to all orders. The appropriate  $p$  is obtained from the  $\overline{\text{MS}}$  OPE by  $p = \text{dimension}(\bar{\theta}_1) - \text{dimension}(\bar{\theta}_0)$ . MSR preserves gauge invariance, Lorentz symmetry, and the simplicity of  $\overline{\text{MS}}$ .

The appropriate values for  $R$  in Eqs. (5), (8), and (11) are constrained by power counting and the structure of large logs in the OPE. The power counting  $\bar{\theta}_1 \sim \Lambda_{\text{QCD}}^p$  implies  $\theta_1 \sim \Lambda_{\text{QCD}}^p$ , so for the matrix element we need  $R = R_0 \sim \mu \gtrsim \Lambda_{\text{QCD}}$  (meaning a larger value where perturbation theory for the OPE still converges), which minimizes  $\ln(\mu/\Lambda_{\text{QCD}})$  and  $\ln(\mu/R)$  terms in  $\theta_1(R, \mu, \Lambda_{\text{QCD}})$ . On the other hand,  $C_0(Q, R, \mu)$  has  $\ln(\mu/Q)$  and  $\ln(\mu/R)$  terms, and for  $R \sim \Lambda_{\text{QCD}}$  no choice of  $\mu$  avoids large logs. For  $R = R_1 \sim \mu \sim Q$  we can minimize the logs in  $C_0(Q, R, \mu)$ , but not in  $\theta_1(R, \mu, \Lambda_{\text{QCD}})$ . When the OPE is carried out in  $\overline{\text{MS}}$  this problem is dealt with using a  $\mu$  RGE to sum large logs between  $Q$  and  $\Lambda_{\text{QCD}}$ . For MSR we must use  $R$  evolution, an RGE in the  $R$  variable [10], to sum logs between  $R_1$  and  $R_0$ . The appropriate  $R$  RGE is formulated with  $\mu = \kappa R$  and  $\kappa \sim 1$  to ensure there are no logs in the anomalous dimension. For  $C_0$  and  $\kappa = 1$ ,

$$R \frac{d}{dR} \ln C_0(Q, R, R) = \bar{\gamma}[\alpha_s(R)] - \left(\frac{R}{Q}\right)^p \gamma[\alpha_s(R)], \quad (12)$$

where  $\bar{\gamma}[\alpha_s] = \sum_{n=0}^{\infty} \bar{\gamma}_n [\alpha_s(R)/4\pi]^{n+1}$  is the familiar  $\overline{\text{MS}}$ -anomalous dimension and

$$\gamma[\alpha_s] = \sum_{n=0}^{\infty} \gamma_n \left[\frac{\alpha_s(R)}{4\pi}\right]^{n+1} \quad (13)$$

is the  $R$ -anomalous dimension with

$$\gamma_{n-1} = p a_{n0} - 2 \sum_{m=1}^{n-1} m a_{m0} \beta_{n-m-1}. \quad (14)$$

Here we are using the  $\overline{\text{MS}}$   $\beta$  function,  $\mu(d/d\mu)\alpha_s(\mu) = -\frac{\alpha_s(\mu)}{2\pi} \sum_{n=0}^{\infty} \beta_n [\alpha_s(\mu)/4\pi]^n$ . The choice in Eq. (10) keeps Eq. (12) simple. In cases where  $\tilde{\gamma}$  is absent, we expect Eq. (12) to converge at lower  $R$  scales than are typical for the  $\mu$  RGE due to the  $(R/Q)^p$  factor multiplying  $\gamma$ . For  $R_1 > R_0$  the solution of Eq. (12) is

$$C_0(Q, R_0, R_0) = C_0(Q, R_1, R_1)U_R(Q, R_1, R_0)U_\mu(R_1, R_0), \quad (15)$$

where  $U_\mu$  is a usual  $\overline{\text{MS}}$  evolution factor and  $U_R$  is the  $R$  evolution. For  $p = 1$ , the complete solution for  $U_R$  was obtained in Ref. [10]. It is straightforward to generalize this to any  $p$ . The RGE solution is independent of a choice  $\kappa \neq 1$  up to higher-order terms. At  $\text{N}^{k+1}\text{LL}$  order the (real) result is

