APPLICATIONS OF LINEAR TRANSFORMATION THEORY TO THE SYNTHESIS OF LINEAR ACTIVE NONBILATERAL NETWORKS

by

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A new approach to the synthesis of active nonbilateral linear networks is presented in this thesis - the linear transformation theory approach. The idea of using linear transformations as an aid in the analysis and synthesis of active nonbilateral linear networks is due to Guillemin(1,2). By analysis of an active nonbilateral network through linear transformations it is meant a method whereby the dynamic variables of the network are expressed in terms of those of passive bilateral network by means of linear transformations. The synthesis procedure is the inverse of the analysis procedure. One starts with a passive bilateral linear network and through the agency of linear transformations of dynamic variables converts the passive bilateral network into an active nonbilateral one with certain desired network properties. In addition to presenting new results on the synthesis of active nonbilateral networks, new results are presented on the analysis of linear networks and the properties of driving point and transfer functions of active nonbilateral linear networks.

A method of analysis of linear networks is presented that is applicable to networks whose elements may have any number of terminals. This analysis method was arrived at as a generalization of a method due to Guillemin(2). It has been found that the method of analysis presented in this thesis is very similar to a special case of Kron's(16) method of analysis which involves "tearing networks". The similarity is evidenced by making a correspondence between "torn networks" in his method and the multiterminal-pair network element of the analysis method of this thesis. Whereas the "torn network" of Kron may contain internal sources, the multiterminal-pair element is assumed to be homogeneous, i.e., to exhibit no terminal-pair voltages and currents when unexcited externally. An advantage of the analysis method of this thesis is its simplicity. No tensor algebra or notation are involved and in fact the method requires little more knowledge in its application than an ability to write conventional loop and node equations when mutual inductance is present. Some attention is given in the section on network analysis to the special situation in which the equilibrium matrix of a network composed of MTP (multiterminal-pair) elements may be found by a simple addition of the parameter matrices of the component MTP elements. The Additive Class of networks is defined. An Additive network has the property that not only loop and node but also mixed equilibrium matrices may be evaluated by addition of matrices describing the behavior of component MTP elements.

Both real and complex (frequency dependent) linear transformations are considered as an aid in obtaining synthesis techniques for active nonbilateral linear networks. Specific attention is given to RC networks containing active nonbilateral resistive MTP elements. Certain difficulties are found in using real linear transformations and final synthesis techniques are developed only with the use of complex linear transformations. As a preliminary to the development of synthesis techniques using complex linear transformations an investigation is made into the complex natural frequencies caused by the introduction of an active nonbilateral three terminal resistive device into a passive bilateral RC network. The approach used is general from the point of view that three terminal (or multiterminal) active nonbilateral devices may be handled that do not have a description on either an impedance or admittance basis but only on a mixed basis. It is shown that the zeroes of a certain Characteristic Determinant are the complex poles of the network. A new expression is given for the driving point impedance of a network consisting of a passive bilateral network with an embedded multiterminal active nonbilateral device.

With the aid of complex linear transformations three new transfer function synthesis techniques are derived. Each technique involves 2 two terminal-pair passive bilateral RC-networks and one three terminal active nonbilateral resistive device. The first two synthesis techniques will synthesize any stable transfer function to within a constant multiplier. The third synthesis technique is somewhat restricted with regard to the complex pole locations of the relevant transfer function. However, this third synthesis technique is of considerable theoretical interest since the three terminal active nonbilateral device involved may, without loss of generality, be specialized to a Gyrator. Since a Gyrator is passive (in fact, lossless), the "activity" of the active nonbilateral resistive device is not a necessary requirement to obtain complex natural frequencies in an RC network. Further support is given to this statement when it is demonstrated that an RC network with embedded active bilateral resistive devices must have its natural frequencies constrained to the σ axis. Thus, in fact, it is the nonbilaterality rather than the activity of the embedded active nonbilateral resistive device that allows the natural frequencies of an "active" RC network to become complex.

An effective analytic approach to the study of the fundamental properties of driving point and transfer functions of passive bilateral networks is based upon expressing the network functions in terms of the energy functions associated with the network. This approach was initially formulated by Brune[17] and further elaborated upon by Guillemin[3,18]. This thesis presents a number of new properties of active nonbilateral networks which are derived by extending the energy function approach to active nonbilateral linear networks.

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TABLE OF CONTENTS

Chapter 1

Introduction

Chapter 2

Analysis of Linear Networks

TABLE OF CONTENTS (Continued)

Chapter 3

Analysis of LLF:R Networks By Linear Transformation Theory

Chapter 4

Applications of Real Transformations to the Synthesis of RC-LLF:R Transfer Functions

Chapter 5

Complex Natural Frequencies of an RC-LLF:R Network

TABLE OF CONTENTS (Continued)

TABLE OF CONTENTS (Continued)

LIST OF FIGURES

LIST OF FIGURES (Continued)

LIST OF FIGURES (Continued)

LIST OF FIGURES (Continued)

CHAPTER I

INTRODUCTION

1.1 Introduction

The work in this thesis is concerned primarily with the application of linear transformation theory to the synthesis of active-nonbilateral linear networks. New results are also presented on the frequency domain properties of such networks and the general problem of linear network analysis. The idea of using linear transformation theory as an aid both in the analysis and synthesis of active-nonbilateral linear networks is due to Guillemin.[1,2] A brief discussion of his results and others in this area will be found in Section 1.4 after a discussion of the class of networks dealt with in this thesis in Section 1.2, and a discussion of recent results in the synthesis of active-nonbilateral networks using nontransformation theory approaches in Section 1.3. In Section 1.5 there is presented the method of approach used in this thesis to synthesize active-nonbilateral networks by means of linear transformation theory and a discussion is given of the resulting types of problems that arise. Finally Section 1.6 gives a brief summary of the thesis.

1.2 The Class of LLF Networks

The active-nonbilateral class of networks considered in this thesis may be regarded as a logical extension of the class of networks commonly designated as LLFPB[3] (linear, lumped, finite, passive, bilateral). An LLFPB network is conveniently defined as any network consisting of an interconnection of resistances, inductances, and capacitances. These elements are all positive and yield symmetric, positive-definite parameter matrices. If the resistances, inductances, and capacitances are allowed to take on negative as well as positive values, the network may lose its passive character but must still remain bilateral. The branch parameter matrices are still symmetrical but they no longer define positive definite quadratic forms. The mathematical significance of a non positive-definite resistance parameter matrix is that under some conditions of network excitation, the net average power into the network may become negative. Physically, this means that the network is delivering average power into the circuit external to itself rather than absorbing average power from it. The mathematical significance of non positive-definite capacitance and inductance parameter matrices is that under some conditions of network excitation the net stored capacitive or inductive energy may become negative. Physically, this means that the inductive or capacitive portion of the network is delivering average energy into the network external to itself rather than absorbing average energy from it. We might properly denote this class of networks as LLFB (linear, lumped, finite, bilateral) dropping the letter P from LLFPB because the network may no longer be called passive, but retaining the letter B since the network is still bilateral.

We may now drop the bilaterality restriction in the following way. Consider that in addition to the resistances, inductances, and capacitances of the LLFPB network we add three types of multiterminal black boxes - capacitive, inductive, and resistive. Thus the capacitive black box would be described by a s.c. admittance matrix equal to the product of a real matrix and s (the complex frequency variable). The real matrix need be neither bilateral nor define a positive definite quadratic form. Entirely analogous statements apply to the inductance and resistance boxes. Through a method suggested by Guillemin[2] these black boxes may be handled as far as network analysis is concerned like ordinary resistances, inductances, and capacitances. When this is done one finds that the inductance, resistance, and capacitance parameter matrices of a network containing such black boxes become non-symmetrical and do not define positive definite quadratic forms. We will use the letters LLF (linear, lumped, finite) to denote such a network, i.e., a network consisting of positive inductances, resistances, and capacitances plus resistance, capacitance, and inductance black boxes of the type described above. It will be convenient to have an abbreviation for each of these types of boxes. The following definitions appear to be appropriate:

R-LLF: Active-Nonbilateral Resistance Box
L-LLF: Active-Nonbilateral Inductance Box
C-LLF: Active-Nonbilateral Capacitance Box

It would be convenient to have a further notation for an LLFPB network containing black boxes of only one or two of the types above. We will use the following notation:

LLF:X: A network consisting of positive resistances, inductances, capacitances and black boxes of the types X-LLF where X may be one of the letters R, L, or C.

LLF:XY: A network consisting of positive resistances, inductances, capacitances and black boxes of the two types X-LLF and Y-LLF when X, Y may be any two of the letters R,L,C.

If all three types of boxes are involved the letters LLF will be used. Thus the class of networks which contain positive R,L,C plus active-nonbilateral resistance boxes would be designated by LLF:R.

The question naturally arises at this point as to the correspondence between the class of active-nonbilateral linear networks defined above and the physical active-nonbilateral networks appearing in practice. It is clear immediately that just as with LLFPB networks, delay lines or any other elements are omitted if they have impedances or admittances that are non-rational functions of the complex frequency variable s.

This follows from the more or less obvious fact that the inclusion of black boxes of the type described in an otherwise LLFPB network leaves the rational character of network functions unchanged. It should be noted also that we have been confining ourselves right from the start, to a discussion of time invarient linear systems, i.e., systems whose behavior is described by linear differential equations with constant coefficients

Excluding the above classes of networks, the remaining possibility for linear active-nonbilateral networks appearing in practice is the class of networks that may be formed by interconnecting resistances, inductances, and capacitances with vacuum tubes and transistors. It is assumed that the latter elements are being operated under small signal conditions so that the incremental behavior of the devices are linear. If one is concerned with a range of operating frequencies that is sufficiently low one may regard the vacuum tube and transistor to be active-nonbilateral resistance boxes and thus members of the class R-LLF. For frequencies above this range one finds that shunt capacitances across terminal pairs come into play. Thus for a large range of frequencies circuits containing vacuum tubes and transistors still belong to the class LLF:R even though the individual vacuum tubes and transistors do not belong to R-LLF. This comes about from the fact that in this range of frequencies, the active-nonbilateral character of these devices is resistive rather than inductive or capacitive in character. Of course if the operating frequency of the vacuum tube or transistor is pushed too high then transit time effects come into play. This causes the vacuum tube or transistor to be no longer representable as an R-LLF device with lumped terminal capacitances.

It appears then that from a practical point of view the class of networks LLF:R warrants our first consideration as far as the development of synthesis methods is concerned. Furthermore it will be demonstrated in Chapter 4 that the black boxes belonging to L-LLF and C-LLF may be synthesized from networks of the class LLF:R. Thus both from a theoretical and a practical point of view there does not appear to be any justification for giving anything but a passing glance at the other classes of LLF networks defined above. This is just the attitude that will be taken in this thesis even though some of the results would require small modification to be applicable to the classes of LLF networks other than LLF:R.

If a network consists of only resistances and capacitances plus R-LLF devices we will designate it an RC-LLF:R network. The extension of this notation to other two element kind combinations is clear.

1.3 Previous Results

In this section we will discuss some previous results in the synthesis of active-nonbilateral linear networks that have been arrived at using a non-transformation theory approach. We will confine our attention to synthesis methods involving a small number of R-LLF devices.

Networks containing vacuum tubes, transistors, resistors, inductors, and capacitors have had a wide variety of applications in communications technology. Some of the important ones are listed below:

1. Amplification of signals
2. Compensation of existing non-ideal characteristics of system such as with feedback amplifiers.
3. Simplification of linear transfer function synthesis due to the isolating property of vacuum tubes
4. Compensation for parasitic dissipation in passive elements. Examples are afforded by the Q-multiplier and to stretch a point - the oscillator
5. The general ability to relax various restrictions imposed on the driving point and transfer functions of LLFPB Networks. In particular the ability to make two-element kind network functions behave as general as three-element kind network functions.

Items 1 through 4 are familiar applications. But item 5 is relatively new and perhaps represents the most fascinating item to a network theorist like the author who has only recently left the warm shelter of LLFPB network theory. It is demonstrated in this latter discipline that the natural frequencies of RC (and RL) networks are confined to the negative-real axis. This is unfortunate since the theoretical capacitive element is far closer to the physical capacitive element than is the theoretical inductance to its physical counterpart. This non-ideal behavior of the physical inductor is especially noticeable at low frequencies. Thus a design of an RC network based on ideal R and C for low frequency applications will be much more likely to yield the results predicted than a similar design involving inductances. In addition the inductors required for low frequency application become bulky and costly.

Linvill[(4)] was apparently the first person to demonstrate the general character obtainable for the transfer function of an active-nonbilateral RC network. He developed a general transfer impedance synthesis method involving two passive bilateral RC networks in a cascade connection separated by a negative impedance converter. By this connection any specified stable transfer function can be realized to within a constant multiplier. The negative impedance converter is an ideal two terminal-pair element that yields at one terminal pair the negative of the impedance connected at the other terminal pair. Subsequent work by others[(5,6)] has produced very good practical negative impedance converters. In the realization of the negative impedance converter two or more vacuum tubes or transistors are required. The total number of elements in the RC networks is of the order of magnitude of the number of elements that would be required to synthesize the specified transfer function by an RLC network and thus the synthesis method does not require an increase in network complexity.

Following Linvill, Horowitz[(7)] modified Linvill's method so that the negative impedance converter was not explicitly used. His method involves the manipulation of dependent sources leading to a realization of the active element directly in terms of at most two transistors. The final result is as general as Linvill's.

Recently Yanagisawa[8] presented a general transfer admittance synthesis method which involves a negative impedance converter plus two RC networks in a parallel type rather than a cascade type of connection. The advantage of this method over Linvills lies in the fact that the RC networks can be so simple that an L type network configuration is sufficient. The final result is that four driving point impedances are required to be synthesized rather than two two terminal-pair RC networks with complex transmission zeroes (in the general case) as with Linvill's method.

We should also mention the work of DeClaris.[9] He has shown that any stable (denominator polynomial Hurwitz) driving point function can be realized with R's, L's, C's, and a two terminal-pair device called a "controlled" source. When considered as a grounded two terminal-pair device, the "controlled" source may be regarded as an ideal vacuum tube. An ideal vacuum tube is defined here as an incremental model of a vacuum tube which has infinite plate resistance and no interelectrode capacitance. (There is a dual controlled source which he mentioned, but this has no realization in terms of an ideal vacuum tube). Some of his synthesis methods require that an ungrounded two-terminal pair "controlled" source be used. In this case one cannot use the ideal vacuum tube alone. A possible theoretical realization for the ungrounded two-terminal pair controlled source is an ideal vacuum tube in cascade with an ideal transformer. In particular he gave a synthesis method for a driving point impedance in terms of R's, C's and one controlled source. However, although it was not explicitly stated, the method will only work if the "controlled" source is an ungrounded two terminal-pair network.

According to well-informed sources Kinnarawalla* has developed a method of synthesizing any p.r. driving point impedance using one negative impedance converter plus associated resistances and capacitances. Unfortunately there appears to be no published record of this method.

The above researchers have clearly demonstrated that an RC network with an embedded R-LLF device (assuming we can call a negative impedance converter an R-LLF device) may have driving point and transfer functions of a general character. However none of these people have investigated the following general question: how do the parameters of an arbitrary embedded R-LLF device influence the locations of the complex natural frequencies of an RC-LLF:R network? In the work mentioned above either the R-LLF device or the network configuration or both are frozen a-priori. Both the negative impedance converter and the "controlled" source have given general results. Is there something special about these devices? Are there other devices that will do as well? Are there other types of network configurations that will yield as general results as those used by Linvill, Horawitz, and Yanagisawa? These questions have been investigated in this thesis using a linear transformation theory approach to the study of active nonbilateral networks. It is believed that considerable light is shed upon the above questions. A synthesis procedure is given involving an ideal vacuum tube plus two two terminal-pair RC networks that allows the synthesis of an arbitrarily

*Bell Telephone Lab., Murray Hill, N.J.

specified stable transfer impedance or voltage ratio to within a constant multiplier (no pole at infinity). Another general synthesis procedure is given for synthesizing any one of the four possible transfer functions (impedance, admittance, voltage, and current). This method involves the same configuration as Linvill used with the negative impedance converter but involves a new R-LLF device. It is also shown that fairly general transfer functions may be realized using a gyrator although complete generality is not obtainable here.

1.4 Linear Transformation Theory and Linear Active-Nonbilateral Networks

The concept of applying linear transformation theory as an aid to the analysis and synthesis of active-nonbilateral linear networks is due to Guillemin.[1,2] His initial impetus for considering the application of linear transformations to the study of active-nonbilateral linear networks came from a consideration arising in the formulation of equilibrium equations for LLFPB networks. He noted that if, for example, one cut-set γ is used to define node-pair voltage variables and another cut-set α is used to formulate Kirchoff current equations, then the equilibrium equations on the node basis become unsymmetrical. If the cut set α is used both for defining an independent set of node pair voltages and for formulating Kirchoff's current equations, the node equilibrium matrix becomes symmetrical. One may readily show that the node-pair voltages for the symmetrical formulation of equilibrium equations are related to those for the unsymmetrical formulation through a real non-singular transformation matrix. Thus we have a situation in which an LLFPB network is characterized by an unsymmetrical admittance matrix by effecting an appropriate linear transformation of node pair voltage variables. The intriguing possibility then suggested itself to Guillemin that if LLFPB networks could be characterized by non-symmetrical impedance or admittance matrices through a linear transformation of dynamic variables, active-nonbilateral linear networks might well be characterizable in terms of symmetrical impedance or admittance matrices through use of a linear transformation of variables. Or, more to the point, perhaps the dynamic variables of an active-nonbilateral network could be expressed either in terms of those of an LLFPB network, or else in terms of those of a simpler active-nonbilateral network. Guillemin has demonstrated this supposition to be true in at least one general sense. He demonstrated the following fact. Let there be given a network of the class LLF:R excited in some particular fashion. To be specific let us apply current sources at a set of independent node pairs. Let the response quantities be node pair voltages. Then we can find an LLFPB network (there are actually an infinite number) with the same set of current sources applied whose node pair voltages are related to those of the LLF:R network through a real non-singular transformation matrix. However, in addition to the current sources applied to the LLFPB network there must also be voltage sources applied in all the links. Moreover these voltage sources are dependent rather than independent. Specifically the link voltage sources are related to the LLFPB node pair voltage through a real transformation matrix. Thus while such a representation allows the dynamic variables of an LLF:R network to be expressed in terms of those of an LLFPB network one must contend with dependent sources in the LLFPB network.

A rather interesting result was derived by Guillemin[2] using a linear transformation of network variables wherein the elements of the transformation matrix are functions of the complex frequency variable s. Specifically, he represented the node pair voltages of a multistage transistor amplifier as a linear transformation of those of a multistage vacuum tube amplifier. This transformation theory approach leads to a synthesis technique wherein one may design a multistage transistor amplifier to have the same transfer impedance as the multistage vacuum tube amplifier. Masenten[10] elaborated upon this result in his Masters Thesis.

Another, more specialized, synthesis method using real transformations has been derived by Nashed[11] and Stockham[12]. This method allows gain to be inserted in the transfer functions of a network, if certain conditions with regard to network configuration are satisfied. Nashed's method of approach is more general than Stockham's. However an examination of his results and Stockham's show an inconsistency. Namely, the network configuration which Stockham proved allows gain insertion is not in the class of networks that Nashed claims allows gain insertion. This inconsistency has been resolved by the author as a byproduct of some general investigations of the application of linear transformation theory to the study of active-nonbilateral linear networks. The result arrived at is more general than those of Nashed and Stockham both with regard to approach and with regard to possible network configurations which allow gain insertion by applying linear transformation theory.

1.5 Linear Transformation Approach Used in This Thesis

1.5.1 General Approach

In this section the general approach will be presented that is used in this thesis for applying linear transformation theory to the synthesis of LLF:R networks. It is an approach which is invariably used when one starts developing design or synthesis methods for any new field. An appropriate name for this approach might be "synthesis through analysis". It proceeds in the following way. First existing methods of analysis are studied or, possibly, new methods of analysis are developed. In general, different techniques of analysis will be found to be particularly effective with different classes of systems. Thus one finds a pairing of analysis techniques with system classes perhaps on the basis of ease of analysis or a simplicity of viewpoint that allows a good understanding of the basic physical mechanisms involved. Having found these pairs an attempt is made to develop synthesis methods by an appropriate inversion of the results of analysis.

Applying the above thougts to using linear transformation theory to synthesize LLF:R networks one must study existing methods of analysis or else develop new methods of analysis. By analysis we mean in our case the process by which the dynamic variables of an LLF:R network are represented in terms of those of an LLFPB network by means of some nonsingular transformation matrix. The transformation matrix may be either real or complex (i.e. a function of $s = \sigma + j\omega$). Next we must attempt to find LLF:R network configurations and techniques of analysis that, in combination, allow as simple an interpretation as possible of the physical processes involved, i.e. of the mechanism by which in retrospect the linear transformation effects a conversion of network passivity and bilaterality into non-passivity and nonbilaterality. Presuming that we have found such analysis techniques we carry through an analysis of some specific configurations. Examination of the results of this analysis will hopefully point to methods whereby one may reverse the process and say, synthesize a transfer function otherwise unobtainable by an LLFPB network.

Exactly how the above ideas were put into use in this thesis will be discussed in this and the following section. First let it be noted that, in the large, no new methods of linear transformation theory analysis of LLF:R networks were developed. Rather existing methods, all due to Professor Guillemin, were either specialized, modified, or extended. In particular a search was made for those methods of analysis which did not lead to dependent sources embedded in the LLFPB reference network, since it was felt that the presence of dependent sources could only occlude an understanding of the basic physical processes involved in the transformation theory approach. Thus Guilemin's general analysis method was studied only with respect to determining the conditions under which dependent sources do not appear. As a result of this investigation a rather general result was discovered. This result is discussed in detail in Chapter 3, Section 3.2. In brief, if an LLF:R network with open circuit impedance matrix Z satisfies certain conditions with regard to topology, then we may express its voltages and currents in terms of those of an LLFPB network with impedance matrix $\hat{Z}$ and no dependent sources will appear.

Further investigations along different channels than the above were carried out to determine linear transformation theory analysis techniques that allowed a representation of an LLF:R network in terms of an LLFPB network without dependent sources. The results of these further investigations are reported in Sections 3.3 and 3.4 of Chapter 3. Both complex and real linear transformations are considered. The complex transformation methods discussed lead to easily interpretable results when the embedded R-LLF device has a small number of terminals. Some interesting results were found with real transformations. It is shown in Section 3.3.1 that the method arrived at by specializing Guillemin's branch transformation analysis method so that no dependent sources appear is a special result of a quite different approach. This latter approach also has resolved the viewpoints of Nashed and Stockham, as discussed in Section 1.4, with regard to inserting gain.

Following the above general investigations of analysis techniques in Chapter 3, analysis of particular network configurations is carried through in Chapters 4 and 6. In particular the methods of analysis were applied to LLF:R networks consisting of an RC-LLFPB network of a general character in conjunction with one three-terminal R-LLF device. The motivation for such a restriction should be apparent from the discussion of Section 1.3. In the following section a general discussion is given of the difficulties that arise in designing synthesis methods through applying the "synthesis through analysis" method to synthesis of active-nonbilateral linear networks by linear transformation theory.

1.5.2 Specific Approach

Let us suppose that we have analyzed an LLF:R network into an LLFPB network and a set of linear transformations. We will confine ourselves here, for illustrative purposes, to discussion of the situation in which current sources are the excitation and the node pair voltages are the responses. Let it now be assumed that through some method of linear transformation analysis we obtain

$$Qi_s = \hat{i}_s$$

$$e = P\hat{e} \tag{1.5.1}$$

where the pairs i_s, e and $\hat{i}_s, \hat{e}$, all column matrices (or vectors), represent the current excitation and node pair voltage response of the LLF:R network and the corresponding quantities for the LLFPB reference network, respectively. The transformation matrices P,Q are assumed to be nonsingular but not necessarily real, i.e., they may be a function of $s = \sigma + j\omega$. If Z and $\hat{Z}$ denote the o.c. impedance matrix of the LLF:R network and the LLFPB reference network, then by definition

$$e = Zi_s \tag{1.5.2}$$

and

$$\hat{e} = \hat{Z}\hat{i}_s \tag{1.5.3}$$

If Eq. 1.5.3 is premultiplied by Q and Eq. 1.5.1 are used we find that

$$e = (P\hat{Z}Q)i_s \tag{1.5.4}$$

from which we deduce that

$$Z = P\hat{Z}Q \tag{1.5.5}$$

Thus the pair of transformations (1.5.1) cause the LLF:R o.c. impedance matrix to be expressed as the result of a premultiplication and postmultiplication of nonsingular matrices upon the o.c. impedance matrix of the LLFPB network. Conversely, if an LLF-R o.c. impedance matrix is expressible in the form (1.5.5) then its current excitation vector $\hat{i}_s$ and voltage response vector $\hat{e}$ are related to those of the LLFPB network with o.c. impedance matrix $\hat{Z}$ through the transformations (1.5.1).

Let us restate the assumptions relating to the specific example we shall discuss. We have available an active-nonbilateral network consisting of an RC-LLFPB network with one embedded multi-terminal RLLF device. This RC-LLF:R network has an o.c. impedance matrix $\hat{Z}$ defined for a certain set of accessible terminal pairs. Through analysis by linear transformation theory it is presumed that we have found an RC-LLFPB network with o.c. impedance matrix $\hat{Z}$ such that Eq. 1.5.5 is satsified.

We may make one general observation at this point. If P and Q are real then the o.c. impedance poles of the RC-LLF:R network are identical to those of the RC-LLFPB network. This comes about from the fact that each element of the Z matrix is expressed by (1.5.5) as linear combinations of the elements of the $\hat{Z}$ matrix. Thus without further manipulation of some type real transformations appear to lead to rather restricted results for RC-LLF:R networks. One possible approach to extend the usefulness of real transformation is based upon the following thought. While real transformations of the type in Eq. 1.5.1 leave the poles of the open circuit driving point impedances constrained to the negative real axis, the zeroes of the impedances are not so constrained. This comes about from the fact that through the agency of real transformations, the RC-LLF:R driving point impedances are expressed not only as linear combinations of RC-LLFPB o.c. driving point impedances but also of RC-LLFPB transfer impedances. Since a short circuit constraint placed at a terminal pair with impedance zeroes at complex frequencies will yield a network with natural frequencies at these same complex frequencies we may achieve through the intermediary of real transformations upon an RC-LLFPB network an RC-LLF:R network with natural frequencies in the complex plane. These ideas are put to use in Chapter 4.

If the transformation matrices P and Q are functions of s then the o.c. impedance poles may lie anywhere. The poles of Z are, in general, the poles of $P, \hat{Z}$, and Q. While the poles of $\hat{Z}$ must lie along the negative real axis the poles of P and Q need not be constrained in this way. In the methods of synthesis derived in this thesis that use complex linear transformations, the elements of P and Q are also functions of the elements of $\hat{Z}$. In such a case the elements of Z will, as a rule, be nonlinear functions of the elements of $\hat{Z}$.* In addition they will be functions of the parameters of the multi-terminal R-LLF device of the RC-LLF:R network. From observation

*It may appear that an inconsistency exists at this point since linear transformations of the network variables have produced RC-LLF:R driving point and transfer functions which are nonlinear functions of those of the RC-LLFPB reference network. However there is no inconsistency here since Eq. 1.5.1 which define analysis by linear transformation theory show that e and i_s are linearly related to e and i_s respectively whether or not P and Q involve the elements Z.

of the elements of Z it is possible to determine driving point and transfer functions which clearly exhibit the possibility of complete generality in the location of zeroes and poles. For purposes of discussion, suppose that examination of Z shows that the o.c. driving point impedance at terminal pair $1(z_{11})$ has the form

$$z_{11} = \frac{az_{34}}{bz_{33} - cz_{12}} \tag{1.5.6}$$

where z_{33}, z_{12}, and z_{34} are open circuit impedances for the LLFPB reference network and a,b,c are constants dependent upon the R-LLF device embedded in the RC-LLF:R network. It is important to note that the specifications on the RC-LLFPB network involve four terminal pairs as shown below.

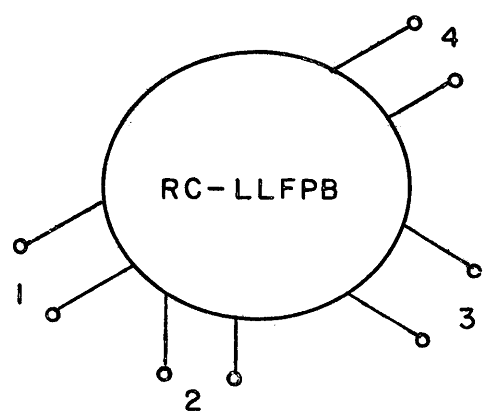


Figure 1.5.1. LLFPB Reference Network

What one would like to do with z_{11} is synthesize it for an arbitrarily prescribed set of poles and zeroes. Having an expression for z_{11} of the form of Eq. 1.5.6 in which it can be recognized that z_{11} has the potentiality of complete generality in location of poles and zeroes is far from having a synthesis method.

Before one can synthesize z_{11} to specification, the following steps must be completed in order.

1. Sufficient conditions for realizability of $\hat{z}_{33}$, $\hat{z}_{12}$, and $\hat{z}_{34}$ must be determined so that one can be sure a specified set of these network functions come from a 4 terminal-pair RC-LLFPB network

2. An Algorithm must be found such that one may go in a step by step procedure from a specified pole-zero pattern for z_{11} to a set of functions $\hat{z}_{33}$, $\hat{z}_{12}$, $\hat{z}_{34}$ that satisfy the sufficient conditions established in item 1.