$$U_R(Q, R_1, R_0) = \exp\left\{\left(\frac{\Lambda_{\text{QCD}}^{(k)}}{Q}\right)^p \sum_{j=0}^k S_j(-p)^j e^{i\pi p \hat{b}_1} p^{(p\hat{b}_1)} \times [\Gamma(-p\hat{b}_1 - j, pt_0) - \Gamma(-p\hat{b}_1 - j, pt_1)]\right\}, \quad (16)$$

with  $\Gamma(c, t)$  the incomplete gamma function and  $t_{0,1} = -2\pi/(\beta_0\alpha_s(R_{0,1}))$ . Here  $\Lambda_{\text{QCD}}^{(0)} = Re^t$ ,  $\Lambda_{\text{QCD}}^{(1)} = Re^t(-t)^{\hat{b}_1}$ , and  $\Lambda_{\text{QCD}}^{(2)} = Re^t(-t)^{\hat{b}_1} e^{-\hat{b}_2/t}$  are evaluated at a very large reference  $R$  with  $t = -2\pi/(\beta_0\alpha_s(R))$  such that they exhibit their  $R$  independence, and

$$\hat{b}_1 = \beta_1/(2\beta_0^2), \quad \hat{b}_2 = (\beta_1^2 - \beta_0\beta_2)/(4\beta_0^4), \quad (17)$$

$$\hat{b}_3 = (\beta_1^3 - 2\beta_0\beta_1\beta_2 + \beta_0^2\beta_3)/(8\beta_0^6).$$

Defining  $\tilde{\gamma}_n = \gamma_n/(2\beta_0)^{n+1}$ , the coefficients of  $U_R$  needed for the first three orders of  $R$  evolution are

$$S_0 = \tilde{\gamma}_0, \quad S_1 = \tilde{\gamma}_1 - (\hat{b}_1 + p\hat{b}_2)\tilde{\gamma}_0, \quad (18)$$

$$S_2 = \tilde{\gamma}_2 - (\hat{b}_1 + p\hat{b}_2)\tilde{\gamma}_1$$

$$+ [(1 + \hat{p}b_1)\hat{b}_2 + p(p\hat{b}_2^2 + \hat{b}_3)/2]\tilde{\gamma}_0.$$

Then Eq. (8) becomes

$$\sigma = C_0(Q, R_1, R_1)U_R(Q, R_1, R_0)U_\mu(R_1, R_0)\theta_0(R_0) + \frac{\theta_1(R_0, R_0)}{Q^p} + \dots, \quad (19)$$

and this result sums logs between  $R_1 \sim Q$  and  $R_0 \sim \Lambda_{\text{QCD}}$ . This gives natural  $R$  scales for the coefficients and matrix elements in the OPE. The use of  $R$  evolution allows us to sum these logs while, at the same time, maintaining the independence of the factorization of scales from the leading renormalon.

In Eq. (15),  $R_0$  is the scale at which renormalon contributions are subtracted from the leading power perturbative series and from a power suppressed matrix element. The variation of  $R_0$  gives an estimate of the size of these power corrections if they are left out of the analysis. If the power corrections are included then the  $R_0$  dependence cancels out between the leading power perturbative term and the power correction matrix element. As explained

above, the choice of  $R_0$  is not arbitrary, we must have  $R_0 \sim \Lambda_{\text{QCD}}$  to avoid enhancing power corrections, and  $R_0 > \Lambda_{\text{QCD}}$  to maintain perturbation theory in  $\alpha_s(R_0)$ . Thus, the natural choice for  $R_0$  is around 1 GeV, which is close to the confinement scale, but still in the perturbative regime. In Eq. (15), the variation of  $R_1$  has the meaning of the usual  $\mu$  variation in the OPE, i.e., its variation gives an estimate of the size of the higher-order perturbative corrections. In order to reliably compute  $C_0(Q, R_1, R_1)$  in fixed order perturbation theory one must take  $R_1 \sim Q$ .