3. The RC-LLF:R network must be synthesized

Let us discuss these problems in order. First we note from Eq. 1.5.6 that if we are to synthesize z_{11} we must be able to specify a set of functions $\hat{z}_{33}$ $\hat{z}_{12}$, and $\hat{z}_{34}$ which may actually come from a 4 terminal-pair RC network. It will be presumed that the RC-network does not contain ideal transformers. The reason for this is that the LLF:R networks considered in this thesis have LLFPB subnetworks intimately related if not identical, to the corresponding reference LLFPB networks. In such a case, the synthesis of the LLF:R network involves also the synthesis of the LLFPB reference network. No attempt is made in this thesis to use ideal transformers in synthesizing RC-LLF:R networks since it is felt that by so doing one has defeated the original purpose of studying such networks. After all, an ideal transformer is a limiting form of a pair of mutually coupled coils. Inclusion of such elements in RC networks makes them in effect RLC networks since any practical realization involving transformers will actually introduce inductance.

To find sufficient conditions for specifying $\hat{z}_{33}$, $\hat{z}_{12}$, and $\hat{z}_{34}$ we must put ourselves in the position of synthesizing the 4 terminal

RC Network of Fig. 1. The general problem of multiterminal-pair synthesis of RC-LLFPB networks without ideal transformers is a presently unsolved problem of network synthesis. Exactly what can be done without ideal transformers is summarized in Fig. 1.5.2 for RC Networks.

	Any RC	Any RC	May Not Be Specified
Specifications	$\hat{y}_{11}(\hat{y}_{22})$	$\hat{y}_{12}$ (to within a constant) multiplier	$\hat{y}_{22}(y_{11})$
	$\hat{z}_{11}(\hat{z}_{22})$	$\hat{z}_{12}$ (to within a constant) multiplier	$\hat{z}_{22}(z_{11})$
	$\hat{y}_{11}$	$\hat{y}_{22}$	$\hat{y}_{12}$
	$\hat{z}_{11}$	$\hat{z}_{22}$	$\hat{z}_{12}$

Figure 1.5.2. Constructible Specifications on Two Terminal-Pair RC Networks Without Ideal Transformers

We will call the specifications of Fig. 1.5.2 constructible specifications for obvious reasons.

Reference to Fig. 1.5.2 indicates that we may synthesize for one driving point admittance (impedance) and a transfer admittance (impedance). There is a maximum constant multiplier[13] that may be specified in the transfer function. The following dilemma thus arises - we need to synthesize a four terminal-pair RC-LLFPB network for certain prescribed driving point and transfer functions but we only have available synthesis methods which allow a synthesis of two terminal-pair networks for the specifications of Fig. 1.5.2. Since it is not the intent of this thesis to develop methods of multiterminal-pair RC-LLFPB network synthesis no significant work in this direction was accomplished. Rather, the following obvious approach to a possible solution of the dilemma is followed. The multiterminal-pair RC-LLFPB network is restricted to a makeup of two terminal-pair RC-LLFPB networks. In our illustrative example we would interconnect a number of two terminal-pair RC networks to form a four terminal-pair RC network. We attempt to arrange the two terminal-pair component networks so that the quantities $\hat{z}_{33}$, $\hat{z}_{12}$, and $\hat{z}_{34}$ become functions of constructible component network functions as indicated in Fig. 1.5.2. Thus, for example, if for some arrangement of component RC networks we find that $\hat{z}_{12}$ is a function of both driving point impedances z^a_{11}, z^a_{22}, and the transfer impedance z^a_{12} of an RC component network, a, we will be in trouble unless we can specialize this latter RC network to a T_π, T, or L type configuration of driving point functions. This latter type of specialization is sometimes effective but frequently such a specialization restricts $\hat{z}_{12}$ considerably in character and prevents a final realization of z_{11} with completely general specifications. Such a method of synthesizing a four terminal-pair RC-LLFPB network is clearly a "cut and try" method which requires a certain degree of ingenuity for its effective use.

Let us suppose that we have managed to subdivide the 4 terminal-pair RC-LLFPB reference network into two terminal-pair component subnetworks such that z_{11} becomes expressed in terms of network functions involving "constructible" specifications. Suppose for example that two subnetworks are involved. If we have completed step 1 one possible form for z_{11} is

$$z_{11} = f(z^a_{11}, z^a_{12}, y^b_{22}, y^b_{12}) \tag{1.5.7}$$

where f is some rational function of its argument. We now turn our attention to step 2. This involves finding a realizable set z^a_{11}, z^a_{12}, y^b_{22}, y^b_{12} when z_{11} is initially specified. It is the first step at which the process of analysis is reversed i.e., the process of synthesis becomes initiated. One must find an Algorithm such that given z_{11} one may find in a step by step procedure a realizable set of RC-LLFPB two terminal-pair driving point and transfer functions $z^a_{11}, z^a_{12}, y^b_{22}, y^b_{12}$.

The solution of the last problem is subdivided into two parts. One part involves synthesizing the RC-LLFPB portion of the RC-LLF:R network and the other part involves synthesizing the R-LLF portion of the RC-LLF:R network. It would be well to recall that we initially started with the RC-LLF:R network. This was "analyzed" into an RC-LLFPB network and a set of linear transformations of dynamic variables. Thus the RC-LLF:R network configuration is known <u>a-priori</u>. The synthesis of the RC-LLFPB portion of the RC-LLF:R network will in general be a difficult task since this will be a multiterminal-pair RC network. Fortunately in the networks discussed in this thesis the RC-LLFPB portion of the RC-LLF:R network is closely related to or identical to the reference RC-LLFPB network. This results in the fortuitous result that a solution of step 1 above is also a solution of the first part of step 3. With regard to the second part of step 3, we note that there is no general practical method available at present for synthesizing a multiterminal-pair R-LLF device for prescribed s.c. conductance matrix. By practical it is meant a synthesis method which involves components which are commercially available transistors and vacuum tubes in addition to positive resistance. In Chapter 4 there is presented a theoretical method which involves positive and negative resistances plus ideal vacuum tubes. From a practical point of view one would prefer to use a synthesis technique which involves an R-LLF device with as few terminals as possible. The minimum number which can produce general results in z_{11} is three. This arises from the fact that a two terminal R-LLF device is only a negative or positive resistance and, as demonstrated in Chapter 7, a network consisting of positive and negative resistances plus positive capacitances cannot have natural frequencies off the real axis. The various synthesis techniques have thus involved only a three-terminal R-LLF device in order to keep the R-LLF portion of the RC-LLF:R network as simple as possible.

1.6 Summary of Thesis Results

In this section we will summarize the new results afforded by the thesis. We will consider subjects in the order in which they appear in the thesis. The work falls into three categories:

(1) Analysis of linear networks

(2) Synthesis of transfer functions through application of linear transformation theory

(3) Properties of driving point and transfer impedances of LLF networks

1.6.1 Analysis

A method of analysis of linear networks is presented that is applicable to networks whose elements may have any number of terminals. The method is conventional in that it involves a formulation of equilibrium equations and their subsequent solution for the desired network properties. This is in contrast to methods like those of Mason[14] and Percival[15] for instance which might be termed "purely" topological in nature. By a "purely topological method" it is meant that the desired network properties are found by operations upon a suitably constructed network graph.

The analysis method described in Chapter 2 was arrived at as a generalization of a method due to Guillemin[2]. In Guillemin's method the network elements are R's, L's, C's and multiterminal active-nonbilateral resistive devices. Each multiterminal-pair device, which is assumed to have node-to-datum terminal pairs assigned, is represented for purposes of analysis by a tree of branches and a set of linear equations with real coefficients relatively the voltage and currents of the branches. Once the multiterminal pair resistive devices are replaced by a tree of branches conventional methods of network analysis are found applicable. In Chapter 2 the network elements are all MTP (multiterminal-pair) devices which are not necessarily resistive in character. The only requirement is that the terminal-pair voltages and currents be related by linear equations. The definition of terminal pairs for each MTP element is arbitrary. To each definition of terminal pairs there corresponds a different branch representation. The concept of the associated MP (multiple) network is introduced as the physical network corresponding to a MTP element but with no assignment of terminal pairs. Since a network with a given number of nodes may have terminal pairs assigned in a large number of ways, each MP network is said to be describable by a large number of MTP elements - one for each different assignment of terminal pairs. The relationship between the impedance and admittance matrices is given for those MTP elements that describe the same associated MP network.

Some attention is given to the special situation in which the equilibrium matrix of a network composed of MTP elements may be found by simple addition of the parameter matrices of the component MTP elements. The Additive class of networks is defined. An Additive network has the property that not only loop and node equilibrium matrices but also mixed equilibrium matrices may be evaluated by addition of matrices describing the behavior of the MTP elements. The Additive class of networks is found in Chapters 5 and 6 to be of particular importance in the application of complex linear transformations to the synthesis of LLF networks.

It has been found that the method of analysis of Chapter 2 is very similar to a special case of Kron's[16] method of analysis which involves "tearing networks" if we make a correspondence between "torn networks" in this method and the MTP element in Chapter 2. Where as the "torn network" of Kron may contain internal sources the MTP element is assumed to be homogenous, i.e., to exhibit no terminal pair voltages and currents when unexcited externally. An advantage of the method of Chapter 2 is its simplicity. No Tensor algebra or notation are involved and in fact the method requires little more knowledge in its application than an ability to write conventional loop and node equations when mutual inductance is present.

1.6.2 Synthesis

In Chapter 3 some particular techniques of LLF:R network analysis through linear transformation theory are presented. The techniques involve both real and complex linear transformations. In Chapters 4 and 6 these analysis procedures are reversed in accordance with the "synthesis through analysis" procedure outlined in Section 1.5. Chapter 4 deals with real transformations and Chapter 6 with complex transformations. The synthesis methods arising from real transformations had the general difficulty of being unable to meet the "constructible" specifications requirement discussed in Section 1.5.2. One particular case was found to meet the constructible specifications requirement. However attempts at finding an Algorithm as required in step 2 of Section 1.5.2 have not been successful.

Chapter 6 considers the use of the complex linear transformation techniques of Chapter 3 in order to use the "synthesis through analysis" method to synthesize transfer functions of RC-LLF:R networks. As groundwork for the material of Chapter 6, Chapter 5 investigates the complex natural frequencies caused by the introduction of an R-LLF three terminal device into an RC-LLFPB network. It is shown that the zeroes of a certain Characteristic Determinant are the complex poles of the network. Attention is given to conditions on the R-LLF device and the RC-LLFPB network such that the Characteristic Determinant involves RC-LLFPB network functions that constitute constructible specifications on two terminal pair networks. This is done as an aid in developing potentially acceptable transfer functions (i.e. those having the possibility of general pole-zero locations) which involve only constructible specifications. The approach used is general from the point of view that R-LLF devices may be handled that do not have a description on either an impedance or admittance basis but only on a mixed basis.

The following question is investigated for some specific R-LLF devices. Can an RC-LLFPB network be found such that when the R-LLF device is embedded in the RC-LLFPB network, the resulting RC-LLF:R network will have a prescribed set of natural frequencies? A number of R-LLF devices are found to allow an arbitrary assignment of complex natural frequencies. The gyrator, a passive R-LLF device, is found to allow a fairly general assignment of complex natural frequencies. Thus the "activity" of the R-LLF device is not a necessary requirement to obtain complex natural frequencies. Support is given to this statement in Chapter 7 where it is demonstrated that an RC-LLFPB network with embedded active bilateral resistive devices is constrained to have σ axis natural frequencies. Thus in fact it is the <u>nonbilaterality</u> of the R-LLF device rather than its <u>activity</u> which allows the natural frequencies of an RC-LLF:R network to become complex.

At the close of Chapter 5 a new expression is given for the driving point impedance of a network consisting of a passive network with an embedded MTP R-LLF device. Particular expressions are given for the cases in which the R-LLF device has three terminals and is only describable in one of four possible ways (impedance, admittance, and two mixed cases).

In Chapter 6 three new general transfer function synthesis techniques are presented that involve two two terminal-pair RC-LLFPB networks and one three terminal R-LLF device. Two of these techniques will synthesize any stable transfer function to within a constant multiplier. The R-LLF device involved in the third technique includes the gyrator as a special case and does not allow a completely general assignment of poles. The R-LLF device involved in the second technique is an ideal vacuum tube and that in the third technique an R-LLF device involving a singular short circuit admittance matrix. Since the primary emphasis of this thesis is to present a new approach to the synthesis of LLF:R networks, the linear transformation approach, no special attention is given to the practical design of the R-LLF devices involved. References are given in the literature to cases where practical realizations are discussed for particular cases of the R-LLF devices of Synthesis Techniques 1 and 3.

1.6.3 Properties

An effective analytic approach to the study of the fundamental properties of driving point and transfer functions of LLFPB networks is based upon expressing the network functions in terms of energy functions associated with the network. This approach was initially formulated by Brune[17] and further elaborated upon by Guillemin.[3,18] Chapter 7 presents a number of new properties of LLF networks which are derived by extending the energy function approach discussed above to LLF networks. In making this extension it is found that the so called energy functions F_o, T_o and V_o of Reference 3 become complex. When they are resolved into real and imaginary parts the interesting result appears that the real parts are a function only of the symmetric portions and the imaginary parts are a function only of the skew-symmetric portions of the parameter matrices of the embedded R-LLF, L-LLF, and C-LLF devices.

Rather than listing here specific properties derived in Chapter 7, the reader is referred to the statement of these properties in Chapter 7.

CHAPTER 2

ANALYSIS OF LINEAR NETWORKS

2.1 Introduction

In this chapter a method of analyzing linear networks is presented that is applicable to networks whose elements may have any number of terminals. The method is conventional in that it involves a formulation of equilibrium equations and their subsequent solution for the desired network properties. This is in contrast to methods like those of Mason(14) and Percival(15), for instance which might be termed "purely topological" in nature. By a "purely topological method" it is meant that the desired network properties are found by operations upon a suitably constructed network graph without the intermediary operation of formulating equilibrium equations.

In the method of analysis proposed here each multi-terminal element is represented, for purposes of defining voltage and current variables, by a tree whose nodes are the terminals of the multi-terminal element. The tree is constructed by creating branches between those terminal pairs at which node pair voltages are defined. By this artifice we see that the number of ways independent node pair voltages may be assigned at the terminals of a multi-terminal element of n nodes is just the number of different trees that may be formed from n nodes. This number is n^{n-2}.

Since each node-pair voltage exists across a branch and since the driving current at a node pair may be identified with the corresponding branch current we may regard the s.c. admittance matrix (or o.c. impedance matrix) as a "generalized" branch parameter matrix of the multi-terminal element. Each of the possible n^{n-2} branch parameter matrices are related by simple congruent transformations. It is clear that a graph of the interconnection of these branches together with the branch parameter matrix is sufficient to characterized the multi-terminal element. In fact, we may say that the multi-terminal element has been represented with regard to terminal behavior by a set of mutually coupled branches. These branches differ from a set of mutually coupled inductors in only two respects: the coupling coefficients are functions of s (the complex frequency variable) and the coupling between branches is not necessarily bilateral. Conventional methods of network solutions are found to be applicable despite these differences. As far as dependent sources are concerned there is no more need to include them in the analysis than there is to include them in the analysis of LLFPB networks with mutually coupled inductances.

After each multi-terminal device has been represented by a set of branches, according to the dictates of necessity or convenience, one may use the conventional methods of defining voltage and current variables to analyze the network. The construction of cut-set and tie-set matrices proceeds as in networks wherein the coupling between branches is purely bilateral. Just as with the purely bilateral case one may write the equilibrium equations at the outset by using the loop method or the node method of analysis. Herein lies the advantage of this method over Shekel's[(19)]. It is well known that some problems are better suited to the loop method than the node method. With

Shekel's method we are not only constrained to use the node basis but we must use the node to datum variables. What is claimed for the method of analysis presented in this section is a much greater degree of flexibility than is present in Shekel's method.

In Section 2.4 some attention is given to the special situation in which equilibrium matrices may be found by addition of branch parameter matrices of component multiterminal network elements. The additive class of networks is defined. An additive network has the property that not only loop and node equilibrium matrices but also mixed equilibrium matrices may be evaluated by addition of component branch parameter matrices.

The flexibility in the proposed method of analysis has been found very useful in problems associated with applying linear transformation theory to the synthesis of LLF:R networks.

In the following section we will discuss the characterization of the multiterminal-pair network element as a set of mutually coupled branches.

2.2 The Multiterminal-Pair Network Element

In this section it will be shown that a multiterminal-pair homogeneous* linear network may be completely characterized, as far as terminal-pair behavior is concerned, by a tree (or a number of trees) of mutually coupled branches. The quickest way to understand this characterization is to start with a tree of mutually coupled inductances and represent it as a multiterminal-pair network.

* A homogeneous multiterminal-pair linear network is defined as a linear network which exhibits zero terminal-pair voltages and currents when not excited externally by voltage or current sources.

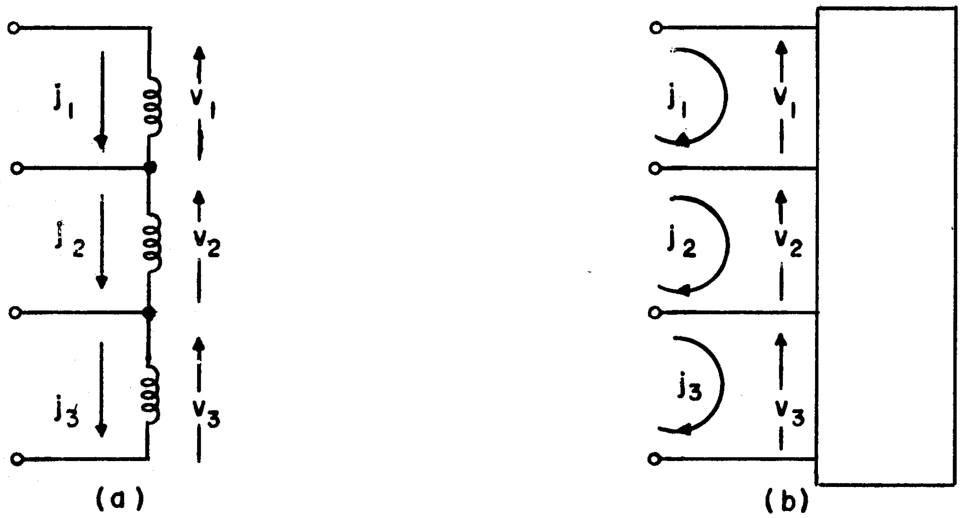


Fig. 2.2.1 Representation Of Tree Of Coupled Inductances As A Multiterminal Pair Network

Figure 2.2.1a depicts three mutually coupled inductances arranged in a tree. The positive reference directions are indicated for the branch voltages (- to + in direction of arrow) v_1, v_2, v_3 and branch currents j_1, j_2, and j_3. These branch voltages and branch currents are related as follows

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} sL_{11} & sL_{12} & sL_{13} \\ sL_{12} & sL_{22} & sL_{23} \\ sL_{13} & sL_{23} & sL_{33} \end{bmatrix} \begin{bmatrix} j_1 \\ j_2 \\ j_3 \end{bmatrix} \qquad (2.2.1)$$

The matrix

$$L = \begin{bmatrix} L_{11} & L_{12} & L_{13} \\ L_{12} & L_{22} & L_{23} \\ L_{13} & L_{23} & L_{33} \end{bmatrix} \qquad (2.2.2)$$

is the branch inductance parameter matrix for the three inductances of Fig. 2.2.1a. If the matrices v and j are defined as

$$v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \quad j = \begin{bmatrix} j_1 \\ j_2 \\ j_3 \end{bmatrix} \tag{2.2.3}$$

then Eq. 2.2.1 becomes

$$v = sLj \tag{2.2.4}$$

Let us suppose that someone has presented us with a three terminal pair black box and desires to know the open circuit impedance matrix. This black box is shown in Fig. 2.2.1b. Terminal pairs 1, 2, and 3 are defined unambiguously by the arrows denoting the positive reference directions for the corresponding terminal pair voltages v_1, v_2, and v_3. A similar claim may be made for the arrows denoting the positive reference directions for the terminal pair currents but this is somewhat more difficult to see because of the sharing of a common node by adjacent terminal pairs. When measurements are made at the terminal-pairs of the black box to determine the relationship between the terminal-pair voltages and currents it is found that

$$v = Zj \tag{2.2.5}$$

where

$$Z = sL \tag{2.2.6}$$

The matrix Z is the open circuit impedance matrix of the three terminal-pair device and L is given by Eq. 2.2.2. It is clear that if the black box of Fig. 2.2.1b were embedded in some linear network we would not disturb the operation of this network if the black box were replaced by the tree of mutually coupled coils of Fig. 2.2.2a.

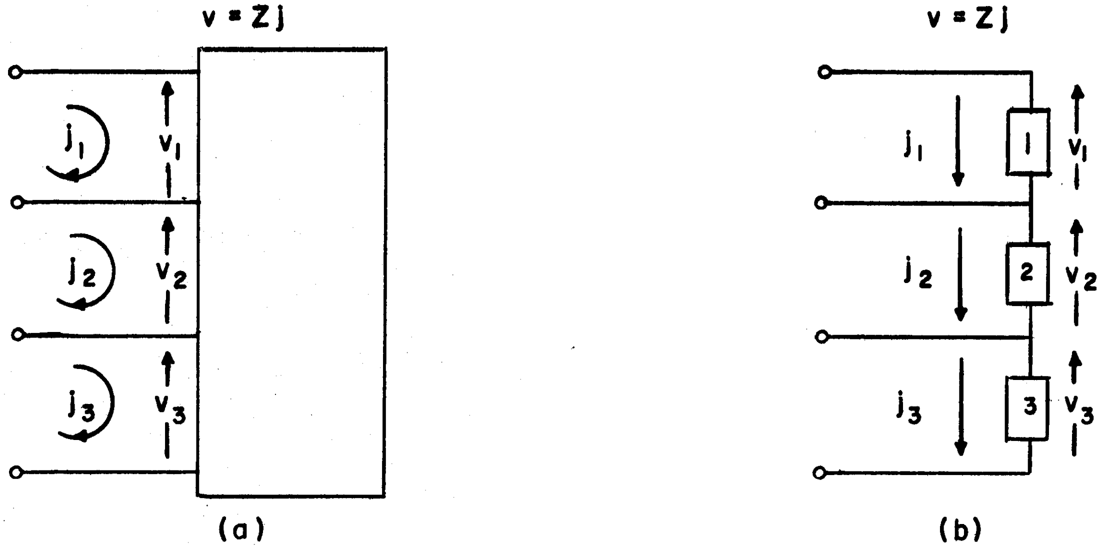


Fig. 2.2.2 Representation of Three Terminal-Pair Device By Three Mutually Coupled Branches

In making this replacement we need only be sure that each coil replaces the correct terminal pair. Then the current through coil s (s = 1,2,3) and the voltage across coil s become identical to the current circulating on terminal pair n and the voltage across terminal pair n respectively.

Generalizing the above ideas, it is proposed that the three terminal-pair device of Fig. 2.2.2a with o.c. impedance matrix

$$Z = \begin{bmatrix} z_{11} z_{12} z_{13} \\ z_{21} z_{22} z_{23} \\ z_{31} z_{32} z_{33} \end{bmatrix} \tag{2.2.7}$$

(where z_{jk} does not necessarily equal z_{kj}) may be replaced in any network by the three coupled branches of Fig. 2.2.2b. The branch impedance parameter matrix of these coupled branches is just the matrix Z. It may be worthwhile to summarize the general procedure followed in replacing the multiterminal-pair element by a group of coupled branches:

1. A branch is created across each terminal-pair of the MTP (multiterminal-pair) element such that branch s is across the s^{th} terminal pair.

2. The voltage and current of the s^{th} branch are defined to be identical to the voltage across the s^{th} terminal pair and the current circulating on the s^{th} terminal-pair, respectively.

3. The MTP network is removed leaving a set of mutually coupled branches with branch impedance parameter matrix defined identical to the open circuit impedance matrix of the MTP network.

A black box with a number of terminals sticking out has been called an MP (multipole) network[20]. By pairing the terminals or nodes of a given MP network one may generate a large number of MTP networks. If the MP network has n + 1 nodes one may select at most n independent terminal pair voltages. We may readily understand this fact from the branch representation of an MTP network, since it is well known that a network containing n + 1 nodes has at most n independent branch voltages which moreover are those belonging to a tree of branches. Thus if an MTP network contains n terminal pairs and n + 1 nodes, the set of mutually coupled branches representing it will form a tree. If the number of nodes is greater than n + 1 the branches will form a group of isolated trees. It will be convenient to regard an MTP network as being formed from an associated MP network which is physically the same device as the MTP network but which has no defined terminal pairs.

The following two definitions will be used:

1. The associated MP network will be said to be <u>completely described</u> by the MTP network if the equivalent branches of the MTP form a single tree.

2. The associated MP network will be said to be <u>partially described</u> by the MTP network if the equivalent branches of the MTP form more than one isolated tree.

It will be recalled that these MTP network elements are to be interconnected in an arbitrary fashion to form a larger network. In general, the final network can be solved only if the MTP elements completely describe their associated MP networks. This should be obvious since if an MTP element only partially describes its associated MP network, there are undefined terminal pairs which may be excited when the MTP element is connected into a larger network.

A common example of an MTP element which partially describes its associated MP network is the ungrounded two terminal-pair as shown in Fig. 2.2.3 with its coupled branch representation.

Fig. 2.2.3 An MTP Element Which Provides A Partial Description of It's Associated MP Network

Unless specifically stated to the contrary it will be assumed in the subsequent discussions that the MTP elements dealt with completely describe their associated MP networks.

The number of MTP elements that may be formed from a single MP network of n nodes is just equal to the number of different trees that may be constructed to connect all n nodes. This number may be shown to be equal to n^{n-2}. Thus in our example of Fig. 2.2.2 the associated MP network has 4 nodes so that 4^2 or 16 different MTP elements may be formed from it. Two such elements and their coupled tree branch representations are shown in Fig. 2.2.4. For clarity of presentation the positive reference directions are shown only for terminal pair voltages. The arrow on a branch indicates the positive direction of branch voltage.

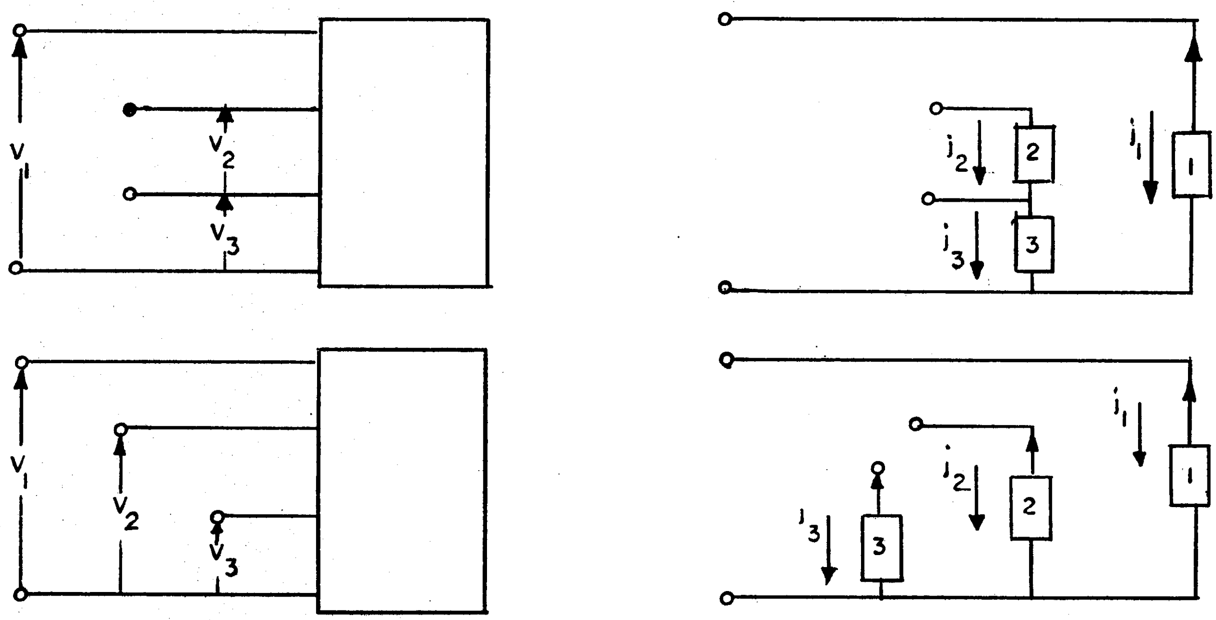


Fig. 2.2.4 Two MTP Elements For the Same Associated MP Network

The question naturally arises as to the relationship between the branch impedance (or admittance) parameters of each of the n^{n-2} MTP elements having the same associated MP network. This question will be answered in Section 2.3.3.

2.3 Formulation of Equilibrium Equations

2.3.1 Systematic Approach

It is presumed that a network is to be analyzed which consists of an interconnection of MTP elements. The first step consists of replacing each MTP element by a tree of coupled branches. When this is done the network becomes composed of two-terminal elements. The selection of an independent set of voltage and current variables proceeds as with LLFPB networks[18]. Let us presume that we are going to formulate equilibrium equations on the loop basis. The network is assumed to have n + 1 nodes, b branches, and ℓ links. By the usual methods a tie set matrix is found which defines an independent set of loop currents. This matrix will be called $\beta_{\ell b}$ with the subscripts ℓb denoting the fact that it has ℓ rows and b columns. Let e_s denote the column matrix of voltage sources in loops

$$e_s = \begin{bmatrix} e_{s1} \\ e_{s2} \\ \cdot \\ \cdot \\ \cdot \\ e_{s\ell} \end{bmatrix} \tag{2.3.1}$$

where, e.g., e_{s1} is the sum of the voltage sources on loop 1, etc. Also let i represent the column vector of resulting loop currents,

$$i = \begin{bmatrix} i_1 \\ i_2 \\ \cdot \\ \cdot \\ \cdot \\ i_\ell \end{bmatrix} \tag{2.3.2}$$

Let us assume that the network consists of an interconnection of m MTP elements. It will also be assumed throughout that these MTP elements are distinct from one another in the sense that there is no coupling between the branches of one MTP element and any other. The MTP elements are ordered in some convenient way by labeling them with the integers 1,2,···m. Then the branches of MTP element 1 are numbered in order. Following this the branches of MTP element 2 are numbered and so forth until all the branches of the network have been labeled. The column matrix of branch currents j and the column matrix of branch voltages v may then be represented in the partitioned form

$$j = \begin{bmatrix} J_1 \\ J_2 \\ \cdot \\ \cdot \\ \cdot \\ J_m \end{bmatrix} \quad v = \begin{bmatrix} V_1 \\ V_2 \\ \cdot \\ \cdot \\ \cdot \\ V_m \end{bmatrix} \tag{2.3.3}$$

where J_k, V_k are column matrices representing the branch currents and voltages of the k^{th} MTP element. The relationship between v and j is then

$$v = Ij \tag{2.3.4}$$

where the branch parameter matrix I is given by

$$I = \begin{bmatrix} z_1 & 0 & 0 & \cdot \\ 0 & z_2 & 0 & \cdot \\ 0 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & z_m \end{bmatrix} \tag{2.3.5}$$

The matrix z_k is the branch parameter matrix of MTP element k, i.e.,

$$V_k = z_k J_k \tag{2.3.6}$$

The zeroes appearing in I are null matrices.

Now the loop source voltage matrix e_s is related to the branch voltage matrix v by

$$\beta_{\ell b} v = e_s \tag{2.3.7}$$

which is a statement of Kirchoffs Voltage Law, while the branch current matrix j is related to the loop current matrix i by

$$\beta^t_{\mathcal{l}b} i = j \tag{2.3.8}$$

where $\beta_{\mathcal{l}b}$ is the tie set matrix and $\beta^t_{\mathcal{l}b}$ is its transpose.* Equations 2.3.7 and 2.3.8 are of course identical to those used in LLFPB network analysis. So far the only analytical difference noticeable between setting up equilibrium equations for LLFPB networks and for the class of networks under study is the character of the branch parameter matrix, I. In LLFPB analysis this customarily takes the form

$$I = \begin{bmatrix} r & 0 & 0 \\ 0 & [sc]^{-1} & 0 \\ 0 & 0 & s\mathcal{l} \end{bmatrix} \tag{2.3.9}$$

where r is a real non-singular diagonal matrix (the branch resistance matrix), c^{-1} is a real non-singular diagonal matrix (the branch elastance matrix), and $\mathcal{l}$ is a real square non-singular symmetrical matrix (the branch inductance matrix). Each of these matrices defines a positive definite quadratic form. Both Equations 2.3.5 and 2.3.9 are of the same form but our submatrices z_1, z_2, - z_m are of a more general character. They may be unsymmetrical and both rational and irrational functions of s. It is only assumed that they are non-singular.

The formulation of equilibrium equations now proceeds just as in the LLFPB case. The expression for the branch voltages in terms of the branch currents Eq. 2.3.4, is used in Kirchoffs Voltage Law Equations, Eq. 2.3.7. This yields an expression relating source voltages and branch currents as follows

* We are assuming that the consistency conditions are fulfilled, i.e. that the same tie set matrix is used for defining loop current variables as is used for writing Kirchoffs Voltage Law. See Ref. (18) page 79.

$$\beta_{\ell b} I j = e_s \tag{2.3.10}$$

Subsequent use of Eq. 2.3.8 which expresses the branch currents in terms of the loop currents yields

$$e_s = \beta_{\ell b} I \beta_{\ell b}^t i = Z_{\ell \ell} i \tag{2.3.11}$$

where

$$Z_{\ell \ell} = \beta_{\ell b} I \beta_{\ell b}^t \tag{2.3.12}$$

is the equilibrium matrix for the network on the loop basis. The loop currents are solved for by inverting $Z_{\ell \ell}$ in the usual fashion. We have used the subscript $\ell \ell$ to denote that it is an $\ell \times \ell$ matrix.

$$i = Z_{\ell \ell}^{-1} e_s = Y_{\ell \ell} e_s \tag{2.3.13}$$

where

$$Y_{\ell \ell} = Z_{\ell \ell}^{-1} \tag{2.3.14}$$

is appropriately called the short circuit admittance matrix of the network.

If it is desired to see how the branch parameter matrices of the individual MTP elements enter into the formation of Z one may proceed as follows. Partition the tie set matrix in the following way

$$\beta_{\ell b} = \left[\beta_1 \middle| \beta_2 \middle| \cdots \middle| \beta_m \right] \tag{2.3.15}$$

The matrix β_k has ℓ rows and as many columns as the k^{th} MTP element has branches. If the expressions for $\beta_{\ell b}$ and I as given by Eq's. 2.3.15 and 2.3.5 respectively are used in Eq. 2.3.12 one finds that the equilibrium matrix Z takes the form

$$Z_{\ell\ell} = \beta_1 z_1 \beta_1^t + \beta_2 z_2 \beta_2^t + \cdots + \beta_m z_m \beta_m^t = \sum_{k=1}^{m} \beta_k z_k \beta_k^t \qquad (2.3.16)$$

If we define

$$Z_k = \beta_k z_k \beta_k^t \qquad (2.3.17)$$

then $Z_{\ell\ell}$ takes the simple form

$$Z_{\ell\ell} = Z_1 + Z_2 + \cdots Z_m = \sum_{1}^{m} Z_k \qquad (2.3.18)$$

It is clear that the matrices Z_k are the analogue of the loop parameter matrices in LLFPB analysis. In fact we will call Z_k the loop parameter matrix of the k^{th} MTP element. Equation 2.3.18 states that the equilibrium matrix of the network on the loop basis is the sum of the loop parameter matrices of the individual MTP elements.

To illustrate the above ideas an example will be given.

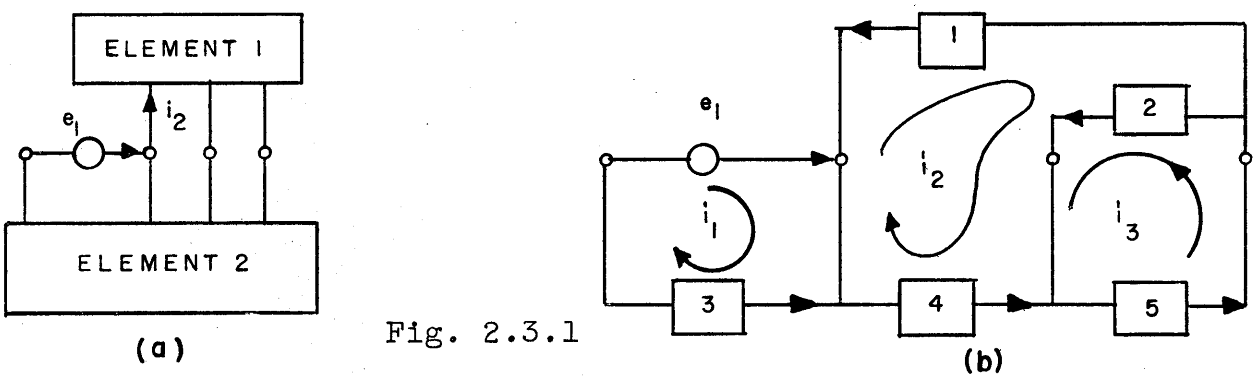


Fig. 2.3.1

Figure 2.3.1a shows a network consisting of an interconnection of two MTP elements. A voltage source e_1 is applied and it is desired to determine the current i_2. The definition of terminal pairs does not appear on this figure but is clear from inspection of Figure 2.3.1b in which each MTP element has been replaced by a set of mutually coupled branches. A set of independent loop currents is indicated on this latter figure. Branches 1 and 2 belong to MTP element 1 and branches 3, 4, and 5 belong to MTP element 2. Inspection of Fig. 2.3.1b shows that the tie set matrix is

$$\beta_{\ell b} = \left[\begin{array}{cc|ccc} 0 & 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & -1 \end{array}\right] \tag{2.3.19}$$

The matrices β_1 and β_2 which are obtained by partitioning $\beta_{\ell b}$ as discussed above are given by

$$\beta_1 = \begin{bmatrix} 0 & 0 \\ 1 & -1 \\ 0 & -1 \end{bmatrix} \quad \beta_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \tag{2.3.20}$$

The branch impedance parameter matrices of MTP elements 1 and 2 are defined below

$$z_1 = \begin{bmatrix} z_{11} z_{12} \\ z_{21} z_{22} \end{bmatrix} \quad ; \quad z_2 = \begin{bmatrix} z_{33} z_{34} z_{35} \\ z_{43} z_{44} z_{45} \\ z_{53} z_{54} z_{55} \end{bmatrix} \tag{2.3.21}$$

Equation 2.3.17 may be used now to calculate the loop parameter matrices

$$Z_1 = \beta_1 z_1 \beta_1^t = \begin{bmatrix} 0 & 0 & 0 \\ 0 & z_{11}+z_{22}-z_{12}-z_{21} & z_{22}-z_{12} \\ 0 & z_{22}-z_{21} & z_{22} \end{bmatrix} \tag{2.3.22}$$

$$Z_2 = \beta_2 z_2 \beta_2^t = \begin{bmatrix} z_{33} & z_{34} & -z_{35} \\ z_{43} & z_{44} & -z_{45} \\ -z_{54} & -z_{54} & z_{55} \end{bmatrix}$$

Then the equilibrium matrix Z is

$$Z = Z_1 + Z_2 = \begin{bmatrix} z_{33} & z_{34} & -z_{35} \\ z_{43} & z_{44}+z_{11}+z_{22}-z_{12}-z_{21} & z_{22}-z_{12}-z_{45} \\ -z_{53} & z_{22}-z_{21}-z_{54} & z_{22}+z_{55} \end{bmatrix} \tag{2.3.23}$$

The rest of the solution is straightforward from this point on and needs no further discussion.

The process of formulating equilibrium equations on the node basis is dual to the procedure discussed above for formulating equilibrium equations on the loop basis and it is felt does not need any extensive elaboration. However the pertinent equations will be summarized. First we define the quantities

$$i_s = \begin{bmatrix} i_{s1} \\ i_{s2} \\ \vdots \\ i_{sn} \end{bmatrix} \quad e = \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{bmatrix} \tag{2.3.24}$$

where i_s is the column matrix of source currents applied at n independent node pairs and e is the column matrix of n resulting node pair voltages. The branch voltages v and branch currents j are defined by Eq. 2.3.3. However the relationship between v and j is specified by the branch admittance matrix A

$$A = I^{-1} = \begin{bmatrix} z_1^{-1} & 0 & 0 & & \vdots \\ 0 & z_2^{-1} & 0 & & \vdots \\ 0 & & & & 0 \\ \cdot & & & & \\ 0 & \cdots & & 0 & z_m^{-1} \end{bmatrix} = \begin{bmatrix} y_1 & 0 & 0 & & \vdots \\ 0 & y_2 & 0 & & \vdots \\ 0 & & & & 0 \\ \cdot & & & & \\ 0 & \cdots & & 0 & y_m \end{bmatrix} \qquad (2.3.25)$$

where

$$y_k = z_k^{-1} \qquad (2.3.26)$$

is the branch admittance matrix of the k^{th} MTP element. Thus

$$j = Av \qquad (2.3.27)$$

It is presumed that a cut set matrix α_{nb} has been selected both for defining node-pair voltages and for writing Kirchoff current equations. This matrix has n rows and b columns. The equations analogous to Eq's. 2.3.7 and 2.3.8 are

$$\alpha_{nb} j = i_s$$
$$\alpha_{nb}^t e = v \qquad (2.3.28)$$

and following the same pattern as for Eq. 2.3.12 we find that the equilibrium equations on the node basis become

$$i_s = Y_{nn} e \qquad (2.3.29)$$

where

$$Y_{nn} = \alpha_{nb} A \alpha_{nb}^t \qquad (2.3.30)$$

By partitioning α_{nb} in the same way as $\beta_{\ell b}$ was we find that

$$Y_{nn} = Y_1 + Y_2 + \cdots Y_m = \sum_{1}^{n} Y_k \tag{2.3.31}$$

where

$$Y_k = \alpha_k y_k \alpha_k^t \tag{2.3.32}$$

is the node parameter matrix of the k^{th} MTP element and

$$\alpha_{nb} = \left[\alpha_1 \,\vdots\, \alpha_2 \,\vdots\, \cdots \,\vdots\, \alpha_n\right] \tag{2.3.33}$$

indicates the partitioning of the cut set matrix. With the open circuit impedance matrix defined as

$$Z_{nn} = Y_{nn}^{-1} \tag{2.3.34}$$

we obtain

$$e = Z_{nn} i_s \tag{2.3.35}$$

as the solution for the set of Eq. 2.3.29.

2.3.2 Formulation By Inspection

The previous section has presented a systematic approach to the formulation of equilibrium equations on the node or loop basis for a network consisting of an interconnection of MTP elements. It is possible to bypass completely all the matrix manipulations required by such a systematic formulation and write down the equilibrium equations by inspection. The procedure followed is identical to that followed when writing equilibrium equations by inspection on the node or loop basis for LLFPB networks containing mutual inductance. There is one new twist which however causes no additional difficulty: the coupling between branches is not necessarily bilateral as for mutually coupled inductances. The procedure is best understood by example. Thus let us write loop equations for the network of Fig. 2.3.1b. We first consider $i_2 = i_3 = 0$ and add up the voltage drops in the direction of positive current for loop 1. This contribution is $i_1 z_{33}$. Next we consider $i_1 = 0$, $i_3 = 0$ and find the voltage drops contributed in loop 1 by i_2. We note that this is just $i_2 z_{34}$. The last subscript 4 denotes the branch, 4, which is inducing voltage into the loop under consideration and the first subscript 3 denotes the branch in the loop which has voltage induced across it by virtue of a coupling from branch 4. The sign of the contribution is readily determined with the aid of the positive reference directions for branch voltage and currents and positive reference directions for loop currents. Note that i_2 passes through branch 4 in the direction of positive branch 4 current (always opposite to the positive direction of branch voltage). Thus the voltage induced across branch 3 is by definition in the direction of the arrow on branch 3. Now let $i_1 = 0$, $i_2 = 0$ and find the voltage induced in loop 1 by loop current i_3. We note that voltage will be induced across branch 3 since branch 5 is coupled to branch 3. However i_3 is in a direction opposite to the positive direction of j_5. Thus the voltage induced in branch 3 will be of a polarity opposite to that indicated by the branch 3 arrow. The contribution to loop 1 is thus $- i_3 z_{35}$ and the first loop equation reads

$$e_1 = i_1 z_{33} + i_2 z_{34} - i_3 z_{35} \tag{2.3.36}$$

The other equations may be written by inspection in the same fashion. Of course if the network is very complicated the systematic formulation of the previous section may be more advisable to use. One may similarly write equations on the node basis by inspection. In theory this is no more difficult and follows an entirely dual pattern. It may be expected that most engineers will need to acquire some practice in formulating equations on the node basis since few people have had practice in writing equilibrium equations on the node basis with mutual inductance present.

2.3.3 Different Representation of MP* Network

As discussed in Section 2.2 one may construct from a given MP network of n nodes, n^{n-2} different MTP* network elements. We may determine the relationship between the branch parameter matrices of two MTP elements having the same associated MP network in the following way. Let MTP element 1 have branch impedance parameter matrix z_1. The definition of terminal-pair voltages for MTP element 2 is given and it is desired to find the relationship between its branch impedance parameter matrix z_2 and that for MTP element 2, represent MTP element 1 as a tree of mutually coupled branches. Excite this network with voltage sources placed and numbered to coincide with the terminal-pair voltages defined for MTP element 2. It is readily seen that the equilibrium matrix which relates this set of source voltages and the resulting response currents is just z_2,

* It will be recalled that an MP (multipole) network is one which is only accessible at a set of nodes. Thus an MP network may be represented as a box with a set of nodes extruding. An MTP (multiterminal-pair) network is an MP network with a set of terminal-pairs assigned.

the branch impedance parameter matrix of MTP element 2. We may apply Eq. 2.3.12 where we identify

$$I = z_1$$

$$Z_{\ell\ell} = z_2$$

$$\beta_{\ell b} = \beta \tag{2.3.37}$$

Here β is the tie set which defines loop currents on MTP element 1 that have been created by inserting the n terminal-pair voltage sources of MTP element 2. Each source inserts a link into MTP element 1 and it is assumed that loop currents have been identified with link currents. The β matrix is clearly square and n x n since there are n loop currents and MTP element 1 has n branches. Moreover it is clearly a nonsingular matrix. Thus we have, using the definitions 2.3.37 in Eq. 2.3.12*

$$z_2 = \beta z_1 \beta^t \tag{2.3.38}$$

Now a dual analysis on the admittance basis would show that*

$$y_2 = \alpha y_1 \alpha^t \tag{2.3.39}$$

where y_k is the branch admittance matrix of the k^{th} MTP network ($k = 1,2$) and α is the cut set matrix used to define the node-pair voltages of MTP element 2 on the tree representation of MTP element 1. Here α is an n x n non-singular matrix. By inverting Eq. 2.3.38 we obtain the following relationship between α and β,

$$\alpha = (\beta^t)^{-1} \tag{2.3.40}$$

i.e. α and β are inverse transposes of one another.

---

* **Assuming the consistency conditions are applicable.**

It is easy to show that the determinants α and β are equal and have the magnitude unity. From Eq. 2.3.40 we obtain

$$\det \alpha = \frac{1}{\det \beta} \tag{2.3.41}$$

But since α and β both contain as elements + 1's or - 1's, their determinants must be integers. The only integers which will satisfy Equation 2.3.41 are +1 and - 1.

An illustrative example will now be given.

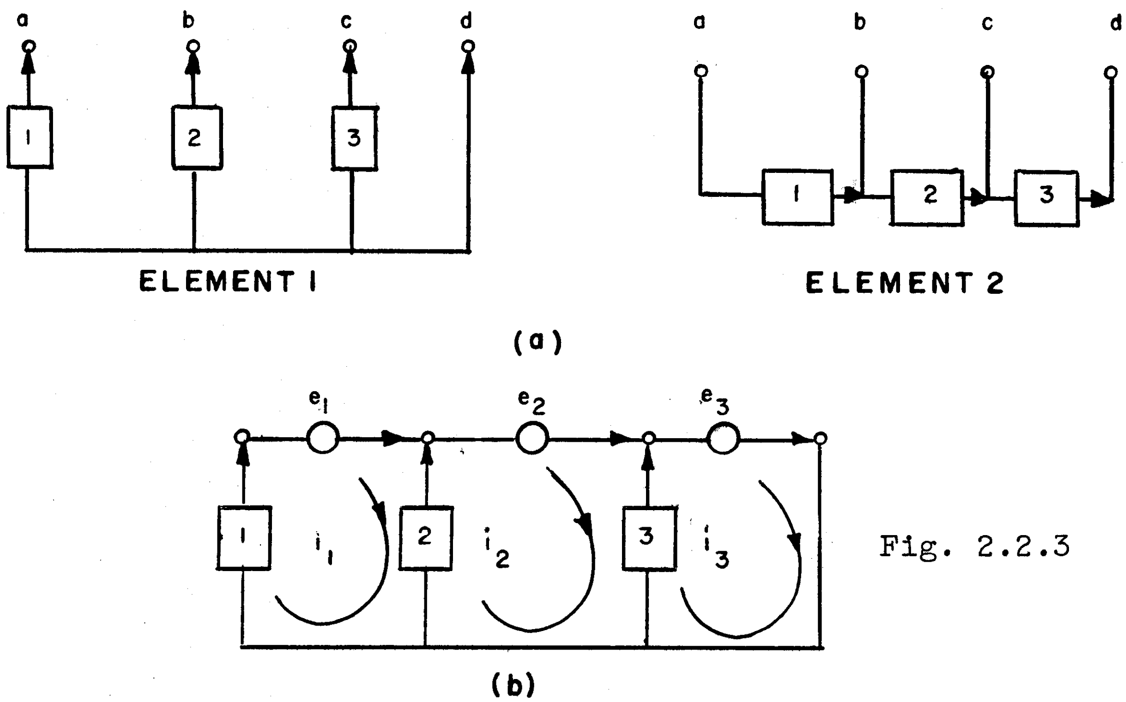


Fig. 2.2.3

Fig. 2.3.2a shows the coupled tree branch representations of MTP element 1 and MTP element 2 with both having the same associated MP network. Suppose that the branch impedance parameter matrix of element 1 is given as

$$z_1 = \begin{bmatrix} z_{11} z_{12} z_{13} \\ z_{21} z_{22} z_{23} \\ z_{31} z_{32} z_{33} \end{bmatrix} \tag{2.3.42}$$

It is desired to determine z_2. In Fig. 2.3.2b, element 1 is excited by three voltage sources with polarity and location to correspond to the terminal pair voltages of element 2. By inspection of this figure we determine that the tie set matrix is

$$\beta = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix} \tag{2.3.43}$$

Thus

$$z_2 = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} z_{11} & z_{12} & z_{13} \\ z_{21} & z_{22} & z_{23} \\ z_{31} & z_{32} & z_{33} \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix} \tag{2.3.44}$$

These matrix products are readily formed yielding

$$z_2 = \begin{bmatrix} z_{11}+z_{22}-z_{12}-z_{21} & z_{23}+z_{12}-z_{13}-z_{22} & z_{13}-z_{23} \\ z_{32}+z_{21}-z_{31}-z_{22} & z_{33}+z_{22}-z_{32}-z_{23} & z_{23}-z_{33} \\ z_{31}-z_{32} & z_{32}-z_{33} & z_{33} \end{bmatrix} \tag{2.3.45}$$

As discussed in Section 2.2, when the associated MP network is only partially described by an MTP element, the branch representation of the MTP element becomes not one tree but a group of isolated trees. If we generate new MTP elements by redefining terminal pairs separately on these isolated trees, then the arguments above may be applied directly to determine the relationship between the parameter matrices of these MTP elements. It will be noted that in the case of a partially described MP network the β matrix may be partitioned into a number of submatrices in the manner indicated for I, Eq. 2.3.5, since loop currents may belong to only one of the isolated trees representing MTP element 1.

2.4 Evaluation of Equilibrium Matrix by Matrix Addition

2.4.1 Impedance and Admittance Matrices

For some special situations one may evaluate the equilibrium matrix by "parameter matrix addition" of the MTP elements. The reason for quotes is that it is not generally the parameter matrices which are added but rather parameter matrices or their principal submatrices* which have been properly augmented by addition of an equal number of rows and columns of zeroes. Whenever the conditions are present for matrix addition one may of course obtain the equilibrium equations very quickly. Examination of Eq's. 2.3.18 and 2.3.31 show that the equilibrium matrix on the loop and node basis is always obtainable by summing directly the loop parameter and node parameter matrices respectively of the component MTP elements. Let us repeat the equation for the loop parameter matrix of the k^{th} MTP element.

$$Z_k = \beta_k z_k \beta_k^t \tag{2.4.1}$$

We recall that z_k is the branch impedance matrix of the k^{th} element and β_k is that submatrix of the tie set matrix $\beta_{\ell n}$ which tells how the loop currents traverse the branches of the k^{th} MTP element. A zero indicates that a branch is not traversed. A + 1 indicates

* A principal submatrix P of a square matrix S is any square matrix formed from S by striking out rows and corresponding columns.

that a branch has been traversed in the direction of positive branch current and a - 1 indicates a traversal in a direction of negative branch current.

When the matrix Z_k (for all k) is either equal to z_k, to a principal submatrix of z_k, or to versions of these which are augmented by rows and columns of zeroes then the conditions will be said to exist whereby the equilibrium matrix may be evaluated by the addition of component element parameter matrices. A loop parameter matrix Z_k which has any one of the four forms indicated above will be called a Simple loop parameter matrix. It is readily seen from Eq. 2.4.1 that the necessary and sufficient condition for Z_k to be Simple (for general z_k) is that the matrix resulting when all null columns and rows have been removed from β_k be either the unit matrix or the negative of the unit matrix. In terms of the actual network variables this may be stated as follows. The loop parameter matrix of the k^{th} MTP element will be Simple if the following three conditions are satisfied.

(1) Each tie set contains (or equivalently each loop current traverses) at most one branch of the MTP element k. Each branch of MTP element k is contained in only one tie set.

(2) The set of tie sets, T_k, which do contain a set of branches, β_k, of MTP element k, all contain the branches in the same algebraic sense. (This is equivalent to the statement that β_k has elements of the same algebraic sign).

(3) The numbering of the elements of T_k is in the same order as the corresponding elements of β_k. That is, if we arrange the numbering of the loop currents defined by T_k in ascending order, then the corresponding branches which they traverse are also numbered in ascending order. Condition (3) is meaningful only if condition (1) is satisfied since only in this case will there be a one-one correspondence between elements of T_k and elements of β_k.

It is readily seen from item (3) that many loop parameter matrices may be made Simple by just renumbering the branches in β_k.

It will be assumed in any subsequent discussion that such renumbering, if applicable, has been carried out. This renumbering simply corresponds to an interchange of columns of β_k to convert it into a diagonal matrix.

All the above statements and definitions can be carried over in dual form to discuss the conditions under which the node equilibrium matrix may be formed by "node parameter matrix" addition. Only the final statements dual to (1), (2), and (3) will be given. The node parameter matrix of the k^{th} MTP element will be Simple if the following three conditions are satisfied.

(1) Each cut-set contains at most one branch of MTP element k. Each branch of MTP element k is contained in only one cut-set.

(2) The set of cut-sets, C_k, which do contain a set of branches, A_k, of MTP element k all contain these branches in the same algebraic sense. (This is equivalent to the statement that α_k has elements of the same algebraic sign).

(3) The numbering of the elements of C_k is in the same order as the corresponding elements of A_k. Condition (3) is meaningful only if Condition (1) is satisfied since only in this case will there be a one-one correspondence between elements of C_k and elements of A_k.

A renumbering of branches in β_k will sometimes allow Condition (3) to be satisfied. This corresponds to interchanging columns of α_k.

It should be noted that if all the MTP elements have node to datum terminal pairs with a common datum then the conditions for formulation of the node equilibrium matrix by node paramater matrix addition are satisfied. However it should be clear that the node to datum assignment with common datum is not a necessary condition for parameter matrix addition.

Two examples will now be given to illustrate the formulation of equilibrium matrices by addition of branch parameter matrices.