## II. HEAVY MESON MASS SPLITTINGS IN MSR

The  $\overline{\text{MS}}$  OPE for the mass splitting of heavy mesons,  $\Delta m_H^2 = m_{H^*}^2 - m_H^2$  for  $H = B, D$ , is given by

$$\Delta m_H^2 = \bar{C}_G(m_Q, \mu)\mu_G^2(\mu) + \sum_i \bar{C}_i(m_Q, \mu) \frac{2\rho_i^3(\mu)}{3m_Q} + \mathcal{O}(\Lambda_{\text{QCD}}^3/m_Q^2), \quad (20)$$

where  $m_Q = m_b$  or  $m_c$ . Here  $\mu_G^2 = -\langle B_v | \bar{h}_v g \sigma_{\mu\nu} G^{\mu\nu} h_v | B_v \rangle / 3$  is the matrix element of the chromomagnetic operator, and  $\rho_i^3$  for  $i = \pi G, A, LS, \bar{\Lambda} G$  are  $\mathcal{O}(\Lambda_{\text{QCD}}^3)$  matrix elements [11], with  $\rho_{\bar{\Lambda} G}^3(\mu) = (3/2)\bar{\Lambda}\mu_G^2(\mu)$ . At the order of our analysis, tree level values for the  $\bar{C}_i$  suffice, so with  $\bar{\Sigma}_\rho(\mu) = 2/3[\rho_{\pi G}^3(\mu) + \rho_A^3(\mu) - \rho_{LS}^3(\mu) + \rho_{\bar{\Lambda} G}^3(\mu)]$ , we have

$$\Delta m_H^2 = \bar{C}_G(m_Q, \mu)\mu_G^2(\mu) + \bar{\Sigma}_\rho(\mu)/m_Q + \dots \quad (21)$$

Taking the ratio of mass splittings  $r = \Delta m_B^2/\Delta m_D^2$  gives

$$r = \frac{\bar{C}_G(m_b, \mu)}{\bar{C}_G(m_c, \mu)} + \frac{\bar{\Sigma}_\rho(\mu)}{\mu_G^2(\mu)} \left( \frac{1}{m_b} - \frac{1}{m_c} \right) + \dots \quad (22)$$

The first term in this OPE gives a purely perturbative prediction for  $r$ .  $\bar{C}_G$  is known to suffer from an  $\mathcal{O}(\Lambda_{\text{QCD}}/m_Q)$  infrared renormalon ambiguity [11], with a corresponding ambiguity in  $\bar{\Sigma}_\rho(\mu)$ . The three-loop computation of Ref. [12] yields,

$$r = 1 - 0.1113|\alpha_s - 0.0780|\alpha_s^2 - 0.0755|\alpha_s^3 \quad (23)$$

at fixed order with  $\mu = m_c$ , and

$$r = (0.8517)_{\text{LL}} + (-0.0696)_{\Delta\text{NLL}} + (-0.0908)_{\Delta\text{NNLL}} \quad (24)$$

where (N)LL refers to (next-to) leading logarithmic order in RG-improved perturbation theory, etc. There is no sign of convergence in either case. In  $\overline{\text{MS}}$  these leading power predictions are unstable due to the  $p = 1$  renormalon in  $\bar{C}_G$ .

Lets examine the analogous result in the MSR scheme

$$\Delta m_H^2 = C_G(m_Q, R, \mu)\mu_G^2(\mu) + \frac{\Sigma_\rho(R, \mu)}{m_Q} + \dots \quad (25)$$

Since  $p = 1$  the MSR definition in Eq. (11) gives



$$C_G(m_Q, R, \mu) \equiv \bar{C}_G(m_Q, \mu)[\bar{C}_G(R, \mu)]^{-R/m_Q}, \quad (26)$$

where  $\bar{C}_G(m, \mu)$  is obtained from Ref. [12] and we expand in  $\alpha_s(\mu)$ . The OPE in MSR at a scale  $R_0 \gtrsim \Lambda_{\text{QCD}}$  gives

$$r = \frac{C_G(m_b, R_0, R_0)}{C_G(m_c, R_0, R_0)} + \frac{\Sigma_\rho(R_0, R_0)}{\mu_G^2(R_0)} \left( \frac{1}{m_b} - \frac{1}{m_c} \right). \quad (27)$$