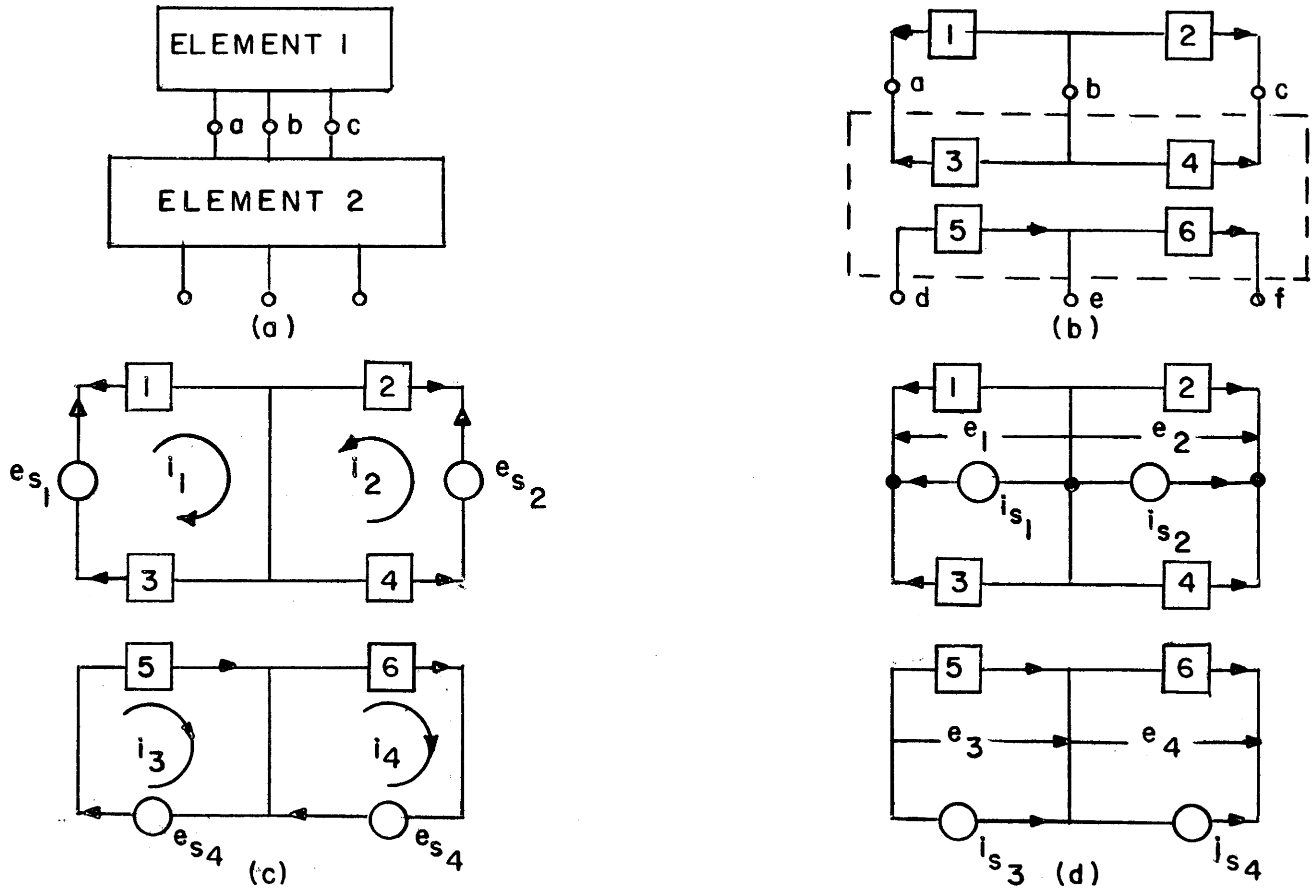


Fig. 2.4.1 Example Illustrating Addition Of Parameter Matrices

In Fig. 2.4.1a a network is shown that consists of an interconnection of two MTP elements. Its equivalent branch representation is shown in Fig. 2.4.1b. It will be noticed that MTP element 2 only partially describes its associated MP network. However this will cause no difficulty so long as the complete network is excited in such a way that the two trees of MTP element 2 are never connected. In Fig. 2.4.1c, the network of Fig. 2.4.1a is shown with 4 voltage sources applied in 4 loops. Inspection of this figure shows that the loop parameter matrices of MTP elements 1 and 2 are Simple. Thus the equilibrium matrix $Z_{\ell\ell}$ relating the loop currents i_1, i_2, i_3, and i_4

to the source voltages e_{s1}, e_{s2}, e_{s3}, e_{s4} may be evaluated by simple addition. To indicate this fact compactly let z_1 be the branch impedance parameter matrix of element 1 and let z_2, the branch impedance parameter matrix of element 2 be partitioned as follows

$$z_2 = \left[\begin{array}{c|c} z_a & z_b \\ \hline z_c & z_d \end{array}\right] \tag{2.4.2}$$

where

$$z_a = \begin{bmatrix} z_{33} z_{34} \\ z_{43} z_{44} \end{bmatrix}, \quad z_b = \begin{bmatrix} z_{35} z_{36} \\ z_{45} z_{46} \end{bmatrix}$$

$$z_c = \begin{bmatrix} z_{53} z_{54} \\ z_{63} z_{64} \end{bmatrix}, \quad z_d = \begin{bmatrix} z_{55} z_{56} \\ z_{65} z_{66} \end{bmatrix} \tag{2.4.3}$$

It is readily seen that

$$Z_1 = \left[\begin{array}{c|c} z_1 & 0 \\ \hline 0 & 0 \end{array}\right] \quad Z_2 = \begin{bmatrix} z_a z_b \\ z_c z_d \end{bmatrix} \tag{2.4.4}$$

so that the loop equilibrium matrix $Z_{\ell\ell}$ is

$$Z_{\ell\ell} = Z_1 + Z_2 = \left[\begin{array}{c|c} z_a + z_1 & z_b \\ \hline z_c & z_d \end{array}\right] \tag{2.4.5}$$

In Fig. 2.4.1b the network of Fig. 2.4.1a is excited by 4 current sources applied at 4 independent node pairs. Inspection of this figure shows that the node parameter matrices of elements 1 and 2 are simple. Thus the equilibrium matrix Y_{nn} relating the node pair voltages e_1, e_2, e_3, e_4 to the source currents i_{s1}, i_{s2}, i_{s3}, i_{s4} may be evaluated by simple addition. Proceeding in the same fashion as for the preceding example let y_1, y_2 be the branch admittance parameter matrices for MTP elements 1 and 2. Let y_2 be partitioned as follows

$$y_2 = \begin{bmatrix} y_a y_b \\ \\ y_c y_d \end{bmatrix} \tag{2.4.6}$$

where

$$y_a = \begin{bmatrix} y_{33} y_{34} \\ \\ y_{43} y_{44} \end{bmatrix} \quad y_b = \begin{bmatrix} y_{35} y_{36} \\ \\ y_{45} y_{46} \end{bmatrix}$$

$$y_c = \begin{bmatrix} y_{53} y_{54} \\ \\ y_{63} y_{64} \end{bmatrix} \quad y_c = \begin{bmatrix} y_{55} y_{56} \\ \\ y_{65} y_{66} \end{bmatrix} \tag{2.4.7}$$

It is readily seen that the node parameter matrices are

$$Y_1 = \left[\begin{array}{c|c} y_1 & 0 \\ \hline 0 & 0 \end{array}\right] \quad Y_2 = \left[\begin{array}{c|c} y_a & y_b \\ \hline y_c & y_d \end{array}\right] \tag{2.4.8}$$

so that the node equilibrium matrix Y_{nn} is given by

$$Y_{nn} = Y_1 + Y_2 = \left[\begin{array}{c|c} y_a + y_1 & y_b \\ \hline y_c & y_d \end{array}\right] \tag{2.4.9}$$

We see that the network of Fig. 2.4.1a has the following interesting attribute. It is possible to define loop currents and node pair voltages such that both the node and loop parameter matrices of the MTP elements are not only Simple but are of the same form for the same MTP element. Note also that MTP element 1 has node and loop parameter matrices which differ from the corresponding branch parameter matrices only in an augmentation by rows and columns of zeroes.

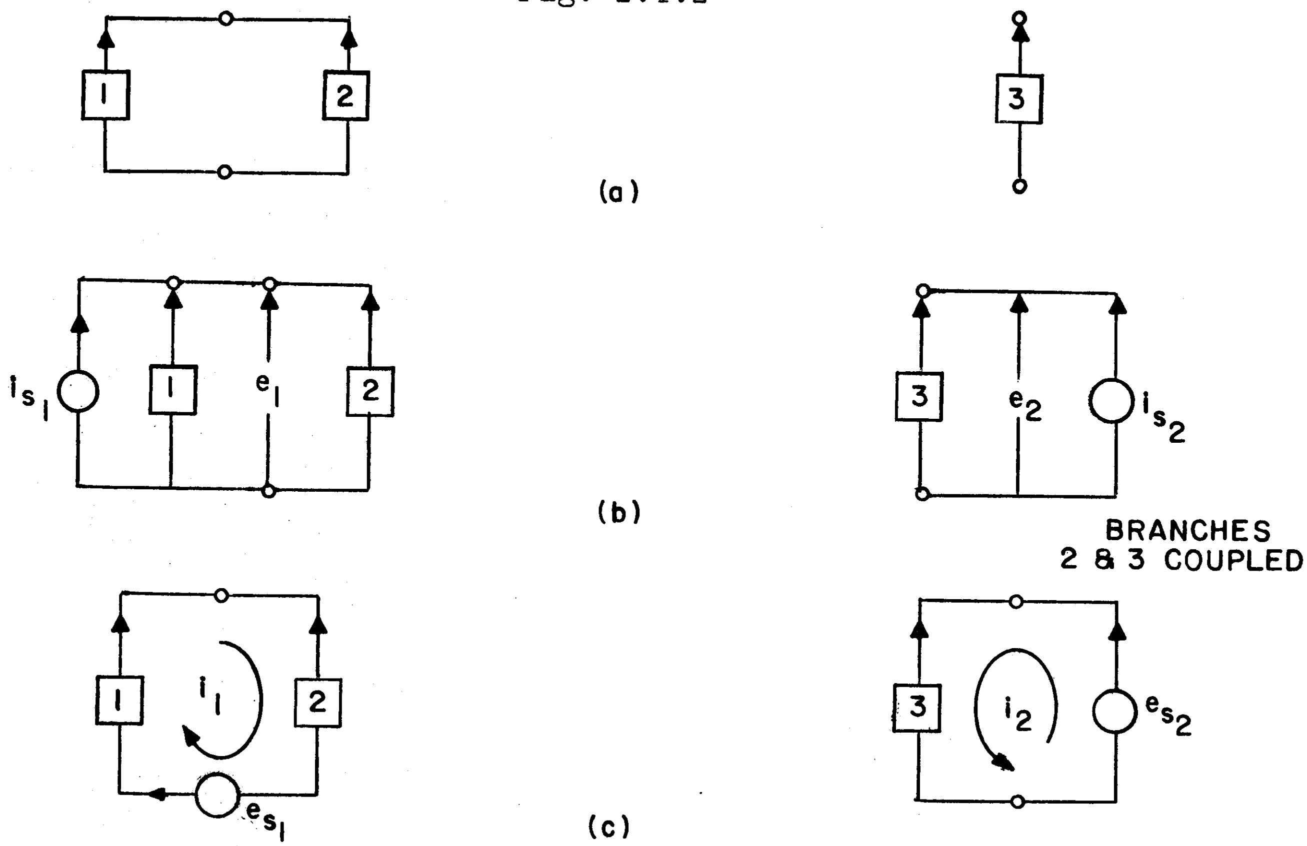
The following definitions will be helpful in subsequent discussions. A Simple loop parameter or node parameter matrix will be called Complete if it is identical to the corresponding branch parameter matrix or differs only by an augmentation with null rows and columns. A network composed of MTP elements will be called an Additive network if it is possible to define loop currents and node pair voltages such that the node and loop parameter matrices of each MTP element are Complete, Simple, and of the same form. The network of Fig. 2.4.1a is an example of an Additive network. It is not difficult to see that the necessary and sufficient conditions for a network to be Additive are that its coupled branch representation be such that

(1) All loops contain one or two branches.

(2) If there are two branches, these branches belong to different MTP elements.

Thus the branches of an Additive network may be divided into two classes: those which are paralleled with another branch and those which are not (isolated branches). If it is desired to have the admittance equilibrium matrix equal to the sum of the branch admittance parameter matrices of the MTP elements then current sources are placed across branches. If it is desired to have the impedance equilibrium matrix equal to the sum of the branch impedance parameter matrices of the MTP elements, voltage sources are placed in loops formed by the paralleled branches and are placed in parallel with the isolated branches. Figure 2.4.2 illustrates the above described manner of exciting an additive network. In Fig. 2.4.2a

Fig. 2.4.2



there is shown a pair of typical paralleled branches and an isolated branch of an additive network. In Fig. 2.4.2b these are excited by current sources and in Fig. 2.4.2c by voltage sources in the correct fashion to make the equilibrium matrices equal to the sum of parameter matrices.

Additive networks have a further interesting and useful property which will be discussed in the following section.

2.4.2 Mixed Matrices

A multiterminal-pair network is usually described either by a short circuit admittance matrix or an open circuit impedance matrix. If voltage sources are applied at terminal pairs the equilibrium matrix is the open circuit impedance matrix. The solution matrix is the short circuit admittance matrix - the inverse of the equilibrium matrix. If current sources are applied at terminal pairs the equilibrium matrix is the s.c. admittance matrix and the solution matrix is the o.c. impedance matrix. There are situations under which one may desire to drive some terminal pairs with voltage sources and the rest with current sources. The response variables are then both terminal-pair voltages and currents. A matrix relating such a mixed excitation and response is called a mixed matrix. If there are n terminal pairs then it is clear that there are 2^n matrices which can be defined such that an excitation or response quantity is either a terminal-pair voltage or current but not both. Of course two of these are the conventional o.c. impedance and s.c. admittance matrix so that there are 2^{n-1} mixed matrices. The various m nodes of excitation are shown in Fig. 2.4.3 for a grounded two terminal-pair network.

An MTP element can be characterized with regard to terminal-pair behavior by what might be called a mixed branch parameter matrix. A network which consists of an interconnection of MTP elements and a number of points of entry (i.e. soldering type insertion of current sources of pliers type insertion of voltage sources) is itself an MTP network. We have discussed the formulation of equilibrium

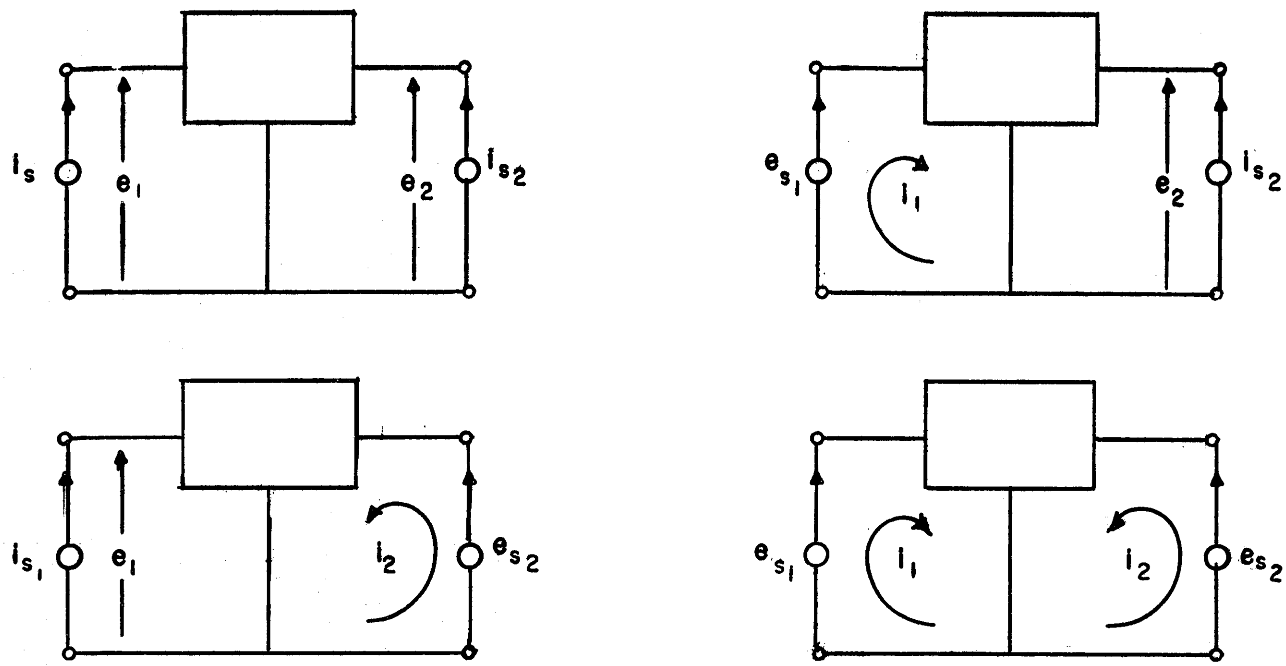


Fig. 2.4.3 Different Ways Of Exciting Two Terminal-Pair Network

equations on the node or loop basis for such a network but such equilibrium equations imply either excitatior by voltage sources in loops or by current sources across node-pairs but not both. Under some conditions it is desirable to formulate equilibrium equations on a mixed basis with some voltage sources and some current sources and with response quantities that are voltages across current sources and currents through voltage source. No detailed discussion of this problem will be given here. Rather a specific situation will be studied which will be of use later on in the thesis. This is the situation in which the network is of the Additive type. We will show that if the network is Additive, mixed equilibrium matrices for the network can, by appropriate choice of loop currents and node pair voltages, be set equal to the sum of mixed branch parameter matrices of the component MTP elements.

Instead of giving a general proof a specific situation will be analyzed and the generalization will be clear to the reader. Consider the network of Fig. 2.4.1 excited as indicated in Fig. 2.4.4 by one current source and three voltage sources.

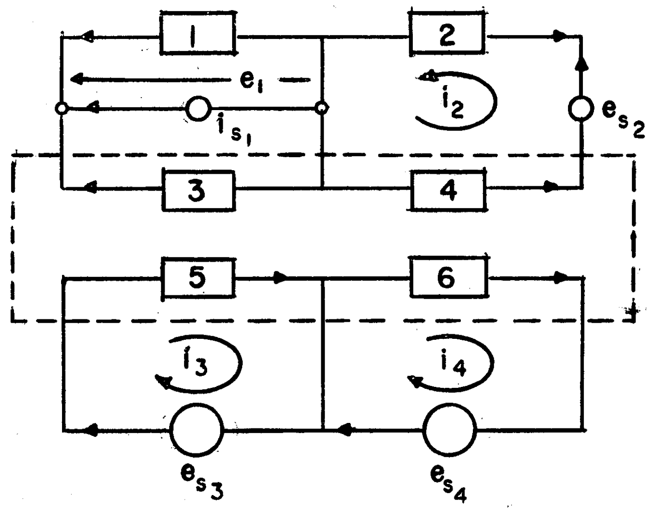


Fig. 2.4.4 Network Of Fig. 2.4.1b Excited With Voltage And Current Sources

The equilibrium equations for this network will be of the form

$$i_{s1} = u_{11}e_1 + b_{12}i_2 + b_{13}i_3 + b_{14}i_4$$

$$e_{22} = a_{21}e_1 - u_{22}i_2 + u_{23}i_3 + u_{24}i_4$$

$$e_{S3} = a_{31}e_1 + u_{32}i_2 + u_{33}i_3 + u_{34}i_4$$

$$e_{s4} = a_{41}e_i + u_{42}i_2 + u_{43}i_3 + u_{44}i_4 \tag{2.4.10}$$

The coefficient a_{j1} is a voltage transfer ratio. It is the ratio of e_{sj} to e_1 with $i_2 = i_3 = i_4 = 0$. The coefficient b_{ij} is a current transfer ratio. It is the ratio of i_{s1} to i_j with $e_1 = i_k = 0$ $(k = j)$. The other coefficients are admittances and impedances. When writing equations on the mixed basis some are applications of Kirchoff's Current Law and some are applications of Kirchoff's Voltage Law. Thus the first equation of 2.4.10 is the result of an application of Kirchoff's Current Law and the last three are applications of Kirchoff's Voltage Law. Let us now write the equilibrium equations for MTP elements 1 and 2. The manner of excitation of these elements is indicated in Fig. 2.4.5.

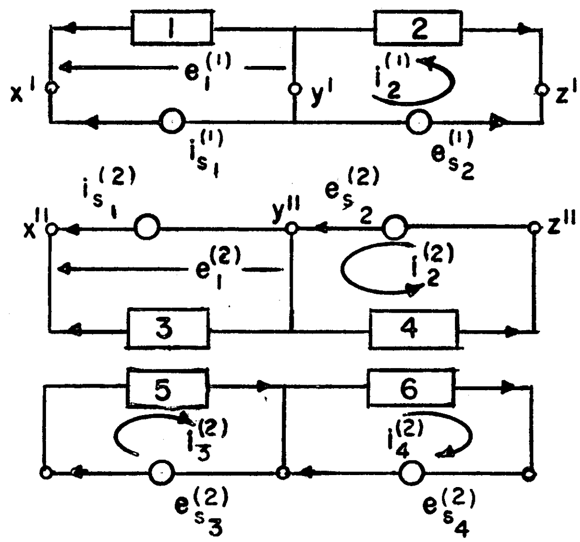


Fig. 2.4.5

The mixed equations for MTP element 1 are

$$
\begin{aligned}
i_{s1}^{(1)} &= u_{11}^{(1)} e_1^{(1)} + b_{12}^{(1)} i_2^{(1)} \\
e_{s2}^{(1)} &= a_{21}^{(1)} e_1^{(1)} + u_{22}^{(1)} i_2^{(1)}
\end{aligned}
\tag{2.4.11}
$$

and those for MTP element 2 are

$$
\begin{aligned}
i_{s1}^{(2)} &= u_{11}^{(2)} e_1^{(2)} + b_{12}^{(2)} i_2^{(2)} + b_{13}^{(2)} i_3^{(2)} + b_{14}^{(2)} i_4^{(2)} \\
e_{s1}^{(2)} &= a_{21}^{(2)} e_1^{(2)} + u_{22}^{(2)} i_2^{(2)} + u_{23}^{(2)} i_3^{(2)} + u_{24}^{(2)} i_4^{(2)} \\
e_{s2}^{(2)} &= a_{31}^{(2)} e_1^{(2)} + u_{32}^{(2)} i_2^{(2)} + u_{33}^{(2)} i_3^{(2)} + u_{34}^{(2)} i_4^{(2)} \\
e_{s3}^{(2)} &= a_{41}^{(2)} e_1^{(2)} + u_{42}^{(2)} i_2^{(2)} + u_{43}^{(2)} i_3^{(2)} + u_{44}^{(2)} i_4^{(2)}
\end{aligned}
\tag{2.4.12}
$$

We adjust the sources until

$$e_1^{(2)} = e_1^{(2)}$$

$$i_2^{(1)} = i_2^{(1)} \qquad (2.4.13)$$

If we now connect the two networks as indicated in Fig. 2.4.4 with

$$i_{s1} = i_s^{(1)} + i_s^{(2)}$$

$$e_{s2} = e_{s2}^{(1)} + e_{s2}^{(2)}$$

$$e_{s3} = e_{s3}^{(2)}$$

$$e_{s4} = e_{s4}^{(2)} \qquad (2.4.14)$$

the operation of the individual elements will be undisturbed since branch voltages and currents will remain the same as before the connection. Using Eq's. 2.4.11 to 2.4.14 we obtain the desired result

$$u_{11} = u_{11}^{(1)} + u_{11}^{(2)} \qquad b_{12} = b_{12}^{(1)} + b_{12}^{(2)}$$

$$a_{21} = a_{21}^{(1)} + a_{21}^{(2)} \qquad u_{22} = u_{22}^{(1)} + u_{22}^{(2)} \qquad (2.4.15)$$

All other coefficients in Eq. 2.4.10 are equal to the corresponding ones in Eq. 2.4.12. It should be noted in closing this section, that the positive reference directions used for source currents and voltages applied to MTP element 2 to define its mixed branch parameter matrix do not coincide with the positive reference directions for its branch voltages and currents. This was necessary in order for Eq. 2.4.15 to hold. If the positive reference directions for the sources are changed then some of the above equations will generally have to be modified by multiplying various coefficients by -1. Assuming that source positive reference directions are defined appropriately we may state the following general conclusion. If for an Additive network source voltages are placed in series with paralleled branches and across isolated branches (see Fig. 2.4.2c) while current sources are placed in parallel with paralleled branches and isolated branches (see Fig. 2.4.2b) then the mixed equilibrium matrix of the network is equal to the sum of the mixed branch parameter matrices of its component MTP elements.

CHAPTER 3

ANALYSIS OF LLF:R NETWORKS BY LINEAR TRANSFORMATION THEORY

3.1 Introduction

In this chapter we will discuss a number of network configurations which allow an LLF:R networks to be analyzed by linear transformation theory such that it may be represented by an LLFPB network and a set of linear transformations relating dynamic variables in the two networks. Since we are confining ourselves to a discussion of LLF:R networks it will always be possible to consider the network under investigation to be composed of MTP elements of two kinds: those that are LLFPB and those that are R-LLF.

As discussed in Section 1.4 of Chapter 1, Guillemin has found a general method of analysis by linear transformation theory which leads to dependent sources in the reference LLFPB networks. This method is presented in Section 3.2. The presentation here differs from Guillemin's in that the network is assumed to be composed entirely of MTP elements which are represented by mutually coupled branches as discussed in Chapter 2. The MTP elements are either LLFPB or R-LLF. Thus the branches of the network may be called LLFPB or R-LLF depending upon whether they are associated with an LLFPB MTP element or an R-LLF element. In Guillemin's presentation, on the other hand, the network consists of MTP elements which are R-LLF and R's, L's, and C's. The MTP elements are assumed to have node to datum terminal-pairs and are replaced by a set of mutually coupled branches. Thus in his presentation also, the branches may be classified as LLFPB or R-LLF, but the LLFPB branches are ordinary resistances, inductances, and capacitances as opposed to the more general LLFPB coupled branches of the presentation of Section 3.2.

Basic to the method of analysis of LLF:R networks through linear transformation theory found by Guillemin is a method of linear network analysis developed by Guillemin some time ago(21). This method is presented in Section 3.2.1 for the case of a network consisting of MTP elements. It is a method of analysis in which it is possible to define generalized cut-set and tie-set matrices which are square and non-singular. In Section 3.3 the general transformation theory method of analysis of Section 3.1 is examined to determine some conditions under which no dependent sources appear in the LLFPB reference network. A rather general result is presented. In brief, if an LLF:R network represented by mutually coupled branches satisfies certain restrictions with regard to topology and with regard to the character of the embedded R-LLF elements, then we may express its voltages and currents in terms of those of an LLFPB network and no dependent sources are required in the LLFPB network.

Section 3.4 considers the possibility of effecting transformations directly upon the equilibrium matrix of the LLF:R network rather than indirectly through the branch parameter matrix as in Section 3.2. In this way the possibility of dependent sources appearing is removed *a priori*.

While Sections 3.2 to 3.4 are concerned with analysis methods involving real linear transformations, Section 3.5 considers particular analysis methods using complex linear transformations. The starting point for the ideas in this section is method of analysis suggested by Guillemin* in which a LLF:R network containing one vacuum tube is analyzed by means of complex linear transformations. The LLFPB reference network is obtained from the LLF:R network by omitting the vacuum tube. This method has been considerably extended and forms the basis for some of the more important results of the thesis presented in Chapters 5 and 6.

3.2 Guillemin's General Method

3.2.1 Network Analysis with Generalized Cut-Set and Tie-Set Matrices

In this Section we will outline a method of analysis of linear networks in which the cut-set and tie-set matrices are square and non-singular. The network to be analyzed is assumed to be composed of MTP elements that have been represented by coupled tree branches. Let there be b branches, n + 1 nodes, and ℓ links. Formulation of node equilibrium equations is summarized by Equations 2.3.28 to 2.3.30, and formulation of loop equilibrium equations is summarized by Equations 2.3.7, 2.3.8, 2.3.10, and 2.3.11. The cut-set matrix α_{nb} contains n rows and b columns while the tie-set matrix $\beta_{\ell b}$ contains ℓ rows and b columns. We will first consider formulating equilibrium equations on the node basis for a slightly modified network for which the cut-set matrix α is non-singular and contains α_{nb} as a submatrix. Then we will consider formulating equilibrium equations on the loop basis for the original network modified in a fashion dual to that for which the cut-set matrix α was defined.

* Unpublished memo.

In this case a non-singular tie-set matrix β will result for which $\beta_{\ell b}$ is a submatrix.

Modify the original network by open circuiting the ℓ independent loops defined by $\beta_{\ell b}$. By this procedure ℓ additional independent terminal pairs are created. Since there are already n independent terminal pairs defined by α_{nb}, the total number of independent terminal pairs is brought to

$$n + \ell = b \tag{3.2.1}$$

Now excite these b terminal pairs with current sources and write the node equilibrium equations. It should be noted that branches in the original network have not been removed but rather current sources have been placed in series with some of them. The cut-set matrix now contains b rows and b columns and must be non-singular since there are as many independent columns in a cut-set matrix as there are independent node pairs. The column matrix of source currents $\bar{I}$ can be arranged to have the form

$$\bar{I} = \begin{bmatrix} i_s' \\ -- \\ i_s \end{bmatrix} \tag{3.2.2}$$

where i_s is the column matrix of source currents applied at the original node pairs and i_s' is the column matrix of source currents applied at the new node pairs. In affect what we have done is allow the loop currents to become current sources - then the voltages across these loop current sources become node pair voltages. If j is the column matrix of branch currents and α is the cut-set matrix for the augmented network, then

$$\alpha j = \bar{i} \tag{3.2.3}$$

is Kirchoff's current low written with the aid of the cut-set schedule. Since the columns of the cut-set schedule are the coefficients in a set of linear equations which relate branch voltages and node pair voltages we have also

$$\alpha^t \, \bar{e} = v \tag{3.2.4}$$

where v is the column matrix of branch voltages and $\bar{e}$, the column matrix of node pair voltages, takes the form

$$\bar{e} = \begin{bmatrix} e' \\ -- \\ e_v \end{bmatrix} \tag{3.2.5}$$

where e' is the column matrix of new node pair voltages and e_v is the column matrix of original node pair voltages. Since α is non-singular it is possible to invert Equations 3.2.3 and 3.2.4 with the result

$$j = \alpha^{-1} \, \bar{i} \tag{3.2.6}$$

$$\bar{e} = \left[\alpha^t\right]^{-1} v$$

Now if we partition α as indicated below

$$\alpha = \begin{bmatrix} \alpha_{\ell b} \\ --- \\ \alpha_{nb} \end{bmatrix} \tag{3.2.7}$$

where α_{jk} has j rows and k columns then Equation 3.2.3 takes the form

$$\alpha_{\ell b} j = i_s' \tag{3.2.8}$$

$$\alpha_{nb} j = i_s$$

The matrix α_{nb} is seen to be just the cut-set matrix defined for the network before augmentation by the new terminal pairs. The matrix $\alpha_{\ell b}$ expresses the loop "source" currents in terms of the branch currents. To formulate the equilibrium equations we note that the relationship between branch voltages and branch currents is specified by Equation (2.3.27). We may use Equation (2.3.30) directly to obtain

$$Y = \alpha A \alpha^t \tag{3.2.9}$$

where Y is the equilibrium matrix on the node basis for the augmented network. Thus

$$\overline{I} = Y \, \overline{e} \tag{3.2.10}$$

We have been assuming that the loop currents are current sources and that the voltages across these current sources are node pair voltages. However, Equation (3.2.10) will still be valid if we assume that these node pair voltages are produced by voltage sources and the currents through these sources are loop currents. In other words if we assume that

$$e_s = e' \tag{3.2.11}$$

$$i_v = i_s'$$

where e_s is now the column matrix of voltage sources and i_v is the column matrix of loop currents, then

$$\begin{bmatrix} i_v \\ -- \\ i_s \end{bmatrix} = Y \begin{bmatrix} e_s \\ -- \\ e_v \end{bmatrix} \tag{3.2.12}$$

In Equation 3.2.12 it is not appropriate to call Y an equilibrium matrix any longer. If we partition Y in the following fashion

$$Y = \left[\begin{array}{c|c} Y_{\ell\ell} & Y_{\ell n} \\ --- & --- \\ Y_{n\ell} & Y_{nn} \end{array}\right] \tag{3.2.13}$$

where Y_{jk} has j rows and k columns, then

$$i_v = Y_{\ell\ell}\, e_s + Y_{\ell n}\, e_v \tag{3.2.14}$$

$$i_s = Y_{n\ell}\, e_s + Y_{nn}\, e_v$$

It will be recognized that the submatrix Y_{nn} is the conventional node equilibrium matrix since it relates source currents to node pair voltages with no voltage sources in the loop ($e_s = 0$).

We may proceed now on an entirely dual basis. Modify the original network by putting short circuits across node pairs defined by the cut-set matrix α_{nb}. By this artifice n additional independent loops are created beyond the ℓ loops already defined by $\beta_{\ell b}$. Thus there are now $n + \ell = b$ independent loops. Now excite these b loops with voltage sources and write the loop equilibrium equations. The tie-set matrix now contains b rows and b columns and is non-singular. The column matrix of source voltages $\tilde{e}$ can be made to take the form

$$\tilde{e} = \begin{bmatrix} e_s \\ -- \\ e_s' \end{bmatrix} \tag{3.2.15}$$

where e_s is the column matrix of source voltages applied in loops of the original network and e_s' is the column matrix of source voltages applied across node pairs. In effect what we have done is allow the node pair voltages to become source voltages - then the currents leaving these node pair voltage sources become loop currents. If v is the column matrix of branch voltages and β is the tie-set for the augmented network, then

$$\beta v = \tilde{e} \tag{3.2.16}$$

is Kirchoff's voltage law written with the aid of the tie-set schedule. Since the columns of the tie-set schedule are the coefficients in a set of linear equations which relate branch currents and loop currents we have also

$$\beta^t \tilde{i} = j \tag{3.2.17}$$

where j is the column matrix of branch currents and $\tilde{i}$, the column matrix of loop currents takes the form

$$\tilde{i} = \begin{bmatrix} i_v \\ -- \\ i' \end{bmatrix} \tag{3.2.18}$$

where i_v is the column matrix of original loop currents and i' is the column matrix of new loop currents. Since β is non-singular it is possible to invert Equations 3.2.16 and 3.2.17 with the result

$$v = \beta^{-1} \tilde{e} \tag{3.2.19}$$

$$\tilde{i} = \left[\beta^t\right]^{-1} j$$

If we partition β as indicated below

$$\beta = \begin{bmatrix} \beta_{\ell b} \\ --- \\ \beta_{nb} \end{bmatrix} \qquad (3.2.20)$$

where β_{jk} has j rows and k columns then Equation 3.2.16 takes the form

$$\beta_{\ell b} v = e_s \qquad (3.2.21)$$

$$\beta_{nb} v = e_s'$$

The matrix $\beta_{\ell b}$ is seen to be just tie-set matrix for the original network. The matrix β_{nb} expresses the node pair voltage sources in terms of the branch voltages. Equilibrium equations on the loop basis are ready formulated now just as indicated by Equations (2.3.11). Thus

$$\tilde{e} = \tilde{Z}i \qquad (3.2.22)$$

where

$$Z = \beta I \beta^t \qquad (3.2.23)$$

and

$$I = A^{-1} \qquad (3.2.24)$$

is the branch impedance parameter matrix.

We have been assuming that node pair voltages are due to voltage sources and currents leaving these voltage sources are loop currents but Equation 3.2.22 will be valid if we assume that current sources are applied at node pairs and that the voltages across these sources are node pair voltages.

In other words we may assume that

$$e'_s = e_v \tag{3.2.25}$$

$$i' = i_s$$

where e_v is the column matrix of node pair voltages and i_s is the column matrix of current sources applied to node pairs. If we partition Z as follows

$$Z = \begin{bmatrix} Z_{\ell\ell} & Z_{\ell n} \\ Z_{n\ell} & Z_{nn} \end{bmatrix} \tag{3.2.26}$$

then with the assumptions (3.2.25), Equation (3.2.22) becomes

$$\begin{bmatrix} e_s \\ \hline e_v \end{bmatrix} = \begin{bmatrix} Z_{\ell\ell} & Z_{\ell n} \\ \hline Z_{n\ell} & Z_{nn} \end{bmatrix} \begin{bmatrix} i_v \\ \hline i_s \end{bmatrix} \tag{3.2.27}$$

It may be recognized that the submatrix $Z_{\ell\ell}$ is the conventional loop equilibrium matrix since it relates source voltages to loop currents with no current sources applied at node pairs ($i_s \equiv 0$). We will define

$$e = \begin{bmatrix} e_s \\ \hline e_v \end{bmatrix} ; \quad i = \begin{bmatrix} i_v \\ \hline i_s \end{bmatrix} \tag{3.2.28}$$

Then Equations 3.2.6 read

$$j = \alpha^{-1}\, i \tag{3.2.29}$$

$$e = \left[\alpha^t\right]^{-1} v$$

and Equations 3.2.19 read

$$v = \beta^{-1} e \tag{3.2.30}$$

$$i = \left[\beta^t\right]^{-1} j$$

A comparison of Equations 3.2.29 and 3.2.30 show that

$$\alpha = (\beta^t)^{-1} \quad , \; \beta = \left[\alpha^t\right]^{-1} \tag{3.2.31}$$

i.e. the generalized cut-set and tie-set matrices are inverse transposes of one another. From Equation 3.2.31 or comparison of Equation 3.2.27 and 3.2.12 it becomes clear that

$$Y = Z^{-1} \tag{3.2.32}$$

If the cut-set α_{nb} and the tie-set $\beta_{\ell b}$ are chosen in the following particular fashion, the matrices α and β take an especially simple form. Choose α_{nb} such that node pair voltages correspond to tree branch voltages and choose $\beta_{\ell b}$ such that loop currents are identified with link currents. In addition, number the branches so that the first ℓ branches are links and the last n are tree branches. Then it is not difficult to see that

$$\beta_{\ell b} = \left[U_{\ell} \,\middle|\, \beta_{\ell n}\right] \tag{3.2.33}$$

$$\beta_{nb} = \left[0 \,\middle|\, U_n\right]$$

$$\alpha_{\ell b} = \left[U_{\ell} \,\middle|\, 0\right]$$

$$\alpha_{nb} = \left[\alpha_{n\ell} \,\middle|\, U_n\right]$$

where U_j is a j x j unit matrix, $\beta_{\ell n}$ is an ℓ x n matrix, and $\alpha_{n\ell}$ is an n x ℓ matrix. From Equation 3.2.31 one may readily deduce that

$$\beta_{\ell n} = - \alpha_{n\ell}^{t} \tag{3.2.34}$$

It should be noted that a numbering of the branches such that the first ℓ are links and the last n are tree branches is not necessarily consistent with the method of numbering suggested in Section 2.3. Thus the branch parameter matrices will not have the simple form of I and A of Equations 2.3.4 and 2.3.25 respectively, although, of course, they can be put in that form by a renumbering of branches.

3.2.2 Application of Linear Transformations

In this Section it will be shown how an LLF:R network with generalized admittance matrix Y (see Equation 3.2.12) may be analyzed by linear transformation theory in terms of an LLFPB reference network with generalized admittance matrix $\hat{Y}$, voltage and current matrices $\hat{e}$ and $\hat{i}$, and a set of real non-singular transformation matrices π and υ such that

$$\begin{aligned} i &= \pi \hat{i} \\ e &= \upsilon^{-1} \hat{e} \end{aligned} \tag{3.2.35}$$

Since the network under consideration is LLF:R its MTP elements are either LLFPB or R-LLF. One may number the R-LLF branches consecutively so that all the R-LLF branch parameter matrices may be grouped as a single real submatrix of A, the branch admittance parameter matrix of the network. By well known techniques one may find real non-singular matrices which upon pre-or-post or both pre- and post-multiplication of A convert the real submatrix into one which comes from the branch parameter matrices of a group of positive resistance (R-LLFPB) boxes. The resulting matrix $\hat{A}$ is then expressible in the form

$$\hat{A} = P^{-1} A Q^{-1} \tag{3.2.36}$$

where P,Q are real non-singular matrices. The matrix $\hat{A}$ may be regarded as the branch parameter matrix of an LLFPB network which differs from the LLF:R network with branch parameter matrix A in that the R-LLF MTP elements of the LLF:R network have become R-LLFPB MTP elements. In fact the relationship 3.2.36 implies that

$$\begin{aligned} j &= P\hat{j} \\ v &= Q^{-1}\hat{v} \end{aligned} \tag{3.2.37}$$

where $\hat{j}$, j are the branch current column matrices of the LLFPB and the LLF:R networks, respectively, with analogous interpretation for v and $\hat{v}$.

It is assumed that the topology and the assignment of voltage and current variables is the same for the LLF:R and the LLFPB network. Then the generalized cut-set and tie-set matrices α and β are the same for both networks. From Equations 3.2.29 and 3.2.30 we deduce that

$$\begin{aligned} \hat{i} &= \alpha\hat{j} \ ; \ i = \alpha j \\ \hat{e} &= \beta\hat{v} \ ; \ e = \beta v \end{aligned} \tag{3.2.38}$$

Using both Equations 3.2.38 and 3.2.37 we find

$$\hat{i} = \alpha P^{-1} j = \alpha P^{-1} \alpha^{-1} i = \pi^{-1} i \tag{3.2. 39}$$

$$\hat{e} = \beta Q v = \beta Q \beta^{-1} v = \upsilon e$$

where the transformation matrices are given by

$$\pi = \alpha P \alpha^{-1} \tag{3.2.40}$$

$$\upsilon = \beta Q \beta^{-1}$$

Now according to definition

$$\hat{i} = \hat{Y} \hat{e} \tag{3.2.41}$$

$$i = Y e$$

The relationship between Y and $\hat{Y}$ is readily determined by premultiplying the first equation in 3.2.41 by π and then using the equalities in Equation 3.2.39. The result is

$$Y = \pi \hat{Y} \upsilon \tag{3.2.42}$$

Equations 3.2.42 and 3.2.39 constitute, in a sense, an analysis of the LLF:R netowrk into an LLFPB network and a set of linear transformation relating voltages and currents in the two networks. However, we note that the matrices i and e or $\hat{i}$ and $\hat{e}$ have elements which are both source quantities and response quantities. Thus, suppose the LLF:R network is excited only by current sources at node pairs. The matrices e and i take the form

$$e = \begin{bmatrix} 0 \\ -- \\ e_v \end{bmatrix} \quad i = \begin{bmatrix} i_v \\ -- \\ i_s \end{bmatrix} \tag{3.2.43}$$

Let the matrix υ be partitioned in the form

$$\upsilon = \left[\begin{array}{c|c} \upsilon_{\ell\ell} & \upsilon_{\ell n} \\ \hline \upsilon_{n\ell} & \upsilon_{nn} \end{array}\right] \tag{3.2.44}$$

where υ_{jk} is a j x k matrix. Then application of the second equation in 3.2.39 shows that

$$\hat{e} = \begin{bmatrix} \hat{e}_s \\ \hline \hat{e}_v \end{bmatrix} = \left[\begin{array}{c|c} \upsilon_{\ell\ell} & \upsilon_{\ell n} \\ \hline \upsilon_{n\ell} & \upsilon_{nn} \end{array}\right] \begin{bmatrix} 0 \\ \hline e_v \end{bmatrix} = \begin{bmatrix} \upsilon_{\ell n} e_v \\ \hline \upsilon_{nn} e_v \end{bmatrix} \tag{3.2.45}$$

i.e.

$$\begin{aligned} \hat{e}_s &= \upsilon_{\ell n} e_v \\ \hat{e}_v &= \upsilon_{nn} e_v \end{aligned} \tag{3.2.46}$$

If we solve for e_v in the second equation in 3.2.45 and use this in the first equation we find that

$$\hat{e}_s = \upsilon_{\ell n} \upsilon_{nn}^{-1} \hat{e}_v \tag{3.2.47}$$

Thus in addition to current sources applied at node pairs, the LLFPB reference network must have voltage sources in loops whose values depend upon node pair voltages - i.e. dependent voltage sources.

Moreover, examination of the first equation in 3.2.39 shows that the current sources in the LLFPB network are not only a function of the current sources in the LLF:R network but also a function of the loop currents of the LLFPB network. Thus the current sources also are dependent current sources.

Of course it is not necessary to use both P and Q to transform an LLF:R A matrix into an LLFPB A matrix. If Q is chosen a unit matrix then υ also becomes a unit matrix. In such a case $e = \hat{e}$ and dependent voltage sources do not appear. However, the current sources in the LLFPB network are still dependent upon loop currents in the LLFPB network. In the following section consideration is given to special conditions which do not lead to dependent sources.

3.3 Special Condition Leading To No Dependent Sources

General conditions for the existence of no dependent sources in the reference network are readily found from Equation 3.2.39. Let π be positioned in the same fashion as υ in 3.2.44. Then we may expand Equation 3.2.39 as follows (we invert the first equation in 3.2.39 to avoid defining new quantities).

$$\begin{aligned} i_v &= \pi_{\ell\ell}\hat{i}_v + \pi_{\ell n}\hat{i}_s \\ i_s &= \pi_{n\ell}\hat{i}_v + \pi_{nn}\hat{i}_s \end{aligned} \tag{3.3.1}$$

and

$$\begin{aligned} \hat{e}_s &= \upsilon_{\ell\ell}e_s + \upsilon_{\ell n}e_v \\ \hat{e}_v &= \upsilon_{n\ell}e_s + \upsilon_{nn}e_v \end{aligned} \tag{3.3.2}$$

It may be seen by inspection of Equations 3.3.1 and 3.3.2 that the current source matrix $\hat{i}_s$ and the voltage source matrix $\hat{e}_s$ of the reference network will

(1) be related to i_s and e_s respectively, by real transformation matrices

(2) be independent of voltages and currents in the LLFPB reference network

only if the matrices $\pi_{n\ell}$ and $\upsilon_{\ell n}$ are null, i.e., only if

$$\pi_{n\ell} \equiv 0 \qquad (3.3.3)$$
$$\upsilon_{\ell n} \equiv 0$$

Application of the conditions 3.3.3 to Equations 3.3.1 and 3.3.2 yield the relationships between i_s and $\hat{i}_s$, and e_s and $\hat{e}_s$ as

$$i_s = \pi_{nn}\hat{i}_s \qquad (3.3.4)$$
$$\hat{e}_s = \upsilon_{nn}e_s$$

Now the equilibrium matrices of interest are the node and loop equilibrium matrices of the network, Y_{nn} and $Z_{\ell\ell}$, respectively. We will confine all discussion in this section to the case in which current sources are applied at node-pairs but no voltage sources are applied in loops. The situation in which voltage sources are applied in loops and no current sources are applied at node pairs is a dual situation and all the arguments and discussion may be carried over in dual form. Thus, in this section we will be concerned with the node equilibrium matrix Y_{nn}. The relationship between Y_{nn} and $\hat{Y}_{nn}$ may be determined from Equation 3.2.42 as

$$Y_{nn} = \pi_{n\ell}\hat{Y}_{\ell\ell}\upsilon_{\ell n} + \pi_{n\ell}\hat{Y}_{\ell n}\upsilon_{nn} + \pi_{nn}\hat{Y}_{n}\,\upsilon_{\ell n} + \pi_{nn}\hat{Y}_{nn}\upsilon_{nn} \qquad (3.3.5)$$

Since we are confining ourselves to the condition in which no dependent sources exist in the LLFPB reference network we apply the restrictions 3.3.3 to Equation 3.3.5 with the result that

$$Y_{nn} = \pi_{nn}\hat{Y}_{nn}\upsilon_{nn} \qquad (3.3.6)$$

Thus we come to the useful result that conditions 3.3.3 which lead to no dependent sources also lead to a simple expression for the relationship between the node-equilibrium matrices of the LLF:R network and the LLFPB reference network. By inverting Equation 3.3.6 we obtain the relationship between the o.c. impedance matrices of these networks as

$$Z_{nn} = \upsilon_{nn}^{-1}\hat{Z}_{nn}\pi_{nn}^{-1} \tag{3.3.7}$$

We will now determine some interesting implications of the requirement that $\pi_{n\ell}$ and $\upsilon_{\ell n}$ be null. To make the algebraic manipulations as simple as possible we will assume that node-pair voltages are identified with tree branch voltages and loop currents with link currents. In this situation the α and β matrices assume the simple forms

$$\beta = \left[\begin{array}{c|c} U_\ell & \beta_{\ell n} \\ \hline 0 & U_n \end{array}\right] \qquad \alpha = \left[\begin{array}{c|c} U_\ell & 0 \\ \hline \alpha_{n\ell} & U_n \end{array}\right] \tag{3.3.8}$$

if the branches are numbered so that the first ℓ are links and the last n are tree branches. It will be further assumed that the LLF:R network contains one imbedded R-LLF MTP element and that the branches of this element are contained in the group of tree branches selected for the network. Let this R-LLF MTP element have s branches and let these branches be the last numbered branches of the LLF:R network. The branch admittance parameter matrix of the LLF:R network then takes the form

$$
A = \left[\begin{array}{ccc}
\hat{y}_{\ell\ell} & \hat{y}_{\ell k} & 0 \\
\hdashline
y_{k\ell} & \hat{y}_{kk} & 0 \\
\hdashline
0 & 0 & g_{ss}
\end{array}\right] \tag{3.3.9}
$$

where the total number of tree branches is given by

$$
n = k + s \tag{3.3.10}
$$

Because we have numbered all the links first the A matrix does not take the simple form of Equation 2.3.25 as discussed at the end of Section 2.3.1. The matrix g_{ss} is the branch admittance parameter matrix of the R-LLF MTP element. The other matrices in Equation 3.3.9 represent the branch admittance parameter matrix of the LLFPB portion of the LLF:R network. To convert our A matrix into the matrix $\hat{A}$ of an LLFPB network we can pre- and post-multiply A as follows to form $\hat{A}$

$$
\hat{A} = \begin{bmatrix}
U_\ell & 0 & 0 \\
0 & U_k & 0 \\
0 & 0 & P_{ss}^{-1}
\end{bmatrix}
\begin{bmatrix}
\hat{y}_{\ell} & \hat{y}_{\ell k} & 0 \\
\hat{y}_{k\ell} & \hat{y}_{kk} & 0 \\
0 & 0 & g_{ss}
\end{bmatrix}
\begin{bmatrix}
U_\ell & 0 & 0 \\
0 & U_k & 0 \\
0 & & Q_{ss}^{-1}
\end{bmatrix} \tag{3.3.11}
$$

$$
= \begin{bmatrix}
\hat{y}_{\ell\ell} & \hat{y}_{\ell k} & 0 \\
\hat{y}_{k\ell} & \hat{y}_{kk} & 0 \\
0 & 0 & \hat{g}_{ss}
\end{bmatrix}
$$

where the submatrices P_{ss}^{-1} and Q_{ss}^{-1} are chosen so that

$$\hat{g}_{ss} = P_{ss}^{-1} g_{ss} Q_{ss}^{-1} \tag{3.3.12}$$

is the branch parameter matrix of an R-LLFPB MTP element, i.e., an s terminal pair black box composed of positive resistances.

We desire to determine expressions for $\pi_{n\ell}$ and $\upsilon_{\ell n}$ in order to find what conditions must be satisfied to make them null. The transformation matrices P and Q are, by inspection of Equation 3.3.11,

$$P = \begin{bmatrix} U_\ell & 0 & 0 \\ 0 & U_k & 0 \\ 0 & 0 & P_{ss} \end{bmatrix} \quad Q = \begin{bmatrix} U_\ell & 0 & 0 \\ 0 & U_k & 0 \\ 0 & 0 & Q_{ss} \end{bmatrix} \tag{3.3.13}$$

By evaluating π and υ according to Equation 3.2.40 we can determine expressions for $\pi_{n\ell}$ and $\upsilon_{\ell n}$ in terms of submatrices of P and Q respectively. To this end let the matrices α and β be partitioned as follows

$$\beta = \begin{bmatrix} U_\ell & \beta_{\ell k} & \beta_{\ell s} \\ 0 & U_k & 0 \\ 0 & 0 & U_s \end{bmatrix} \quad \alpha = \begin{bmatrix} U_\ell & 0 & 0 \\ \alpha_{k\ell} & U_k & 0 \\ \alpha_{s\ell} & 0 & U_s \end{bmatrix} \tag{3.3.14}$$

where the matrices $\beta_{\ell n}$ and $\alpha_{n\ell}$ have been partitioned as follows

$$\beta_{\ell n} = \begin{bmatrix} \beta_{\ell k} & \beta_{\ell s} \end{bmatrix}$$

$$\alpha_{n\ell} = \begin{bmatrix} \alpha_{k\ell} \\ \hline \alpha_{s\ell} \end{bmatrix} \tag{3.3.15}$$

Note that the matrix $\beta_{\ell s}$ is the last s columns of $\beta_{\ell b}$ the tie-set schedule for the network and $\alpha_{s\ell}$ is the last s rows of α_{nb} the cut-set schedule of the network. Since $\alpha_{n\ell}$ and $\beta_{\ell n}$ are negative transposes of one another

$$\begin{bmatrix} \alpha_{k\ell} \\ \hline \alpha_{s\ell} \end{bmatrix} = \begin{bmatrix} -\beta^t_{\ell k} \\ \hline -\beta^t_{\ell s} \end{bmatrix} \tag{3.3.16}$$

or

$$\alpha_{k\ell} = -\beta^t_{\ell k} \tag{3.3.17}$$

$$\alpha_{s\ell} = -\beta^t_{\ell s}$$

Applying Equation 3.2.40 we readily determine the following expressions for $\pi_{n\ell}$ and $\upsilon_{\ell n}$.

$$\pi_{n\ell} = \begin{bmatrix} 0 \\ \hline \left\{U_s - P_{ss}\right\}\alpha_{s\ell} \end{bmatrix} \tag{3.3.18}$$

$$\upsilon_{\ell n} = \left[\begin{array}{c|c} 0 & \beta_{\ell s}\left\{Q_{ss} - U_s\right\} \end{array}\right]$$

Examination of these equations indicates that if

$$\left\{U_s - P_{ss}\right\}\alpha_{s\ell} = 0 \tag{3.3.19}$$

$$\beta_{\ell s}\left\{Q_{ss} - U_s\right\} = 0$$

where the 0 indicates a null matrix then $\pi_{n\ell}$ and $\upsilon_{\ell n}$ will be null. By taking the transpose of the first equation in 3.3.19 and using Equation 3.3.17 this becomes

$$\beta_{\ell s}\left\{P_{ss} - U_s\right\} = 0 \tag{3.3.20}$$

Thus if $\beta_{\ell s}$ is orthogonal to both the matrix $\left[Q_{ss} - U_s\right]$ and the matrix $\left[P_{ss} - U_s\right]$, $\pi_{n\ell}$ and $U_{\ell n}$ will become null. If we desire to have complete freedom in the choice of P_{ss} and Q_{ss} then $\beta_{\ell s}$ must be a null matrix for Equation 3.3.19 to be satisfied. Let us consider this latter situation, i.e.,

$$\beta_{\ell s} = 0 \tag{3.3.21}$$

Equation 3.3.21 states that the last s columns of the tie-set matrix $\beta_{\ell b}$, contain only zeroes. If the last s columns of the tie-set matrix contain only zeroes, then no loop currents can circulate on the last s branches of the network. It is then clear that these branches must be an isolated set of tree branches, so to speak, waving in the breeze. But the last s tree branches are by construction, just the mutually coupled branch representation of the R-LLF MTP element embedded in the LLF:R network. We have then arrived at a trivial result, namely, to perform arbitrary transformations P_{ss} and Q_{ss} upon an arbitrary g_{ss} according to Equation 3.3.12 in order to convert the R-LLF MTP element into a positive resistance box, the R-LLF device must be completely isolated from the LLFPB portion of the LLF:R network.

Let us suppose that we wish only to transform a portion of g_{ss}, i.e., let the matrices P_{ss} and Q_{ss} take the form

$$P_{ss} = \begin{bmatrix} U_r & 0 \\ 0 & P_{dd} \end{bmatrix} \quad Q_{ss} = \begin{bmatrix} U_r & 0 \\ 0 & Q_{dd} \end{bmatrix} \tag{3.3.22}$$

where P_{dd}, Q_{dd} are arbitrary real nonsingular d x d matrices and U_r is an r x r unit matrix (s = r + d). If we use these restricted expressions for P_{ss} and Q_{ss} in Equation 3.3.19 then they take the form

$$\beta\ell_d \left\{Q_{dd} - U_d\right\} = 0 \qquad (3.3.23)$$

$$\beta\ell_d \left\{P_{dd} - U_d\right\} = 0$$

where the matrix $\beta\ell_s$ has been partitioned as follows

$$\beta\ell_s = \left[\beta\ell_r \,\middle|\, \beta\ell_d\right] \qquad (3.3.24)$$

Thus $\beta\ell_d$ represents the last d columns of the tie-set matrix $\beta\ell_b$. If we desire P_{dd}, Q_{dd} to be arbitrary, then

$$\beta\ell_d = 0 \qquad (3.3.25)$$

if Equation 3.3.23 is to be satisfied.

Whereas Equation 3.3.21 leads to trivial results, Equation 3.3.25 does not, as will now be shown. Equation 3.3.25 implies that the last d branches of the R-LLF MTP element form a set of tree branches on which no loop currents circulate. However, the saving grace here is that remaining branches of the R-LLF element, r in number, are not restricted in this way. Thus these r branches or terminal-pairs may be connected with LLFPB branches or terminal-pairs to form the LLF:R network. Such an interconnection of an MTP R-LLF element with an MTP LLFPB element to form an LLF:R network is shown in Figure 3.3.1

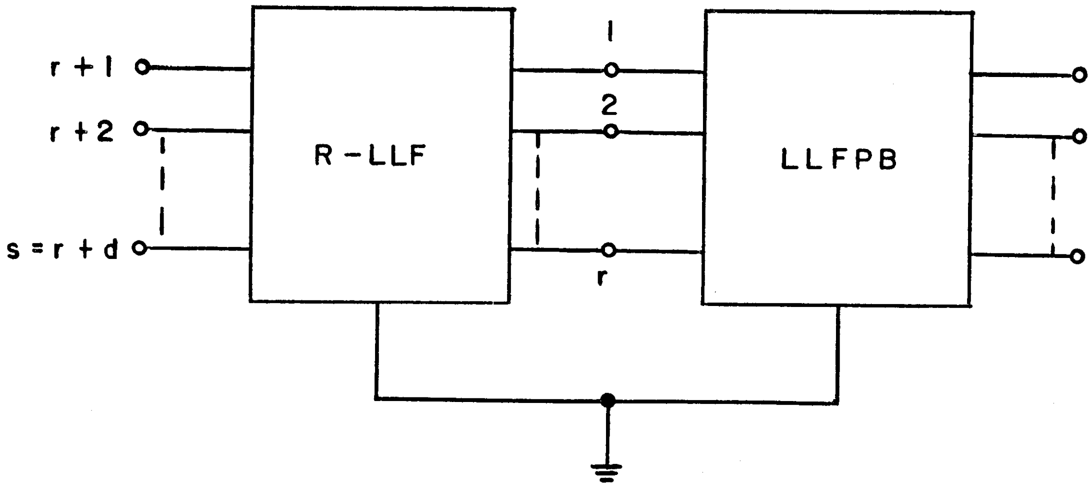


Figure 3.3.1 An LLF:R Network With $\beta_{\ell d} = 0$

In this figure the R-LLF element and the LLFPB element have node to datum terminal-pairs assigned although this need not be done.

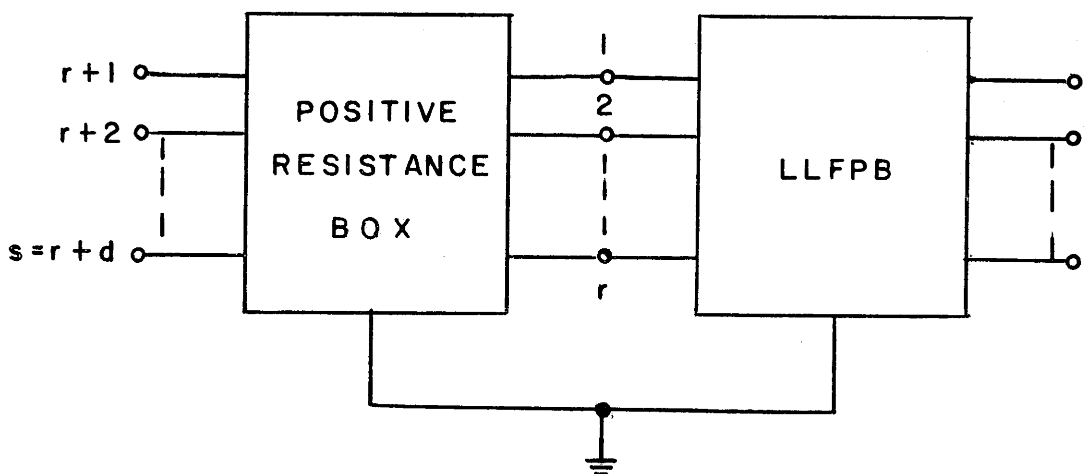


Figure 3.3.2 Reference LLFPB Network for LLF:R Network of Figure 3.3.1

The reference LLFPB network has exactly the same form except that the R-LLF box becomes a resistance box. Figure 3.3.2 illustrates the form of this reference network. The branch parameter matrix g_{ss} must be restricted in form since only then can one expect to produce an LLFPB branch parameter matrix $\hat{g}_{ss}$ with the restricted transformation matrices of Equation 3.3.22. To understand the restrictions on g_{ss} we use the expressions for P_{ss} and Q_{ss} given by Equation 3.3.22 and carry out the matrix products shown in Equation 3.3.12 to evaluate $\hat{g}_{ss}$*. First partition g_{ss} as follows

$$g_{ss} = \begin{bmatrix} g_{rr} & g_{rd} \\ g_{dr} & g_{dd} \end{bmatrix} \tag{3.3.26}$$

Then

$$\hat{g}_{ss} = \begin{bmatrix} U_r & o \\ o & P_{dd}^{-1} \end{bmatrix} \begin{bmatrix} g_{rr} & g_{rd} \\ g_{dr} & g_{dd} \end{bmatrix} \begin{bmatrix} U_r & o \\ o & Q_{dd}^{-1} \end{bmatrix} \tag{3.3.27}$$

$$= \left[\begin{array}{c|c} g_{rr} & g_{rd}\, Q_{dd}^{-1} \\ \hline P_{dd}^{-1}\, g_{dr} & P_{dd}^{-1}\, g_{dd}\, Q_{dd}^{-1} \end{array}\right] = \left[\begin{array}{cc} \hat{g}_{rr} & \hat{g}_{rd} \\ \hline \hat{g}_{dr} & \hat{g}_{dd} \end{array}\right]$$

Given a g_{ss}, it would be very difficult to determine whether it were of the form to permit transformation to an LLFPB branch parameter matrix $\hat{g}_{ss}$ in the manner indicated in Equation 3.3.27. However one may of course generate permissible g_{ss} matrices by reversing the procedure, i.e., starting with a $\hat{g}_{ss}$ and forming g_{ss} by

---

* It is worth reminding the reader at this point that the parameter matrix $\hat{g}_{ss}$ need not be diagonal as is required for the conventional branch conductance parameter matrix. Of course, among other restrictions, it must be symmetrical and define a positive definite quadrative form.

$$g_{ss} = P_{ss}\,\hat{g}_{ss}\,Q_{ss} \tag{3.3.28}$$

To determine the relationship between the LLF:R o.c. impedance matrix Z_{nn} and the reference LLFPB o.c. impedance matrix $\hat{Z}_{nn}$ we must employ Equation 3.3.7. To this and we must evaluate π_{nn} and υ_{nn}. These are readily found to be

$$\pi_{nn} = \begin{bmatrix} U_k & o & o \\ o & U_r & o \\ o & o & P_{dd} \end{bmatrix} ; \quad \upsilon_{nn} = \begin{bmatrix} U_k & o & o \\ o & U_r & o \\ o & o & Q_{dd} \end{bmatrix} \tag{3.3.29}$$

If we partition Z_{nn} and $\hat{Z}_{nn}$ in the same manner as π_{nn} and υ_{nn} we find that

$$Z_{nn} = \upsilon_{nn}^{-1}\,\hat{Z}_{nn}\,\pi_{nn}^{-1} \tag{3.3.30}$$

$$= \begin{bmatrix} z_{kk} & z_{kr} & z_{kd} \\ z_{rk} & z_{rr} & z_{rd} \\ z_{dk} & z_{dr} & z_{dd} \end{bmatrix} = \left[\begin{array}{c|c|c} \hat{z}_{kk} & \hat{z}_{kr} & \hat{z}_{kd}\,P_{dd}^{-1} \\ \hline \hat{z}_{rk} & \hat{z}_{rr} & \hat{z}_{rd}\,P_{dd}^{-1} \\ \hline Q_{dd}^{-1}\,\hat{z}_{dk} & Q_{dd}^{-1}\,\hat{z}_{dr} & Q_{dd}^{-1}\,\hat{z}_{dd}\,P_{dd}^{-1} \end{array}\right]$$

Observation of Equation 3.3.30 indicates that the transfer impedances among the first k + r terminal pairs of the reference LLFPB network are identical to those of the LLF:R network.