Large logs in  $C_G(m_Q, R_0, R_0)$  can be summed with the  $R$  RGE in Eqs. (15)–(18). For simplicity, we integrate out the  $b$  and  $c$  quarks simultaneously at a scale  $R_1 \approx \sqrt{m_b m_c} \gg R_0 \approx \Lambda_{\text{QCD}}$ . This scale for  $R_1$  keeps  $\ln(R_1/m_{b,c})$  small. With  $R$  evolution and  $U_R$  from Eq. (16) we have

$$r = \frac{C_G(m_b, R_1, R_1)U_R(m_b, R_1, R_0)}{C_G(m_c, R_1, R_1)U_R(m_c, R_1, R_0)} + \frac{\Sigma_\rho(R_0, R_0)}{\mu_G^2(R_0)} \left( \frac{1}{m_b} - \frac{1}{m_c} \right). \quad (28)$$

This expression is independent of  $R_1$  and  $R_0$ . Order-by-order, varying  $R_1$  about  $\sqrt{m_b m_c}$  yields an estimate of higher-order perturbative uncertainties, much like varying  $\mu$  in  $\overline{\text{MS}}$ . For  $R_0$  the dependence cancels between the first term in  $r$  and the  $\Sigma_\rho$  power correction. In MSR, the term  $\Sigma_\rho(R_0, R_0)$  is  $\sim \Lambda_{\text{QCD}}^3$  and can have either sign. There can also be a  $R_0$  value where  $\Sigma_\rho(R_0, R_0)$  vanishes. Thus, keeping only the first term in Eq. (28) and varying  $R_0 \sim 1$  GeV by an amount  $\gtrsim \Lambda_{\text{QCD}}$  yields an estimate for the size of this power correction. This technique goes beyond the dimensional analysis estimates used in  $\overline{\text{MS}}$ .

Figure 1 gives perturbative predictions for  $r$  at different orders using the first terms in Eqs. (22) and (28) with  $m_b = 4.7$  GeV,  $m_c = 1.6$  GeV,  $\alpha_s(\sqrt{m_b m_c}) = 0.2627$ , and the four-loop  $\beta$  function. The solid lines are from the MSR scheme, plotted as functions of  $R_0$ . The dashed lines are the fixed order  $\overline{\text{MS}}$  results with  $\mu = \sqrt{m_b m_c}$ . The MSR results exhibit a dramatic improvement in convergence over  $\overline{\text{MS}}$  for a wide range of  $R_0$  values. Varying  $R_1 = \sqrt{m_b m_c}/2$  to  $2\sqrt{m_b m_c}$  at  $\text{N}^3\text{LL}$  (MSR) gives  $\Delta r \approx \pm 0.008$ , which is a significant improvement over  $\mu$  variation in the same range for  $\text{N}^3\text{LO}(\overline{\text{MS}})$  where  $\Delta r \approx \pm 0.068$ . (An alternate estimate is varying  $0.8 < \kappa < 3$  to get  $\Delta r = \pm 0.009$ . This is consistent.) The MSR results converge to an  $R_0$  dependent curve, whose dependence cancels against  $\Sigma_\rho(R_0, R_0)$ , so the residual  $R_0$  dependence provides a method to estimate the size of this power correction. The range  $R_0 = 0.7$  to 1.2 GeV keeps  $R_0$  below  $m_c$  and above  $\Lambda_{\text{QCD}}$  and yields

$$r = 0.860 \pm (0.065)_{\Sigma_\rho} \pm (0.008)_{\text{pert.}} \quad (29)$$

This estimate for the  $\Sigma_\rho$  power correction in MSR is in good agreement with experiment,  $r_{\text{expt}} = 0.886$  ( $D_{u,d}^{(*)}, B_{u,d}^{(*)}$ ) and 0.854 ( $D_s^{(*)}, B_s^{(*)}$ ). MSR achieves a convergent perturbative prediction for  $r$  at leading order in the OPE, and a  $1/m_Q$  power correction of moderate size,  $\sim 0.065$ , significantly smaller than the dimensional analysis estimate of  $\Lambda_{\text{QCD}}(1/m_c - 1/m_b) \sim 0.200$  in  $\overline{\text{MS}}$ .