Equation 3.3.19 may be pursued further but this will not be done here.

3.4 Real Transformations Directly Upon Equilibrium Matrix

In the previous Section there is presented a method of analysis of LLF:R networks by linear transformation theory that begins by effecting real transformations upon the branch parameter matrix of the network to convert it into the branch parameter matrix of an LLFPB network. These linear transformations are shown to imply a second set of linear transformations which relate the dynamic variables in the LLF:R network and those in the reference LLFPB network. A difficulty with this approach, beyond the appearance of dependent sources in the reference network is the quite indirect correspondence between transformations effected upon the branch parameter matrix and the resulting modification of the conventional o.c. impedance matrix Z_{nn} or short circuit admittance matrix $Y_{\ell\ell}$. An alternative approach is conceivable in which one attempts to effect transformations directly upon the equilibrium matrix. In this way the appearance of dependent sources in the reference network is ruled out *a priori*. However the difficulty with this approach is that there is no simple way of determining the structure of the reference network from inspection of its equilibrium matrix. The approach to this latter problem followed in this Section is to find conditions under which transformations directly upon the equilibrium matrix of the LLF:R network can be interpreted simply in terms of transformations upon R-LLF MTP elements of the LLF:R network. We will only concern ourselves with the node-equilibrium matrix since an entirely dual argument follows on the loop basis.

Let us assume that the branches of the network are numbered not in the manner of Sections 3.2 and 3.3 but as in Section 2.3.

The branch admittance matrix then takes the form indicated in Equation 2.3.25. Suppose that the LLF:R network consists of two LLFPB MTP elements and one R-LLF MTP element. The branch admittance matrix A may be represented in this case as

$$A = \begin{bmatrix} \hat{y}_1 & o & o \\ o & \hat{y}_2 & o \\ o & o & g_3 \end{bmatrix} \tag{3.4.1}$$

where $\hat{y}_1$, $\hat{y}_2$ are branch admittance matrices of LLFPB elements 1 and 2 and g_3 is the branch admittance matrix of the R-LLF element. The node equilibrium matrix Y_{nn} is then given by (See Section 2.3)

$$Y_{nn} = \hat{Y}_1 + \hat{Y}_2 + G_3 \tag{3.4.2}$$

where $\hat{Y}_1$ and $\hat{Y}_2$ are the node parameter matrices of LLFPB elements 1 and 2 and G_3 is the node parameter matrix of the R-LLF element. The node parameter matrices are as follows

$$\hat{Y}_1 = \alpha_1 \hat{y}_1 \alpha_1^t \tag{3.4.3}$$

$$\hat{Y}_2 = \alpha_2 \hat{y}_2 \alpha_2^t$$

$$G_3 = \alpha_3 g_3 \alpha_3^t$$

where α_1, α_2, α_3 are submatrices of the cut-set matrix α_{nb} as indicated below

$$\alpha_{nb} = \left[\alpha_1 \mid \alpha_2 \mid \alpha_3\right] \tag{3.4.4}$$

As discussed above we desire to perform transformations upon Y_{nn} directly and to find the conditions under which these transformations affect only the R-LLF element in the LLF:R network. Group $\hat{Y}_1$ and $\hat{Y}_2$ together as follows

$$\hat{Y} = \hat{Y}_1 + \hat{Y}_2 \tag{3.4.5}$$

Then premultiply Y_{nn} by a real nonsingular matrix P_{nn} and postmultiply it by a real nonsingular matrix Q_{nn}. We then find with the aid of Equations 3.4.2 and 3.4.5 that

$$P_{nn}Y_{nn}Q_{nn} = P_{nn}\hat{Y}Q_{nn} + P_{nn}\alpha_3 g_3 \alpha_3^t Q_{nn} \tag{3.4.6}$$

We are searching for conditions under which

$$P_{nn}\hat{Y}Q_{nn} = \hat{Y} \tag{3.4.7}$$

and

$$P_{nn}\alpha_3 g_3 \alpha_3^t Q_{nn} = \alpha_3 \tau_1 g_3 \tau_2 \alpha_3^t \tag{3.4.8}$$

where τ_1 and τ_2 are real nonsingular matrices. In order to avoid dealing with the cut-set submatrices it was decided to specialize to the case where the node parameter matrices were Simple and Complete. (See Section 2.4 for definitions.) In this way the transformations P_{nn} and Q_{nn} upon G_3 can be directly interpreted in terms of transformations upon g_3. At first scrutiny one might conclude that Equation 3.4.7 can only be satisfied in the trivial case in which P_{nn} and Q_{nn} are unit matrices. However this is not the case. Thus let $\hat{Y}$ take the particular form

$$\hat{Y} = \begin{bmatrix} \hat{y}_1 & o & o \\ o & \hat{y}_2 & o \\ o & o & o \end{bmatrix} \tag{3.4.9}$$

This form for $\hat{Y}$ indicates that elements 1 and 2 are isolated. Thus in the LLF:R network, MTP elements 1 and 2 are coupled only through resistive coupling provided by the R-LLF element.

Let P_{nn} and Q_{nn} be given by

$$P_{nn} = \begin{bmatrix} aU_k & o & o \\ o & bU_k & o \\ o & o & P_{dd} \end{bmatrix} \quad Q_{nn} = \begin{bmatrix} a^{-1}U_k & o & o \\ o & b^{-1}U_r & o \\ o & o & Q_{dd} \end{bmatrix} \tag{3.4.10}$$

where a, b are real numbers, U_r is r x r unit matrix which is assumed to have the same number, r, of rows and columns as $\hat{y}_2$, U_k is a k x k unit matrix with the same number of rows and columns as $\hat{y}_1$, and P_{dd}, Q_{dd} are arbitrary real nonsingular d x d matrices. Then one may readily verify that Equation 3.4.7 is satisfied. Let

$$G_3 = g_3 = \begin{bmatrix} g_{kk} & g_{kr} & g_{kd} \\ g_{rk} & g_{rr} & g_{rd} \\ g_{dk} & g_{dr} & g_{dd} \end{bmatrix} \tag{3.4.11}$$

where g_{mn} has m rows and n columns. Then

$$P_{nn}G_3Q_{nn} = \begin{bmatrix} g_{kk} & \frac{a}{b}\, g_{kr} & a\ g_{kd}Q_{dd} \\ \frac{b}{a}\, g_{rk} & g_{rr} & b\ g_{rd}\ Q_{dd} \\ \frac{1}{a}\, P_{dd}g_{dk} & \frac{1}{b}\, P_{dd}g_{dr} & P_{dd}g_{dd}Q_{dd} \end{bmatrix} \qquad (3.4.12)$$

For some R-LLF elements the branch admittance parameter matrix g_3 can be of such a form that numbers a, b and matrices P_{dd}, Q_{dd} can be found such that

$$\hat{g}_3 = \hat{G}_3 = P_{nn}G_3Q_{nn} = P_{nn}g_3Q_{nn} \qquad (3.4.13)$$

is the branch admittance parameter matrix of an LLFPB MTP element. Although it may be difficult to determine whether an arbitrarily selected g_3 has this property one may always generate g_3 matrices with this property by starting with an LLFPB matrix $\hat{g}_3$ and forming g_3 by

$$g_3 = P_{nn}^{-1}\hat{g}_3Q_{nn}^{-1} \qquad (3.4.14)$$

It would be well at this point to make a few comments about the topological restrictions implied by the form of $\hat{Y}$ and G_3 (Equations 3.4.9 and 3.4.11). First we note that the network contains n = r + k + d terminal pairs. The d x d null submatrix in the lower righthand corner of $\hat{Y}$ indicates that the LLFPB portion of the LLF:R network contains only r + k terminal pairs. Examination of G_3 shows that the R-LLF MTP element is assumed to contain n = r + k + d terminal pairs. However, because of the d x d null submatrix in $\hat{Y}$ only the first r + k terminal pairs of the R-LLF element are connected with the LLFPB elements to form the LLF:R network. Thus d terminal pairs of the R-LLF element are isolated.

Let us assume that g_3 has been selected so that $\hat{g}_3$ as given by Equation 3.4.14 is an LLFPB branch parameter matrix. Then

$$\hat{Y}_{nn} = P_{nn}Y_{nn}Q_{nn} \tag{3.4.15}$$

is an LLFPB node equilibrium matrix. If we let a = b = 1 and evaluate the o.c. impedance matrix

$$Z_{nn} = Y_{nn}^{-1} = P_{nn}^{-1}\hat{Y}_{nn}^{-1}Q_{nn}^{-1} = Q_{nn}\hat{Z}_{nn}P_{nn} \tag{3.4.16}$$

we find that it is identical to Equation 3.3.30. Thus we have arrived at a result of the same form as the previous section which considered transformations directly upon the network branch parameter matrix.

On the other hand we get a new result of a restricted nature if we let P_{dd} and Q_{dd} be unit matrices but a and b be unrestricted real numbers, namely,

$$Z_{nn} = \begin{bmatrix} a^{-1}U_r & o & o \\ o & b^{-1}U_k & o \\ o & o & U_d \end{bmatrix} \begin{bmatrix} \hat{z}_{rr} & \hat{z}_{rk} & \hat{z}_{rd} \\ \hat{z}_{kr} & \hat{z}_{kk} & \hat{z}_{kd} \\ \hat{z}_{dr} & \hat{z}_{dk} & \hat{z}_{dd} \end{bmatrix} \begin{bmatrix} aU_r & o & o \\ o & bU_k & o \\ o & o & U_d \end{bmatrix} \tag{3.4.17}$$

$$= \begin{bmatrix} \hat{z}_{rr} & \frac{b}{a}\,\hat{z}_{rk} & \frac{1}{a}\,\hat{z}_{rd} \\ \frac{a}{b}\,\hat{z}_{kr} & \hat{z}_{kk} & \frac{1}{b}\,\hat{z}_{kd} \\ a\,\hat{z}_{dr} & b\,\hat{z}_{dk} & \hat{z}_{dd} \end{bmatrix}$$

Thus in this case the transfer impedances between certain groups of terminal pairs of the LLF:R network are related to the corresponding group of transfer impedances of the reference LLFPB network by a real constant multiplier. In particular we note that if a transfer impedance in one direction is multiplied by a constant c then the transfer impedance in the opposite direction is multiplied by 1/c.

3.5 Complex Transformations

In this section we will confine ourselves to a study of networks of the Additive type (See Section 2.4). It will be recalled that for networks of this type, the equilibrium matrices can be selected so that they are the sum of the parameter matrices of the component MTP elements. This statement applies to a formulation of equilibrium equations on any basis - admittance, impedance, or mixed. We will demonstrate a simple method whereby an Additive LLF:R network may be analysed into an LLFPB network and a set of complex linear transformations relating the dependent variables in the LLF:R network and in the reference LLFPB network. The reference network is identical to the LLFPB portion of the LLF:R network with appropriate short-circuit or open-circuit constraints applied at terminal pairs. We will keep the following discussion general in the sense that it may be applied to equilibrium equations formulated on any basis.

Thus let $\mathcal{E}$ be the equilibrium matrix of the network. We will assume that the LLF:R network consists of an interconnection of an LLFPB MTP element with an embedded R-LLF MTP element of a smaller number of terminal pairs. Then we can represent $\mathcal{E}$ by

$$\mathcal{E} = P + E \tag{3.5.1}$$

where P the parameter matrix of the LLFPB element and E the parameter matrix of the R-LLF element take the forms

$$P = \left[\begin{array}{c|c} P_{ss} & P_{sr} \\ \hline P_{rs} & P_{ss} \end{array}\right] \qquad E = \left[\begin{array}{c|c} e_{ss} & o \\ \hline o & o \end{array}\right] \tag{3.5.2}$$

A subscript mn on a submatrix denotes an m x n matrix. It is clear from 3.5.2 that the R-LLF element is assumed to have s branches or terminal pairs while the LLFPB element has r + s terminal pairs. The equilibrium matrix of the reference network, $\hat{\mathcal{E}}$, will be assumed identical to P, i.e.,

$$\hat{\mathcal{E}} = P \tag{3.5.3}$$

This matrix may be obtained from $\mathcal{E}$ by letting e_{ss} become null. If e_{ss} were a branch admittance matrix then letting e_{ss} become null would correspond in the LLF:R network to removing the R-LLF element and placing open circuit constraints across those terminal pairs of LLFPB element which were connected to the R-LLF element. On the other hand if e_{ss} were a branch impedance matrix, letting it become null would correspond to placing short circuit constraints across terminal pairs of the LLFPB element. If e_{ss} were a mixed parameter matrix some terminal pairs of the LLFPB element necessitate open circuit constraints and other terminal pairs would necessitate short circuit constraints when e_{ss} becomes null.

In order for the reference network equilibrium matrix to be nonsingular, P must be nonsingular. Assuming this to be true we may perform the following manipulation on Equation 3.5.1.

$$\mathcal{E} = P\left[U + P^{-1}E\right] = \hat{\mathcal{E}}\tau^{-1} \tag{3.5.4}$$

where

$$\tau^{-1} = U + \hat{\mathcal{E}}^{-1}E \tag{3.5.5}$$

and U is an $(r + s) \times (r + s)$ unit matrix.

By this artifice the equilibrium matrix of the LLF:R network becomes expressed in terms of that of an LLFPB network by means of the transformation matrix τ^{-1}. We are interested primarily in the solution matrices, the inverses of the equilibrium matrices,

$$\begin{aligned} S &= \mathcal{E}^{-1} \\ \hat{S} &= \hat{\mathcal{E}}^{-1} \end{aligned} \tag{3.5.6}$$

These are related by

$$S = \tau\hat{S} \tag{3.5.7}$$

where

$$\tau = \left[U + \hat{S}E\right]^{-1} \tag{3.5.8}$$

We will now evaluate the transformation matrix τ. To this end partition $\hat{S}$ in the same form as P in Equation 3.5.2,

$$\hat{S} = \begin{bmatrix} \hat{S}_{ss} & \hat{S}_{sr} \\ \hat{S}_{rs} & \hat{S}_{rr} \end{bmatrix} \tag{3.5.9}$$

Then carrying out the operations to form τ^{-1}

$$\tau^{-1} = \begin{bmatrix} U_s & o \\ o & U_r \end{bmatrix} + \begin{bmatrix} \hat{S}_{ss} & \hat{S}_{sr} \\ \hat{S}_{rs} & \hat{S}_{rr} \end{bmatrix} \begin{bmatrix} e_{ss} & o \\ o & o \end{bmatrix} \tag{3.5.10}$$

$$= \begin{bmatrix} U_s + \hat{S}_{ss} e_{ss} & o \\ \hat{S}_{rs} e_{ss} & U_r \end{bmatrix}$$

We may invert τ^{-1} to form τ by writing the matrix equation implied by τ^{-1} and then by algebraic operations forming the inverse set of equations. Thus τ^{-1} implies the set of matrix equations

$$\left\{U_s + \hat{S}_{ss} e_{ss}\right\} x_1 + o = y_1 \tag{3.5.11}$$

$$\left\{S_{rs} e_{ss}\right\} x_1 + x_2 = y_2$$

From the first equation we solve for x_1. Using this value of x_1 in the second equation we solve for x_2. The inverse set of equations reads

$$\left\{U_s + \hat{S}_{ss} e_{ss}\right\}^{-1} y_1 + o = x_1 \tag{3.5.12}$$

$$-\left\{\hat{S}_{rs} e_{ss}\right\}\left\{U_s + \hat{S}_{ss} e_{ss}\right\}^{-1} y_1 + y_2 = x_2$$

Thus if we define

$$T = \left\{U_s + \hat{S}_{ss} e_{ss}\right\}^{-1} \tag{3.5.13}$$

Then the transformation matrix τ is given by

$$\tau = \begin{bmatrix} T & o \\ -\left\{\hat{S}_{rs} e_{ss}\right\} T & U_r \end{bmatrix} \tag{3.5.14}$$

Examination of this matrix indicates that T postmultiplies the first column. Thus we may express τ as the product of two simpler matrices

$$\tau = \begin{bmatrix} U_s & o \\ -\left\{\hat{S}_{rs} e_{ss}\right\} & U_r \end{bmatrix} \begin{bmatrix} T & o \\ o & U_r \end{bmatrix} \tag{3.5.15}$$

Since τ is a complex transformation matrix the natural frequencies of S will be generally both the natural frequencies of $\hat{S}$ and τ. Evaluating the determinant of S we find that

$$\det S = \det\tau \cdot \det\hat{S} \tag{3.5.16}$$

But inspection of Equation 3.5.15 shows that the determinant of the first matrix is unity and of the second matrix is T. Thus

$$\det S = \det T \cdot \det\hat{S} \tag{3.5.17}$$

We may then state that the natural frequencies of S caused by the introduction of the R-LLF device are the poles of detT or the zeroes of $\det T^{-1}$. Thus these natural frequencies are roots of the equation

$$\det\left[U_s + \hat{S}_{ss} e_{ss}\right] = o \tag{3.5.18}$$

It should be noted that $\hat{S}_{ss}$ is the solution matrix of that portion of the LLFPB reference network "seen" from the s terminal pairs connected to the R-LLF device. The matrix e_{ss} is the branch parameter matrix of the R-LLF device. If e_{ss} is a branch admittance matrix then $\hat{S}_{ss}$ will be an o.c. impedance matrix. If e_{ss} is a branch impedance matrix then $\hat{S}_{ss}$ will be a s.c. admittance matrix. If e_{ss} is mixed then $\hat{S}_{ss}$ will also be mixed.

CHAPTER 4

APPLICATIONS OF REAL TRANSFORMATIONS TO THE SYNTHESIS OF RC-LLF:R TRANSFER FUNCTIONS

4.1 Introduction

In Chapter 3 some particular techniques of LLF:R network analysis through linear transformation theory were presented. The techniques involved both real and complex linear transformations. In this and succeeding chapters these analysis procedures will be reversed. We will start with an LLFPB network and through the agency of linear transformations convert it into an LLF:R network. Our primary interest will be to generate RC-LLF:R transfer functions of a general character by starting with an RC-LLFPB network. This chapter will deal with the application of real linear transformations to the generation of such transfer functions. Thus in Section 4.2 we consider the inversion of the analysis procedure of Section 3.3. The possible synthesis methods arising in this manner appear to be undesirable both from the point of view of the complexity of the R-LLF device required and the difficulty of synthesizing the LLFPB portion of the LLF:R network. The latter difficulty arises from the fact that in the synthesis procedure a complete s.c. admittance matrix must be synthesized for a grounded two terminal pair RC network. In the phraseology of Chapter 1, Section 1.5.2, it was not possible to meet the "constructible" specifications requirement.

In Section 4.3 a method of inserting gain in the transfer functions by means of real linear transformations is considered. This method arises as an inverse of an analysis procedure of Section 3.4. It has already been stated that this gain insertion result has previously been found by Nashed and Stockham. Section 4.4 considers a synthesis method which involves making a congruent transformation of the s.c. admittance matrix after gain has been inserted by the method of Section 4.3. The congruent transformation is of the type arising when a new definition of terminal pairs is made for a MTP element. By using a simple congruent transformation in conjunction with gain insertion an RC-LLF:R voltage transfer function of a general character is found. The network consists of one three-terminal RC-LLFPB network and one three-terminal R-LLF device. A final synthesis algorithm was not developed for this voltage transfer function since the expression for this transfer function involved a specification of the complete s.c. admittance matrix of an RC-LLFPB grounded two terminal-pair network. Thus just as in Section 4.2 it was not possible to meet the "constructible" specifications requirement. It is found however that if this network is specialized to a π configuration and placed in parallel with an RC-LLFPB grounded two terminal pair, the constructible requirement can be met and still have a potentially general voltage transfer function. However attempts at finding an algorithm as required in Step 2 of Section 1.5.2 have not been successful.

4.2 Synthesis Through Transformation of the Branch Parameter Matrix

4.2.1 General Approach

In this Section we consider the problem of reversing the analysis procedure of Section 3.3. Such a reverse procedure consists of starting with an LLFPB network that has a specific type of configuratio as exemplified by the network of Figure 3.3.2. The application of real linear transformations converts the positive resistance box to an R-LLF device but leaves the LLFPB sub-network unchanged. As a result the o.c. impedance matrix $\hat{Z}_{dd}$ seen from the set of terminal pairs r + 1 through r + d for the LLFPB network and the corresponding o.c. impedance matrix Z_{dd} for the LLF:R network become related through pre- and post-multiplication by real transformation matrices as indicated below

$$Z_{dd} = Q_{dd}^{-1} \hat{Z}_{dd} P_{dd}^{-1} \tag{4.2.1}$$

The s.c. admittance matrix seen from these terminal pairs then takes the form

$$Y_{dd} = P_{dd} \hat{Y}_{dd} Q_{dd} \tag{4.2.2}$$

where

$$Y_{dd} = Z_{dd}^{-1}$$

$$\hat{Y}_{dd} = \hat{Z}_{dd}^{-1} \tag{4.2.3}$$

are the s.c. admittance matrix of the LLF:R and the LLFPB network, respectively. We will assume that the LLFPB network is RC. Then we readily see that the s.c. admittance poles and the o.c. impedance poles of the LLF:R network are the same as the corresponding ones for the LLFPB network. Thus if we wish to achieve complex natural frequencies for the RC-LLF:R network we may not have either all o.c. constraints or all s.c. constraints at all of terminal pairs. By selecting the transformation matrices P_{dd} and Q_{dd} appropriately we should be able to form a driving point admittance y_{jj}, say, at terminal pair j of the RC-LLF:R network which has zeroes at specified locations in the complex plane. This comes about from the fact that Equation 4.2.2 implies that not only transfer but driving point admittances of the RC-LLF:R network are expressed as linear combinations of the driving point and transfer admittances of the RC-LLFPB network. If we open circuit terminal-pair j and leave the other terminal pairs short-circuited, the resulting network has complex poles where y_{jj} had zeroes. Thus voltage transfer functions between the other d - 1 terminal pairs and terminal pair j or transfer admittances among the d - 1 terminal pairs should have zeroes and poles which may be placed quite generally in the complex plane

To illustrate the above ideas let d = 2 and r = 2 in Figures 3.3.1 and 3.3.2. The resulting reference RC-LLFPB network and the corresponding RC-LLF:R network are shown in Figures 4.2.1a and 4.2.1b respectively.

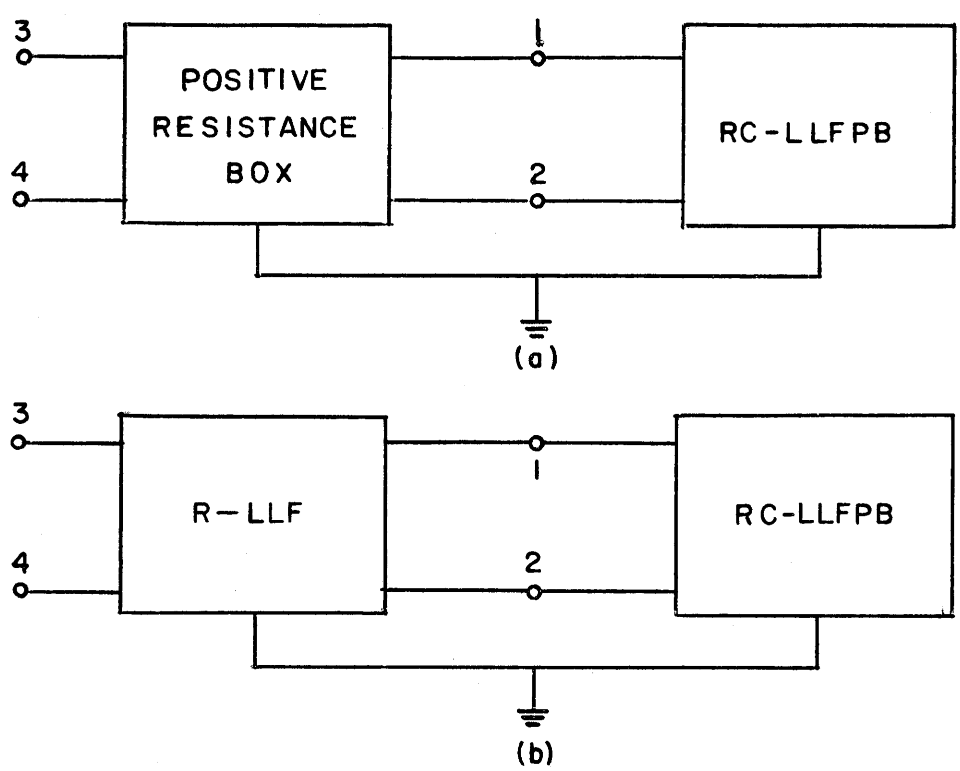


Figure 4.2.1. Networks for Example of Section 4.2

Since d = 2, the matrices Y_{dd} and Y_{dd} are 2 x 2 and for the networks of Figure 4.2.1 correspond to s.c. admittance matrices of grounded two terminal-pair networks. Let the transformation matrices P_{dd} and Q_{dd} be given by

$$P_{dd} = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} \qquad Q_{dd} = \begin{bmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{bmatrix} \tag{4.2.4}$$

and the s.c. admittance matrices Y_{dd}, $\hat{Y}_{dd}$ by

$$Y_{dd} = \begin{bmatrix} y_{33} & y_{34} \\ y_{43} & y_{44} \end{bmatrix} \qquad Y_{dd} = \begin{bmatrix} \hat{y}_{33} & \hat{y}_{34} \\ \hat{y}_{34} & \hat{y}_{44} \end{bmatrix} \tag{4.2.5}$$

Then Equation 4.2.2 takes the more detailed form

$$\begin{bmatrix} y_{33} & y_{34} \\ y_{43} & y_{44} \end{bmatrix} = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} \begin{bmatrix} \hat{y}_{33} & \hat{y}_{34} \\ \hat{y}_{34} & \hat{y}_{44} \end{bmatrix} \begin{bmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{bmatrix} \tag{4.2.6}$$

By appropriate choice of P_{dd} and Q_{dd} we can form an expression for y_{33} that may have zeroes anywhere in the complex plane. Thus from Equation 4.2.6 we find that y_{33} has the general expression

$$y_{33} = \hat{y}_{33}p_{11}q_{11} + \hat{y}_{34}\left[p_{11}q_{21} + q_{11}p_{12}\right] + \hat{y}_{44}p_{12}q_{21} \tag{4.2.7}$$

Now as a general rule we can say that the zeroes of transfer functions or the zeroes of the difference between two driving point functions can have complex plane zeroes of unrestricted nature. Thus we should specialize the coefficients in Equation 4.2.7 so that $\hat{y}_{33}$ is either directly proportional to $\hat{y}_{34}$ or to the difference of $\hat{y}_{33}$ and $\hat{y}_{44}$. Consider the following possibilities

$$(a)\ p_{12} = q_{11} = 0 \qquad \text{then } y_{33} = p_{11}q_{21}\hat{y}_{34}$$

$$(b)\ p_{11} = q_{21} = 0 \qquad \text{then } y_{33} = q_{11}p_{1q}\hat{y}_{34} \tag{4.2.8}$$

$$(c)\ p_{11}q_{21} + q_{11}p_{12} = 0 \text{ then } y_{33} = \hat{y}_{33}p_{11}q_{11} + \hat{y}_{44}\ p_{12}\ q_{21}$$

Actually case (c) is more general than cases (a) or (b) since the difference between two driving point functions has zeroes which may be placed arbitrarily while the zeroes of y_{34} may not be placed arbitrarily since it is a grounded transfer function. Let us then consider case (c) with the further specialization

$$p_{11} = q_{11} = 1 \tag{4.2.9}$$

If we let

$$p_{12} = -q_{21} = x \tag{4.2.10}$$

Then y_{33} takes the form

$$y_{33} = \hat{y}_{33} - x^2\hat{y}_{44} \tag{4.2.11}$$

If we now open circuit terminal pair 3 and leave terminal pair 4 short circuited, the poles of the RC-LLF:R network will be determined by the zeroes of $\hat{y}_{33} - x^2\hat{y}_{44}$. The voltage transfer function from terminal pair 4 to 3 is given by

$$a_{34} = -\frac{y_{34}}{y_{33}} \tag{4.2.12}$$

and the general expression for y_{34} obtained from Equation 4.2.6 is

$$y_{34} = \hat{y}_{33}p_{11}q_{12} + \hat{y}_{34}\left[p_{11}q_{22} + p_{12}q_{12}\right] + \hat{y}_{44}p_{12}q_{22} \tag{4.2.13}$$

With the parameter values of Equations 4.2.9 and 4.2.10, y_{34} becomes

$$y_{34} = -\hat{y}_{33} + \hat{y}_{34}\left[q_{22} - xq_{12}\right] + \hat{y}_{44}xq_{22} \tag{4.2.14}$$

The expression for y_{34} may be simplified by letting

$$q_{22} = 0 \ ; \ q_{12} = 1 \tag{4.2.15}$$

Our final expression for the voltage transfer ratio a_{34} becomes

$$a_{34} = \frac{\hat{y}_{33} + x^2\hat{y}_{34}}{\hat{y}_{33} - x^2\hat{y}_{44}} \tag{4.2.16}$$

We have thus arrived at an RC-LLF:R voltage transfer function which is potentially capable of having considerable generality in the location of poles and zeroes. But, as discussed in Chapter 1, Section 1.5.2, three additional steps must be completed before one can specify a set of poles and zeroes for a_{34} and obtain the RC-LLF:R network which exhibits this transfer function. We are unable to complete the first step since the expression for a_{34} involves a specification of the complete s.c. admittance matrix of the reference RC-LLFPB network from terminal pairs 3 and 4. In fact it has not been found possible to specialize the elements of P_{dd} and Q_{dd} to alter this situation and at the same time obtain an a_{34} of a sufficiently general character. Thus the results of this section will be only of academic interest until further results are available on the necessary and sufficient conditions for realization of s.c. admittance matrices of grounded two-terminal-pair RC-LLFPB networks (containing no ideal transformers).

4.2.2 Example

In this section we will consider a simple choice for the positive resistance box of Figure 4.2.1a and then obtain the branch admittance matrix g_{ss} of the R-LLF device required in the network of Figure 4.2.1b such that the voltage transfer ratio of this network has the form of Equation 4.2.16.

The general expression for g_{ss} is obtained from Equations 3.3.27 and 3.3.28 as

$$g_{ss} = \begin{bmatrix} U_r & 0 \\ 0 & P_{dd} \end{bmatrix} \begin{bmatrix} \hat{g}_{rr} & \hat{g}_{rd} \\ \hat{g}_{dr} & \hat{g}_{dd} \end{bmatrix} \begin{bmatrix} U_r & 0 \\ 0 & Q_{dd} \end{bmatrix}$$

$$= \begin{bmatrix} \hat{g}_{rr} & \hat{g}_{rd}Q_{dd} \\ P_{dd}\hat{g}_{dr} & P_{dd}\hat{g}_{dd}Q_{dd} \end{bmatrix} \tag{4.2.17}$$

In our example d = r = 2 so that g_{ss} and $\hat{g}_{ss}$ are 4 x 4 branch parameter matrices. In Figure 4.2.2 a simple positive resistance box is shown.

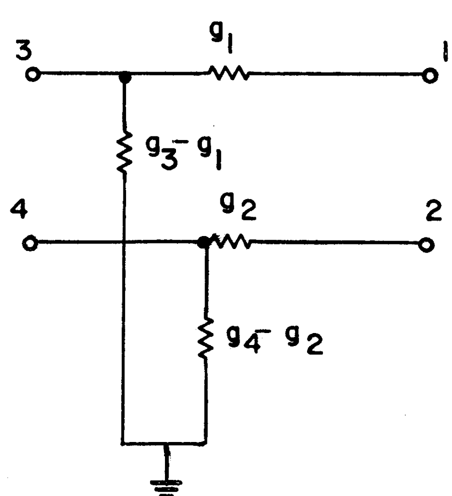


Figure 4.2.2. A Simple Positive Resistance Box

The resulting RC-LLFPB reference network is shown in Figure 4.2.3.

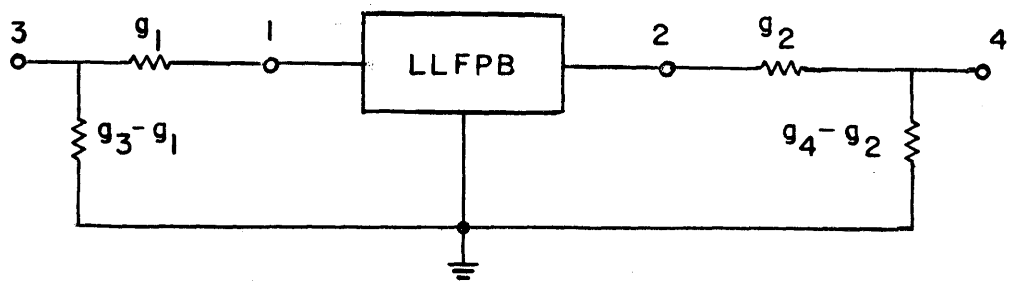


Figure 4.2.3. LLFPB Network With Simple Resistance Box

It should be noted that the positive resistance box consists of nothing more than resistance voltage dividers hung on the terminals of the RC-LLFPB subnetwork. Thus it may be expected that the synthesis for a prescribed set of driving point and transfer functions at terminal pairs 3 and 4 should not be essentially any more difficult than synthesizing for a set of driving point and transfer functions at terminal pairs 1 and 2. It should be noted that the driving point and transfer functions at terminal pairs 1 and 2 are those for the general grounded RC network with resistance termination at both terminal pairs.

The branch admittance parameter matrix of the positive resistance box is readily found to be

$$g_{ss} = \left[\begin{array}{cc|cc} g_1 & 0 & -g_1 & 0 \\ 0 & g_2 & 0 & -g_2 \\ \hline -g_1 & 0 & g_3 & 0 \\ 0 & -g_2 & 0 & g_4 \end{array}\right] \tag{4.2.18}$$

If the transformation parameter values given by Equations 4.2.9, 4.2.10, and 4.2.15 are used in Equation 4.2.4, the transformation matrices assume the form

$$P_{dd} = \begin{bmatrix} 1 & x \\ p_{21} & p_{22} \end{bmatrix} \qquad Q_{dd} = \begin{bmatrix} 1 & 1 \\ -x & 0 \end{bmatrix} \tag{4.2.19}$$

The transfer function a_{34} does not depend upon the coefficients p_{21} and p_{22} so we are free to choose them according to convenience. We will choose

$$p_{21} = 1$$

$$p_{22} = 0 \tag{4.2.20}$$

Since the resulting g_{ss} has an especially simple form when this is done. Following Equation 4.2.17

$$g_{rd} = \hat{g}_{rd}Q_{dd} = \begin{bmatrix} -g_1 & 0 \\ 0 & -g_2 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -x & 0 \end{bmatrix} = \begin{bmatrix} -g_1 & -g_1 \\ xg_2 & 0 \end{bmatrix}$$

$$g_{dr} = P_{dd}\hat{g}_{dr} = \begin{bmatrix} 1 & x \\ 1 & 0 \end{bmatrix}\begin{bmatrix} -g_1 & 0 \\ 0 & -g_2 \end{bmatrix} = \begin{bmatrix} -g_1 & -g_2x \\ -g_1 & 0 \end{bmatrix} \tag{4.2.21}$$

$$g_{dd} = P_{dd}\hat{g}_{dd}Q_{dd} = \begin{bmatrix} 1 & x \\ 1 & 0 \end{bmatrix}\begin{bmatrix} g_3 & 0 \\ 0 & g_4 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -x & 0 \end{bmatrix} = \begin{bmatrix} g_3-x^2g_4 & g_3 \\ g_3 & g_3 \end{bmatrix}$$

Thus the branch admittance matrix of the R-LLF MTP element is given by

$$g_{ss} = \left[\begin{array}{cc|cc} g_1 & 0 & -g_1 & -g_1 \\ 0 & g_2 & xg_2 & 0 \\ \hline -g_1 & -g_2 x & g_3 - x^2 g_4 & g_3 \\ -g_1 & 0 & g_3 & g_3 \end{array}\right] \qquad (4.2.22)$$

There are at present no general practical methods of synthesizing a multiterminal-pair R-LLF black box for prescribed s.c. admittance matrix. By a practical method it is meant a synthesis method which involves a realization in terms of practical devices such as vacuum tubes and transistors. In Section 4.2.3 a theoretical method is presented which involves a realization in terms of ideal vacuum tubes (or gyrators) and positive and negative resistances. Using the results of Section 4.2.3 it is shown in Section 4.2.4 how an L-LLF or C-LLF multiterminal-pair black box may be realized in terms of R-LLF devices plus positive capacitances and inductances.

4.2.3 Synthesis of R-LLF Element

We consider here the problem of synthesizing an R-LLF MTP element for prescribed branch admittance parameter matrix. It is assumed that the associated MP network is completely described by the MTP element. Thus if the MTP element has n terminal pairs it also has n + 1 nodes.

The first step in the synthesis procedure consists of changing the synthesis specifications to a node to datum branch admittance parameter matrix. This is readily accomplished with the results of

Section 2.3.3. Thus let g'_{ss} be the original specified branch admittance matrix. It is assumed that the coupled tree branch representation of the desired network is given together with g'_{ss}. Then we select any node as datum and determine the branch admittance matrix g_{ss} that accompanies this selection of node to datum variables. According to Section 2.3.3 the relationship between g'_{ss} and g_{ss} is

$$g_{ss} = \alpha g'_{ss} \alpha^t \tag{4.2.23}$$

where α is a cut-set matrix which defines the node-to-datum variables of the second MTP element upon the coupled branch representation of the first originally specified MTP element. Clearly a synthesis of g_{ss} for the node-to-datum case automatically synthesizes g'_{ss} for the original case.

The second step in the synthesis procedure consists in showing how the n x n node-to-datum admittance matrix g_{ss} can be synthesized from node-to-datum MTP elements with 2 x 2 admittance matrices. Let N denote the MTP element with parameter matrix g_{ss}. Since node to datum variables are assigned for N, it has the coupled tree branch representation of Figure 4.2.4.

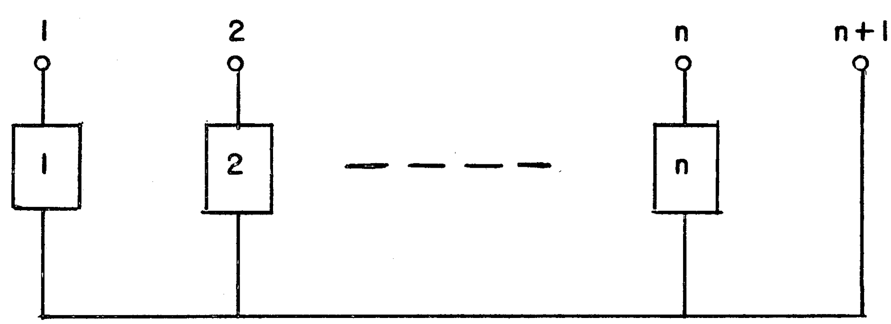


Figure 4.2.4. Coupled Tree Branch Representation of N

Consider each branch of N to be divided into n - 1 sub-branches. The sub-branches of a particular branch are not coupled to one another but each sub-branch of a branch k is coupled to exactly one sub-branch of some other branch j and to no other sub-branches in the network. Suppose the pair of coupling coefficients between branches j and k is $\left[g_{jk}, g_{kj}\right]$. Then the coupled sub-branches of branches j and k, denoted as j' and k', are assumed to have the pair of coupling coefficients $\left[g_{jk}, g_{kj}\right]$. The self admittances of the sub-branches of a particular branch j of N are constrained only in the respect that their sum must equal the self admittance of the original branch j. If we remove a pair of coupled sub-branches j', k' from N, it is readily seen that the coupling between branches j and k of the resultant network becomes zero and the self admittances of branches j and k are reduced by the self admittances of j' and k'. We note that this pair of coupled sub branches is an MTP element with three terminals and node to datum variables assigned. If becomes clear that successive removal of all pairs of coupled sub branches will leave a network with a null parameter matrix. If now the three-terminal MTP elements are reinserted in the same locations from which they were removed one obtains the network N with parameter matrix g_{ss}.

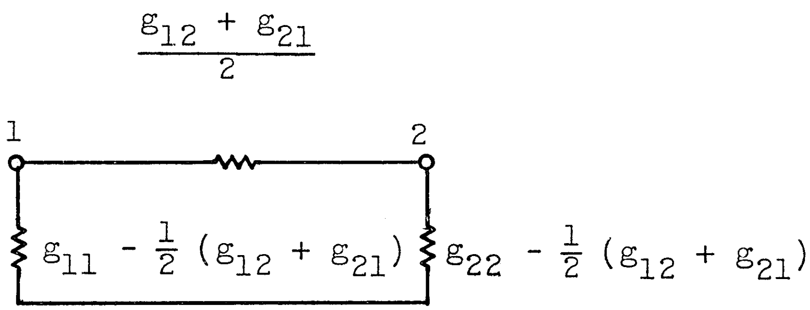
The last step consists of synthesizing the three terminal building blocks. As is well known each building block may be constructed from ideal vacuum tubes (or gyrators) and resistances (positive and negative). This is readily done as follows. Let

$$g = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \qquad (4.2.24)$$

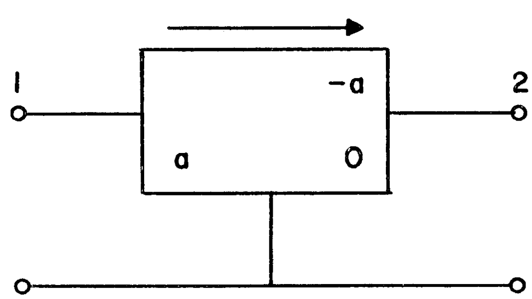
be the branch admittance matrix of a typical three terminal building block. The matrix g is separated into the sum of a symmetric and a skew symmetric matrix as follows

$$\begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} = \begin{bmatrix} g_{11} & \dfrac{[g_{12} + g_{21}]}{2} \\ \dfrac{[g_{12} + g_{21}]}{2} & g_{22} \end{bmatrix} + \begin{bmatrix} 0 & \dfrac{-[g_{21} - g_{12}]}{2} \\ \dfrac{[g_{21} - g_{12}]}{2} & 0 \end{bmatrix} \tag{4.2.25}$$

The first matrix can be constructed from a network containing positive and negative resistances as indicated below



The second matrix is that of a gyrator[22]. We will use the following circuit symbol to denote a gyrator with transfer admittance a from terminal pair 1 to terminal pair 2. The arrow



denotes the fact that a is the transfer admittance from the left hand to the right hand terminal pair.

A parallel connection yields the desired branch parameter matrix as shown in Figure 4.2.5.

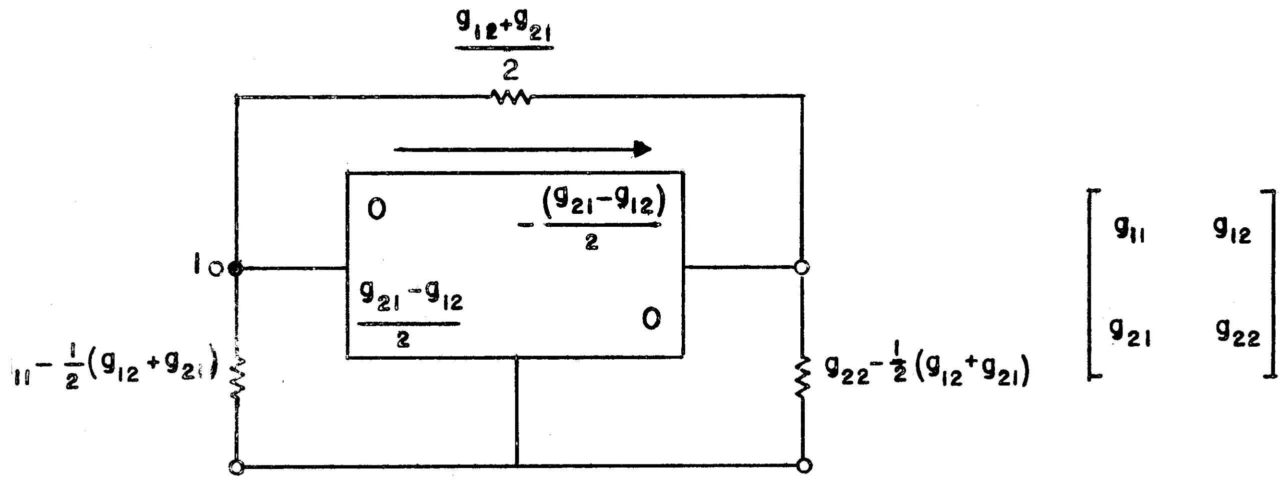
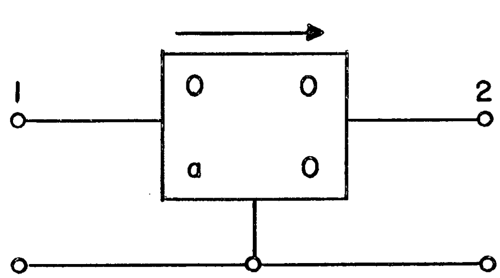


Figure 4.2.5. Realization of Building Block With Gyrator and Resistances

There is an alternate realization of g in terms of an ideal vacuum tube and resistances. To obtain this realization we need only find a realization of a gyrator in terms of an ideal vacuum tube and resistances. An ideal vacuum tube has the branch admittance matrix and circuit symbol below,

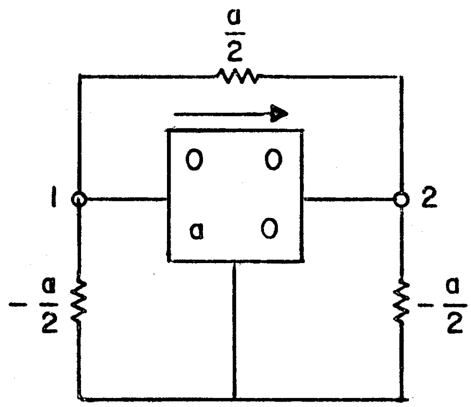
$$G = \begin{bmatrix} 0 & 0 \\ a & 0 \end{bmatrix}$$



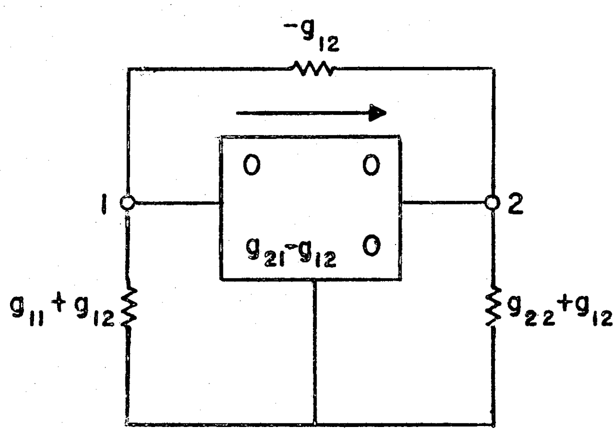
where $a > 0$. The typical gyrator admittance matrix can be regarded as the sum of the ideal VT admittance matrix and another matrix as follows

$$\begin{bmatrix} 0 & -\frac{a}{2} \\ \frac{a}{2} & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ a & 0 \end{bmatrix} + \begin{bmatrix} 0 & -\frac{a}{2} \\ -\frac{a}{2} & 0 \end{bmatrix} \tag{4.2.26}$$

Thus a gyrator has the realization indicated below



Consequently the typical building block may have the realization indicated in Figure 4.2.6.

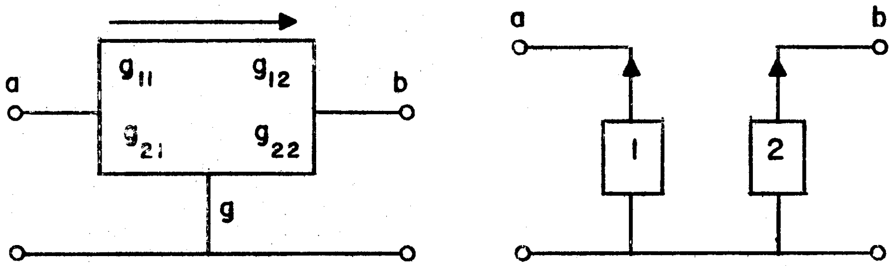


$$\begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix}$$

Figure 4.2.6. Realization of Building Block With Ideal Vacuum Tube and Resistances

The particular realization shown assumes that $g_{21} > g_{12}$. If this is not true then the appropriate realization is found by interchanging the numbers 1 and 2 on the diagram.

It will be convenient to define a circuit symbol for the general three terminal R-LLF device with a prescribed s.c. admittance matrix. This symbol is shown in Figure 4.2.7 together with the corresponding set of coupled branches and its branch admittance parameter matrix.



$$\begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix}$$

Figure 4.2.7. The General Grounded Two Terminal Pair
R-LLF Circuit Element

The arrow denotes that the lower left transconductance g_{21} is the s.c. transfer admittance from terminal-pair a-ground to terminal pair b-ground, i.e., a source to sink relationship in the arrow direction.

As an example of the application of the R-LLF MTP element synthesis technique of this section we may consider the synthesis of the matrix g_{ss} of Equation 4.2.22. Since the procedure is straightforward only the final result is given. This is shown in Figure 2.4.8. Note that only one three-terminal non-bilateral element is required. This can be deduced by inspection of g_{ss} which shows that non-bilaterality exists only between branches 2 and 3.

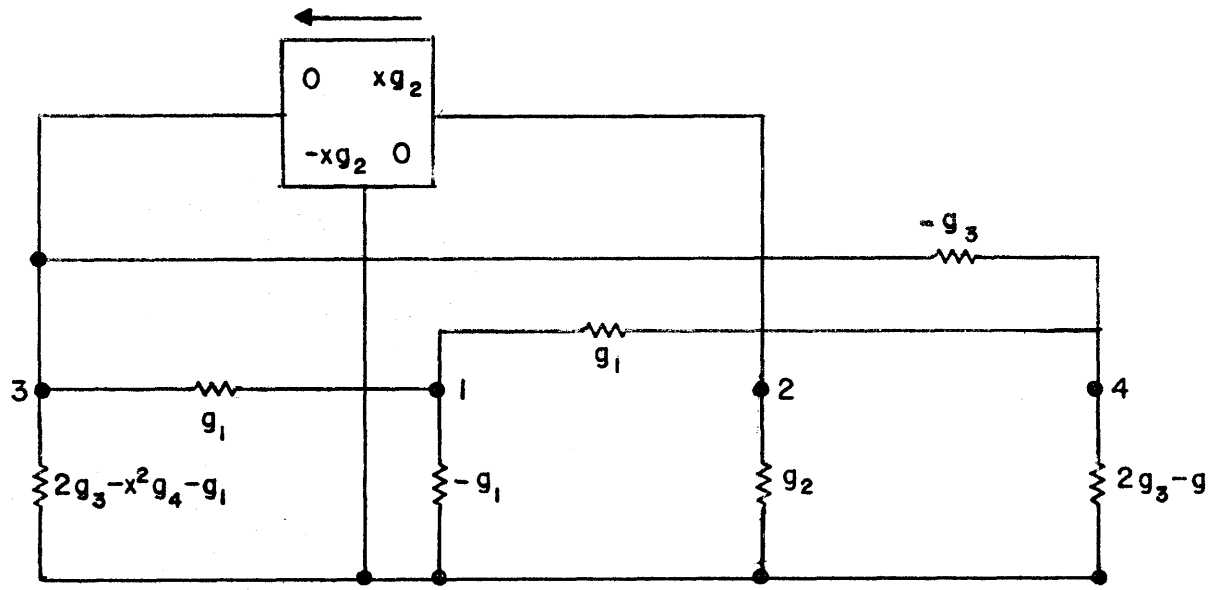


Figure 4.2.8. Synthesis of R-LLF MTP Element For Example of Section 4.2.2

4.2.4 Synthesis of L-LLF and C-LLF Elements

From the results of Section 4.2.3 it is readily seen that an L-LLF or C-LLF MTP element can be synthesized for prescribed branch admittance matrix if the corresponding typical three terminal building block can be constructed. For a capacitive MTP element the building block has the parameter matrix

$$y_c = s \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \tag{4.2.27}$$

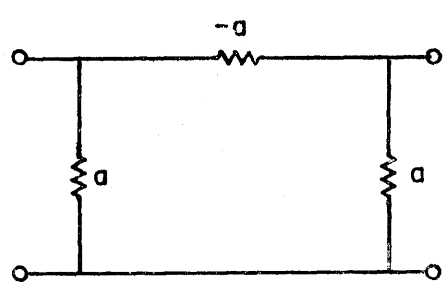
while for an inductive MTP element the building block has the parameter matrix

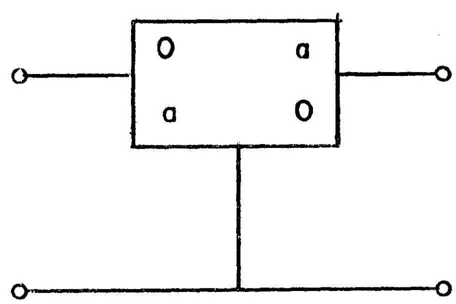
$$y_\gamma = \frac{1}{s} \begin{bmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{bmatrix} \tag{4.2.28}$$

It is also clear from the discussion of the previous section that these typical blocks can be built if negative capacitors and inductors plus the counterpart of the ideal vacuum tube (IVT) are available. We shall now demonstrate that a negative capacitance (inductance) can be built from a positive inductance (capacitance) and positive and negative resistances. Then we shall show that the counterpart of the IVT for the capacitive case (inductive case) is constructible from two IVT's and one inductance (capacitance) apart from a change in sign of the transfer admittance which does not affect the synthesis procedure. Let the three terminal element with branch parameter matrix

$$\begin{bmatrix} 0 & a \\ a & 0 \end{bmatrix}$$

be called an activator. This element has the realization and circuit symbol shown below.





One may readily demonstrate that a unit (a = 1) activator is a negative admittance inverter. Thus if a capacitance of C farads is connected across one terminal pair, the admittance seen at the other terminal pair is $-\frac{1}{sc}$. This latter admittance is precisely that of a negative inductance of C henries. A similar argument follows for the construction of a negative capacitance. Figure 4.2.9 shows the realization of a unit negative inductance and a unit negative capacitance is the fashion just described.

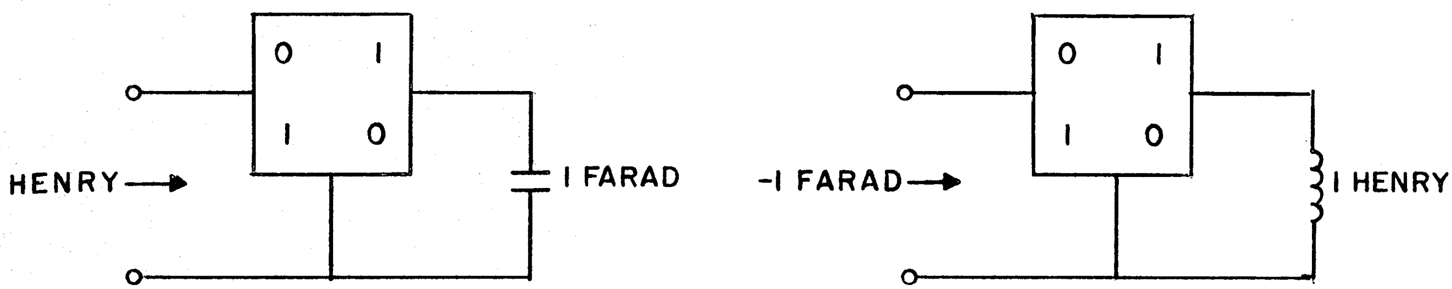


Figure 4.2.9. Realization Of Negative Inductance And Capacitance

Other values of inductance and capacitance are obtained by impedance leveling. The realization of capacitive and inductive versions of the IVT are demonstrated in Figure 4.2.10.