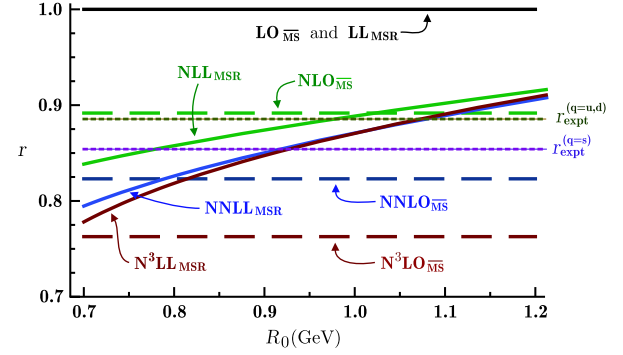


FIG. 1 (color online). Perturbative predictions at leading order in  $1/m_Q$  for the ratio  $r$  of the  $B$ - $B^*$  and  $D$ - $D^*$  mass splittings in the MSR scheme (solid curves) versus  $\overline{\text{MS}}$  (dashed curves). The  $R_0$  dependence of the solid red curve provides an estimate for the power correction, independent of the comparison with the experimental data. Neither  $R_1$  nor  $\mu$  variation is shown in the figure.

### III. ELLIS-JAFFE SUM RULE IN MSR

In  $\overline{\text{MS}}$ , the Ellis-Jaffe sum rule [13] for the proton in deep inelastic scattering with momentum transfer  $Q$  is

$$M_1(Q) = \left[ \bar{C}_B(Q, \mu)\theta_B + \bar{C}_0(Q, \mu)\frac{\hat{a}_0}{9} \right] + \frac{\bar{\theta}_1(\mu)}{Q^2}. \quad (30)$$

$\bar{C}_{B,0}(Q, \mu)$  are known at three loops [14]. Despite the  $\mu$  arguments displayed here, the two leading order terms are being written so that both coefficients and matrix elements are separately  $\mu$  independent:  $\theta_B = g_A/12 + a_8/36$  is given by the axial couplings  $g_A = 1.2694$  and  $a_8 = 0.572$  for the nucleon and hyperon, while  $\hat{a}_0$  is a  $Q$  and  $\mu$  independent  $\overline{\text{MS}}$  matrix element.  $\bar{\theta}_1$  denotes all  $1/Q^2$  power corrections with their Wilson coefficients at tree level. The  $\overline{\text{MS}}$  coefficients are affected by a  $p = 2$  renormalon [15], which is removed in the MSR scheme. Switching to MSR with Eq. (11) gives  $[i = B, 0]$

$$C_i(Q, R, R) \equiv \bar{C}_i(Q, R)[\bar{C}_i(R, R)]^{-R^2/Q^2}. \quad (31)$$

With  $R$  evolution, the MSR OPE prediction is

$$M_1(Q) = \left[ C_B(Q, R_1, R_1)U_R^B(Q, R_1, R_0)\theta_B + C_0(Q, R_1, R_1)U_R^0(Q, R_1, R_0)\frac{\hat{a}_0}{9} \right] + \frac{\theta_1(R_0, R_0)}{Q^2}, \quad (32)$$

where  $U_R^{B,0}$  are given by Eq. (16) with  $p = 2$  and the corresponding  $(a_{n0})^{B,0}$  determine the appropriate  $(\gamma_n)^{B,0}$ .

Figures 2 and 3 show perturbative predictions for the Ellis-Jaffe sum rule at leading power in  $1/Q$ , compared with proton data from Ref. [16]. We use  $\alpha_s(4 \text{ GeV}) = 0.2282$ , and the four-loop  $\beta$  function with four flavors. In Fig. 2, we show order-by-order results for the  $\overline{\text{MS}}$  scheme at  $\mu = Q$ , and for the resummed MSR scheme with

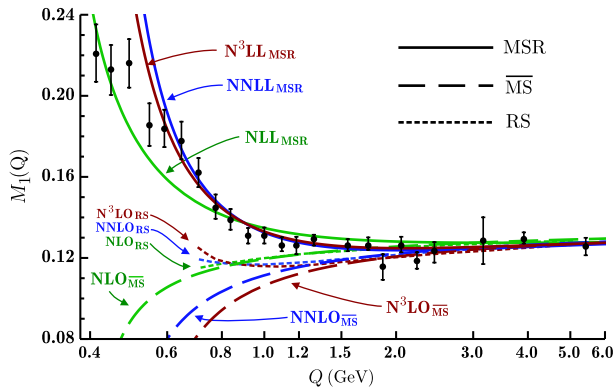


FIG. 2 (color online). Perturbative results for the Ellis-Jaffe sum rule in the MSR, RS, and  $\overline{\text{MS}}$  schemes, at leading order in  $1/Q$ . For all curves, the one parameter,  $\hat{a}_0$ , is fixed by data at  $Q \simeq 5$  GeV.

$R_1 = Q$  and  $R_0 = 0.9$  GeV. We fix  $\hat{a}_0 = 0.141$  so that both  $\overline{\text{MS}}$  and MSR agree with the data for  $Q \simeq 5$  GeV.  $\overline{\text{MS}}$  agrees well with the data for large  $Q$ , but turns away at  $Q \lesssim 2$  GeV and no longer converges. In contrast, the MSR results still converge quickly and exhibit excellent agreement with the data over a wide range of  $Q$ 's. The NLL MSR result already has the right curvature and, at NNLL and  $\text{N}^3\text{LL}$  the agreement for  $Q \geq 0.6$  GeV improves. We also display predictions in the RS scheme with subtraction scale  $\nu_f = 1.0$  GeV from Fig. 3(d) of Ref. [8], which improve slightly over the  $\overline{\text{MS}}$  results, but may not be capturing the dominant power law dependence on the factorization scale.

In Fig. 3, we show uncertainties for the three-loop results in the  $\overline{\text{MS}}$  and MSR schemes for the OPE. The dashed red curve is the  $\overline{\text{MS}}$  prediction, and the blue band estimates the higher-order perturbative uncertainties varying  $\mu$  in the range  $\mu^{\min}(Q) < \mu < 2Q$ . For  $Q > 1.5$  GeV,  $\mu^{\min} = Q/2$ , while for  $Q < 1.5$  GeV,  $\mu^{\min} = 1.3Q/(1.1 + Q/(1 \text{ GeV}))$ . The red solid line is the MSR prediction, the red band is the perturbative uncertainty from varying

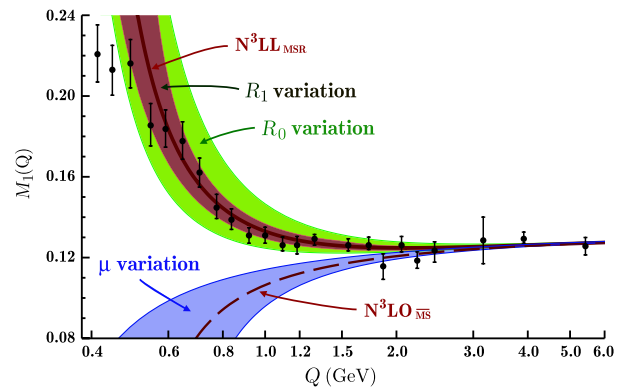


FIG. 3 (color online). Uncertainty estimates in the MSR scheme and  $\overline{\text{MS}}$  scheme for the Ellis-Jaffe sum rule at leading order in  $1/Q$ .

$R_1$  in the same range as was done for  $\mu$  in  $\overline{\text{MS}}$ , and the green band estimates the  $1/Q^2$  power correction by varying  $R_0 = 0.7$  to  $1.2$  GeV. (Varying  $0.8 < \kappa < 3$  gives  $\Delta M_1/M_1 \lesssim 0.8\%$  at  $\text{N}^3\text{LL}$  for  $Q \geq 0.8$  GeV, which is smaller than varying  $R_1$ .) Figure 3 implies

$$-0.01 \text{ GeV}^2 \lesssim \theta_1(R_0, R_0) \lesssim 0.01 \text{ GeV}^2 \quad (33)$$

in MSR, which is a much smaller power correction than the  $\sim 0.10 \text{ GeV}^2$  estimate from naive dimensional analysis in  $\overline{\text{MS}}$ .

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