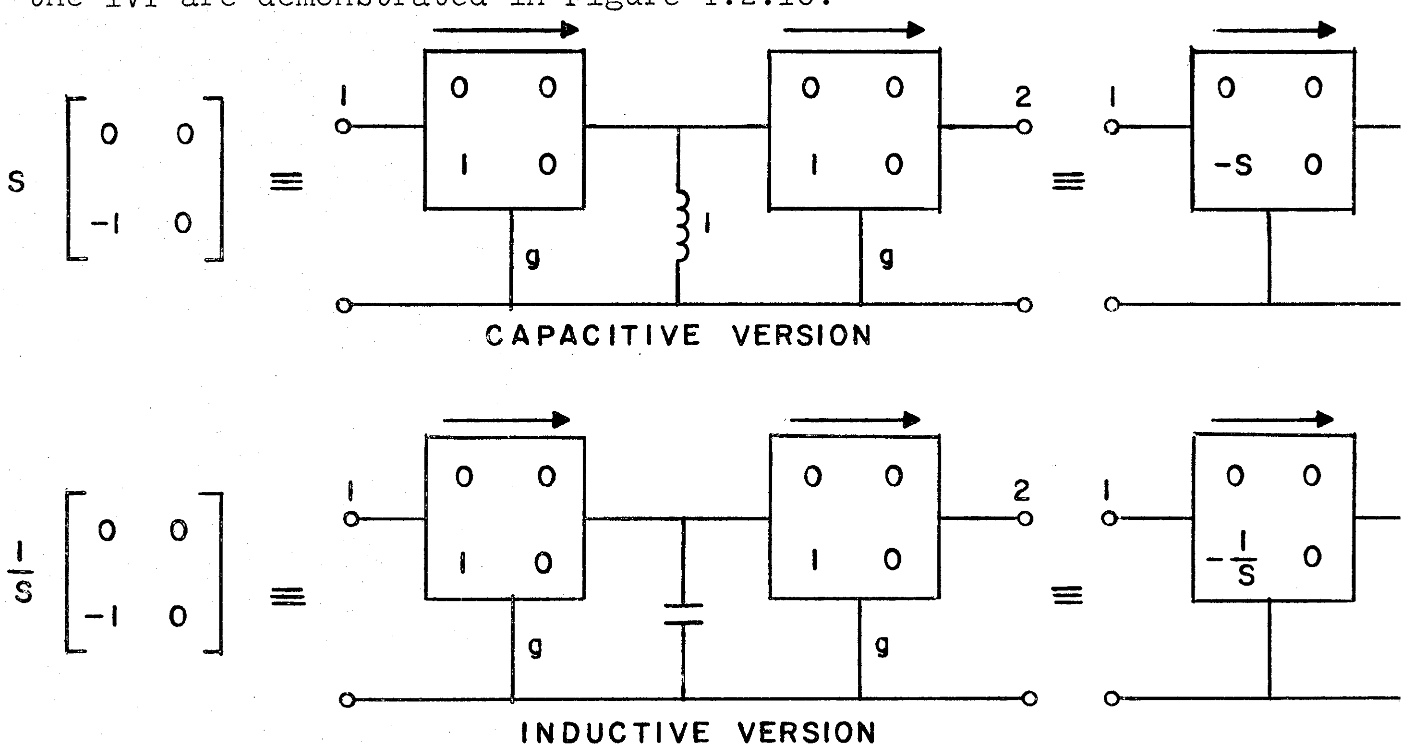


Figure 4.2.10. Realization Of Capacitive And Inductive Versions Of Ideal Vacuum Tube

Note that the sign of the transfer admittances are negative. If positive signs are desired they may be obtained in a variety of ways. But from a theoretical point of view this is not necessary since it is easy to see that the three terminal building block may be constructed with devices of either algebraic sign.

4.3 Gain Insertion

In this section we consider the inversion of a very special linear transformation theory analysis procedure which is found in Section 3.4. For this analysis procedure the o.c. impedance matrix of the LLF:R network and that of the LLFPB reference network are related according to Equation 3.4.17. Examination of this equation indicates that this analysis procedure will lead to a method whereby gain may be inserted in transfer impedances. We note, however, that the LLFPB network which is to have gain inserted is restricted with regard to topology as is clearly indicated by the form of $\hat{Y}$ in Equation 3.4.9. From the discussion centering around this equation one may readily deduce that gain may be inserted between two terminal pairs of an LLFPB network in the fashion indicated by Equation 3.4.17 only if these terminal pairs can be associated with MTP elements which have at most one node in common. Thus if these MTP elements are to be coupled they must be coupled by resistance coupling. It may be deduced from Equations 3.4.11 to 3.4.14 that it is this passive resistance coupling which becomes active and nonbilateral when gain is inserted. To illustrate these ideas we may assume that the equilibrium matrix of the LLFPB network, $\hat{Y}_{nn}$, has the form

$$\hat{Y}_{nn} = \hat{Y} + \hat{G} \tag{4.3.1}$$

where

$$\hat{Y} = \begin{bmatrix} \hat{y}_1 & 0 & 0 \\ 0 & \hat{y}_2 & 0 \\ 0 & 0 & \hat{y}_3 \end{bmatrix} \tag{4.3.2}$$

is the combined Simple and Complete node parameter matrix of three isolated LLFPB MTP elements and

$$\hat{G} = \begin{bmatrix} \hat{g}_{kk} & \hat{g}_{kr} & \hat{g}_{kd} \\ \hat{g}_{rk} & \hat{g}_{rr} & \hat{g}_{rd} \\ \hat{g}_{dk} & \hat{g}_{dr} & \hat{g}_{dd} \end{bmatrix} \tag{4.3.3}$$

is the Simple and Complete node parameter matrix of a positive resistance box which provides coupling between elements 1, 2 and 3. We may then write $\hat{Y}_{nn}$ in the form

$$\hat{Y}_{nn} = \begin{bmatrix} \hat{y}_{kk} & \hat{g}_{kr} & \hat{g}_{kd} \\ \hat{g}_{rk} & \hat{y}_{rr} & \hat{g}_{rd} \\ \hat{g}_{dk} & \hat{g}_{dr} & \hat{y}_{dd} \end{bmatrix} \tag{4.3.4}$$

where

$$\begin{aligned} \hat{y}_{kk} &= \hat{y}_1 + \hat{g}_{kk} \\ \hat{y}_{rr} &= \hat{y}_2 + \hat{g}_{rr} \\ \hat{y}_{dd} &= \hat{y}_3 + \hat{g}_{dd} \end{aligned} \tag{4.3.5}$$

Gain can be inserted between two terminal pairs so long as they cannot both be associated with the same LLFPB element (y_1, y_2, y_3). This gain insertion is accomplished by pre- and post-multiplying Y_{nn} by transformation matrices as indicated below to form Y_{nn} - the node equilibrium matrix of an LLF:R network with the required gain.

$$Y_{nn} = \begin{bmatrix} a^{-1}U_k & 0 & 0 \\ 0 & b^{-1}U_r & 0 \\ 0 & 0 & c^{-1}U_d \end{bmatrix} \begin{bmatrix} \hat{y}_{kk} & \hat{g}_{kr} & \hat{g}_{kd} \\ \hat{g}_{rk} & \hat{y}_{rr} & \hat{g}_{rd} \\ \hat{g}_{dk} & \hat{g}_{dr} & \hat{y}_{dd} \end{bmatrix} \begin{bmatrix} aU_k & 0 & 0 \\ 0 & bU_r & 0 \\ 0 & 0 & cU_d \end{bmatrix}$$

$$= \begin{bmatrix} \hat{y}_{kk} & \frac{b}{a}\hat{g}_{kr} & \frac{c}{a}\hat{g}_{kd} \\ \frac{a}{b}\hat{g}_{rk} & \hat{y}_{rr} & \frac{c}{b}\hat{g}_{rd} \\ \frac{a}{c}\hat{g}_{dk} & \frac{b}{c}\hat{g}_{rr} & \hat{y}_{dd} \end{bmatrix} \qquad (4.3.6)$$

To see how the o.c. impedances of the LLF:R network are related to those of the LLFPB network we invert Equation 4.3.6 to obtain

$$Z_{nn} = Y_{nn}^{-1} = \begin{bmatrix} a^{-1}U_k & 0 & 0 \\ 0 & b^{-1}U_r & 0 \\ 0 & 0 & c^{-1}U_d \end{bmatrix} \begin{bmatrix} \hat{z}_{rr} & \hat{z}_{rk} & \hat{z}_{rd} \\ \hat{z}_{kr} & \hat{z}_{kk} & \hat{z}_{kd} \\ \hat{z}_{dr} & \hat{z}_{dk} & \hat{z}_{dd} \end{bmatrix} \begin{bmatrix} aU_k & 0 & 0 \\ 0 & bU_r & 0 \\ 0 & 0 & cU_d \end{bmatrix}$$

$$= \begin{bmatrix} \hat{z}_{rr} & \frac{b}{a}\hat{z}_{rk} & \frac{c}{a}\hat{z}_{rd} \\ \frac{a}{b}\hat{z}_{kr} & \hat{z}_{kk} & \frac{c}{b}\hat{z}_{kd} \\ \frac{a}{c}\hat{z}_{dr} & \frac{b}{c}\hat{z}_{dk} & \hat{z}_{dd} \end{bmatrix} \qquad (4.3.7)$$

where

$$\hat{Z}_{nn} = \hat{Y}_{nn}^{-1} = \begin{bmatrix} \hat{z}_{rr} & \hat{z}_{rk} & \hat{z}_{rd} \\ \hat{z}_{kr} & \hat{z}_{kk} & \hat{z}_{kd} \\ \hat{z}_{dr} & \hat{z}_{dk} & \hat{z}_{dd} \end{bmatrix} \tag{4.3.8}$$

is the o.c. impedance matrix of LLFPB network. We note that the transformation effected upon the s.c. admittance matrix is the same as that effected upon its inverse, the o.c. impedance matrix. This arises from the fact that $\hat{Y}_{nn}$ in Equation 4.3.6 is subjected to a collinear transformation. Examination of Equation 4.3.7 shows that if gain is inserted between two terminal pairs associated with different MTP elements then the same gain is inserted between all terminal pairs.

The topology of the LLF:R network (coupled branch topology) is the same as the original LLFPB network. Also the first three MTP elements with parameter matrices $\hat{y}_1, \hat{y}_2$, and $\hat{y}_3$ are the same in both networks. However the positive resistance MTP element of the LLFPB network with parameter matrix $\hat{G}$ is replaced in the LLF:R network by an R-LLF box with parameter matrix

$$G = \begin{bmatrix} \hat{g}_{kk} & \frac{b}{a}\hat{g}_{kr} & \frac{c}{a}\hat{g}_{kd} \\ \frac{a}{b}\hat{g}_{rk} & \hat{g}_{rr} & \frac{c}{b}\hat{g}_{rd} \\ \frac{a}{c}\hat{g}_{dk} & \frac{b}{c}\hat{g}_{dr} & \hat{g}_{dd} \end{bmatrix} \tag{4.3.9}$$

It has been stated previously that Nashed and Stockham have also arrived at essentially the same method of gain insertion. It was also mentioned that there was a contradiction between statements made by Nashed with regard to the type of LLFPB network configuration permitting gain insertion and an actual gain insertion method given by Stockham. The reason for Nashed's incorrect conclusion with regard to the type of network configuration permitting gain insertion is that his LLFPB equilibrium matrix was only a special case of Equation 4.3.4. In his equilibrium matrix the submatrices of the equilibrium matrix of Equation 4.3.4 became single elements. Since he assumed node to datum variables he interpreted his equilibrium matrix as arising from a network of the configuration indicated in Figure 4.3.1. This type of network is more restricted than need be for gain insertion.

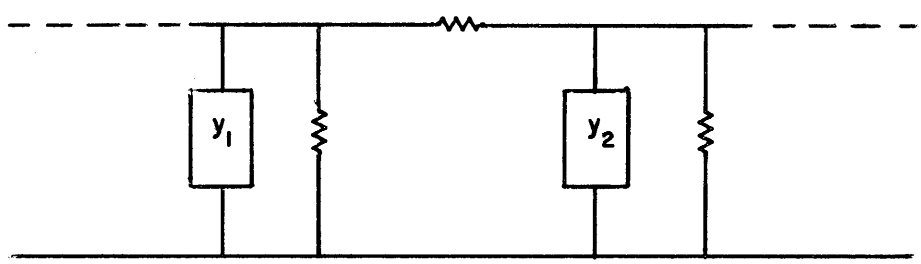


Figure 4.3.1. A Restricted Network Allowing Gain Insertion

To illustrate Stockham's particular result, specialize the $\hat{Y}_{nn}$, $\hat{Y}$, and $\hat{G}$ matrices to the form shown below

$$\hat{Y} = \left[\begin{array}{cc|cc} \hat{y}_{11} & \hat{y}_{12} & 0 & 0 \\ \hat{y}_{12} & \hat{y}_{22} & 0 & 0 \\ \hline 0 & 0 & \hat{y}_{33} & \hat{y}_{34} \\ 0 & 0 & \hat{y}_{34} & \hat{y}_{44} \end{array}\right] = \begin{bmatrix} \hat{y}_1 & 0 \\ 0 & \hat{y}_2 \end{bmatrix} \qquad \hat{G} = \left[\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & \hat{g}_{55} & -\hat{g} & 0 \\ \hline 0 & -\hat{g} & \hat{g}_{66} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] = \begin{bmatrix} \hat{g}_{rr} & \hat{g}_{rk} \\ \hat{g}_{kr} & \hat{g}_{kk} \end{bmatrix}$$

$$\hat{Y}_{nn} = \left[\begin{array}{cc|cc} \hat{y}_{11} & \hat{y}_{12} & 0 & 0 \\ \hat{y}_{12} & \hat{y}_{22} + \hat{g}_{55} & -\hat{g} & 0 \\ \hline 0 & -\hat{g} & \hat{y}_{33} + \hat{g}_{66} & \hat{y}_{34} \\ 0 & 0 & \hat{y}_{34} & \hat{y}_{44} \end{array}\right] = \begin{bmatrix} \hat{y}_{rr} & \hat{g}_{rk} \\ \hat{g}_{kr} & \hat{y}_{kk} \end{bmatrix} \qquad (4.3.10)$$

We will assume that MTP elements 1 and 2 are ungrounded two terminal-pair networks. Thus these elements will only provide a partial description of their associated MP networks. However, these elements will be connected with the positive resistance box to form the LLFPB network in such a way that these partial descriptions are satisfactory. In Figure 4.3.2 the three elements are shown with their coupled branch representations.

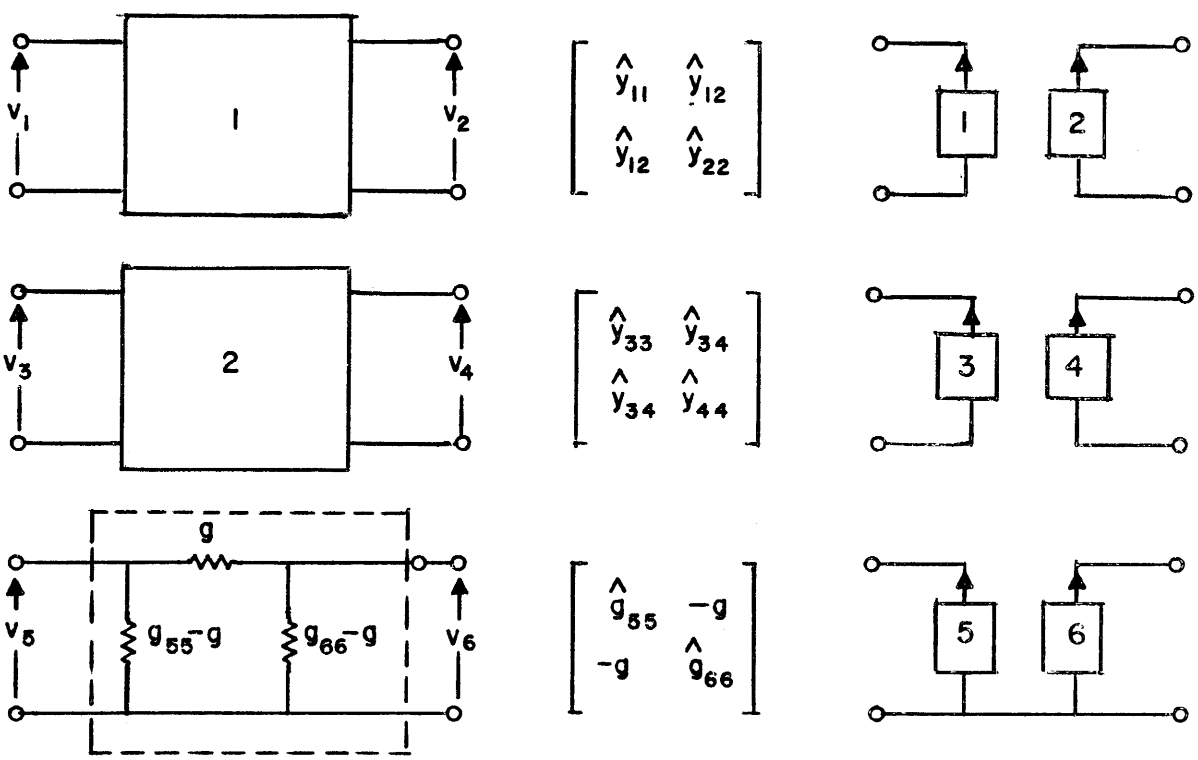


Figure 4.3.2. LLFPB MTP Elements For Network Of Figure 4.3.3

Figure 4.3.3 shows the interconnection of these elements to form an LLFPB network with equilibrium matrix $\hat{Y}_{nn}$ given by Equation 4.3.10.

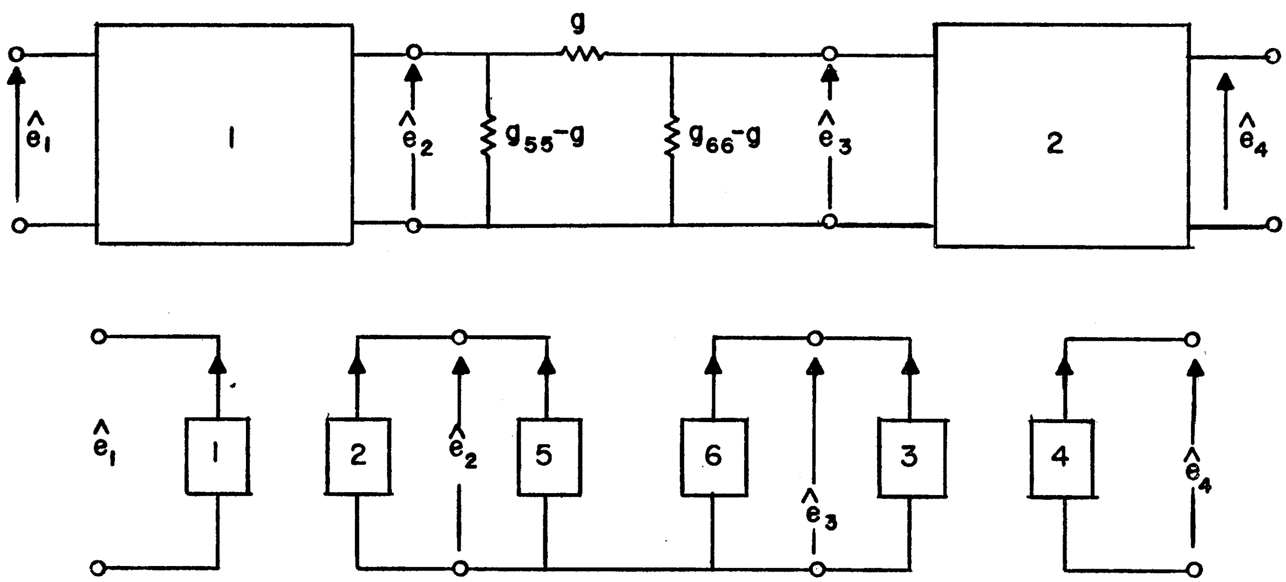


Figure 4.3.3. LLFPB Network To Have Gain Inserted

Now according to the general discussion given above we can insert gain between the following groups of terminal pairs (1,2), (2,3), (3,4) but not between 1 and 2 or between 3 and 4. We insert gain by forming Y_{nn} as follows

$$Y_{nn} = \begin{bmatrix} a^{-1}U_2 & 0 \\ 0 & b^{-1}U_2 \end{bmatrix} \begin{bmatrix} \hat{y}_{rr} & \hat{g}_{rk} \\ \hat{g}_{kr} & \hat{y}_{kk} \end{bmatrix} \begin{bmatrix} aU_2 & 0 \\ 0 & bU_2 \end{bmatrix} = \begin{bmatrix} \hat{y}_{rr} & \frac{b}{a}\hat{g}_{rk} \\ \frac{a}{b}\hat{g}_{kr} & \hat{y}_{kk} \end{bmatrix}$$

$$\begin{bmatrix} \hat{y}_1 & 0 \\ 0 & \hat{y}_2 \end{bmatrix} + \begin{bmatrix} \hat{g}_{rr} & \frac{b}{a}\hat{g}_{rk} \\ \frac{a}{b}\hat{g}_{kr} & \hat{g}_{kk} \end{bmatrix} \qquad (4.3.11)$$

The node parameter matrix of the R-LLF device is readily obtained as

$$G = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & g_{55} & -\frac{b}{a}\hat{g} & 0 \\ 0 & -\frac{a}{b}\hat{g} & g_{66} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad (4.3.12)$$

The open circuit impedance matrices Z_{nn} and $\hat{Z}_{nn}$ are related by

$$Z_{nn} = \begin{bmatrix} a^{-1}U_2 & 0 \\ 0 & b^{-1}U_2 \end{bmatrix} \begin{bmatrix} \hat{z}_{rr} & \hat{z}_{rk} \\ \hat{z}_{kr} & \hat{z}_{kk} \end{bmatrix} \begin{bmatrix} aU_2 & 0 \\ 0 & bU_2 \end{bmatrix} = \begin{bmatrix} \hat{z}_{rr} & \frac{b}{a}\hat{z}_{rk} \\ \frac{a}{b}\hat{z}_{kr} & \hat{z}_{kk} \end{bmatrix}$$

$$= \begin{bmatrix} z_{11} & z_{12} & z_{13} & z_{14} \\ z_{21} & z_{22} & z_{23} & z_{24} \\ z_{31} & z_{32} & z_{33} & z_{34} \\ z_{41} & z_{42} & z_{43} & z_{44} \end{bmatrix} = \begin{bmatrix} \hat{z}_{11} & \hat{z}_{12} & \frac{b}{a}\hat{z}_{13} & \frac{b}{a}\hat{z}_{14} \\ \hat{z}_{12} & \hat{z}_{22} & \frac{b}{a}\hat{z}_{23} & \frac{b}{a}\hat{z}_{24} \\ \frac{a}{b}\hat{z}_{13} & \frac{a}{b}\hat{z}_{23} & \hat{z}_{33} & \hat{z}_{34} \\ \frac{a}{b}\hat{z}_{14} & \frac{a}{b}\hat{z}_{24} & \hat{z}_{34} & \hat{z}_{44} \end{bmatrix} \qquad (4.3.13)$$

Thus the transfer impedances are changed by a factor $\frac{a}{b}$ when going from terminal pairs (1,2) to terminal pairs (3,4) but are multiplied by the reciprocal $\frac{b}{a}$ when going in the oppositve direction. The LLF:R network with the gain inserted is shown in Figure 4.3.4.

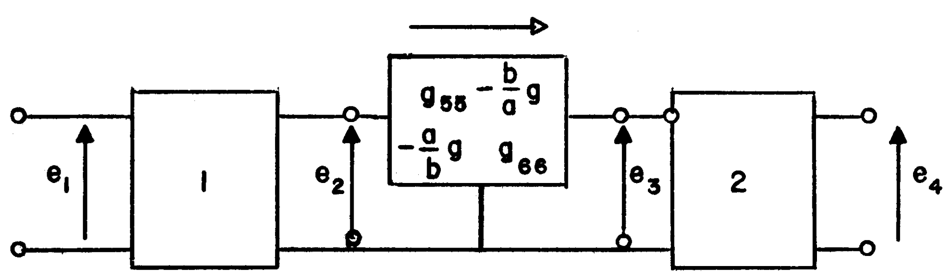


Figure 4.3.4. LLF:R Network With Gain Inserted

The circuit symbol for the R-LLF element is in accordance with the general definition of Figure 4.2.4.

4.4 Application of Congruent Transformation After Gain Insertion

While the synthesis method of the previous section allows gain to be inserted into R-C networks it does not change the locations of the poles or zeroes of the elements of $\hat{Y}_{nn}$ or $\hat{Z}_{nn}$. We will show that if a simple congruent transformation is applied after gain insertion, the zeroes of driving point impedances and admittances may be caused to become complex. This congruent transformation if of the type arising when a new definition of terminal pairs is made for an MTP element. The general idea is as follows. Consider an LLFPB network with equilibrium matrix $\hat{Y}_{nn}$ as given by Equation 4.3.4, i.e., a network which allows gain insertion. Let us suppose gain has been inserted forming an LLF:R network with o.c. impedance matrix given by Equation 4.3.7. If some of the terminal pairs of this network are brought out and the remainder of the network is enclosed in a black box one obtains a multiterminal-pair network. With the available terminals one may define new terminal pairs with associated o.c. impedance matrices. As discussed in Section 2.3.3 the various o.c. matrices for new definitions of terminal pairs are related by simple congruent transformations. Thus suppose Z_1 is the o.c. impedance matrix of a multiterminal-pair LLF:R network N_1 which has arisen by the gain-insertion method of the previous section. The number of terminal pairs of N_1 can be less than n, i.e., although there are n terminal pairs defined to form the equilibrium matrix Y_{nn}, we may bring out less than n terminal pairs to form N_1. In such a case Z_1 is found from $Z_{nn} = Y_{nn}^{-1}$ by striking out rows and columns corresponding to terminal pairs that are not used. It should be noted that the resulting MTP network N_1 will in general only provide a partial description of its associated MP network. Thus in defining terminal pairs on N_1 one must observe the precautions discussed at the end of Section 2.3.3. Assuming these precautions have been observed we form a new MTP network N_2 from N_1 by defining a new set of terminal pairs. The o.c. impedance matrix of N_2,Z_2, is related to that of N_1 by

$$Z_2 = \beta Z_1 \beta^t \tag{4.4.1}$$

where β is the tie set matrix which defines the branch-loop currents of N_2 upon the mutually coupled branch representation of N_1. From the results of the previous section we may relate Z_1 to an LLFPB o.c. impedance matrix $\hat{Z}$ by means of a simple collinear transformation is follows

$$Z_1 = C\hat{Z}C^{-1} \tag{4.4.2}$$

where the transformation matrix C is diagonal. The elements of Z_1 and $\hat{Z}$ differ only by constant gain factors. Thus if Z is an RC-LLFPB o.c. matrix then N_1 is an RC-LLF:R network whose o.c. driving point and transfer impedances are individually identical in character to those of an RC-LLFPB network. Thus the zeroes of driving point impedances lie on the negative real axis. This situation becomes changed by application of the congruent transformation of Equation 4.4.1. The expression for Z_2 in terms of $\hat{Z}$ is then

$$Z_2 = \beta C\hat{Z}C^{-1}\beta^t \qquad (4.4.3)$$

To illustrate the above ideas we will consider a simple example in some detail.

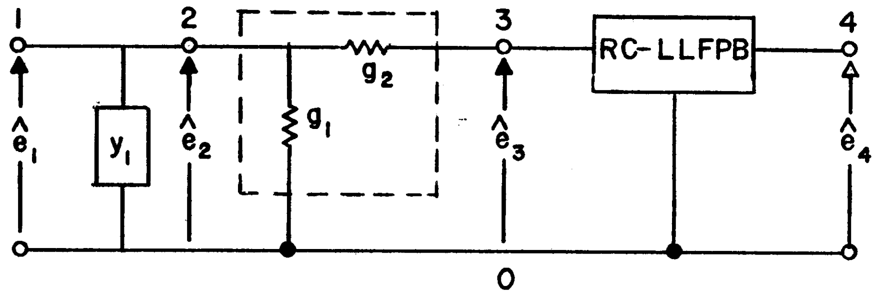


Figure 4.4.1. Simple LLFPB Network Allowing Gain Insertion

Consider the RC-LLFPB network of Figure 4.4.1. It will be recognized that this network is a special case of that of Figure 4.3.3 in which network 1 is just a shunt admittance and

$$g = g_2$$

$$g_{55} = g_1 + g_2 \qquad (4.4.4)$$

If we insert gain according to the method of the previous section we obtain the network of Figure 4.4.2 where we have let

$$a = 1;\ b = x \qquad (4.4.5)$$

without any loss in generality.

Now let us enclose the networks of Figures 4.4.2 and 4.4.1 in boxes and bring out only terminal pairs 1 and 4. In this way we form two grounded two terminal-pair networks. The o.c. impedance matrix of the RC-LLF:R network, Z_1, is related to the o.c. impedance matrix $\hat{Z}$ of the RC-LLFPB network (see Figure 4.3.13) by

$$Z_1 = \begin{bmatrix} z_{11} & z_{14} \\ z_{41} & z_{44} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & x^{-1} \end{bmatrix} \begin{bmatrix} \hat{z}_{11} & \hat{z}_{14} \\ \hat{z}_{14} & \hat{z}_{44} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & x \end{bmatrix} = \begin{bmatrix} \hat{z}_{11} & x\hat{z}_{14} \\ x^{-1}\hat{z}_{14} & \hat{z}_{44} \end{bmatrix}$$

(4.4.6)

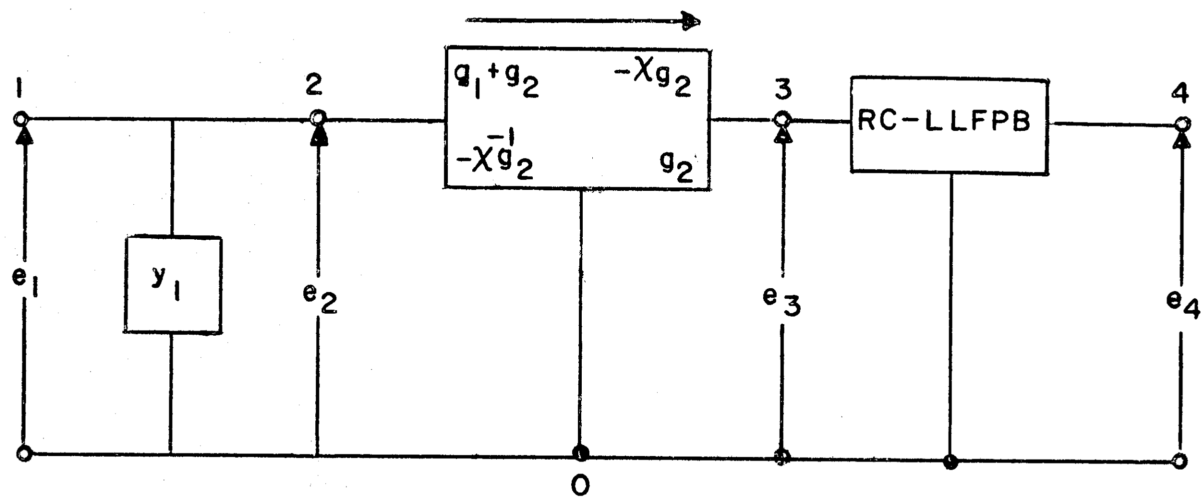


Figure 4.4.2. Network Of Figure 4.4.1 With Gain Inserted

The network N_1 with o.c. impedance matrix Z_1 is shown in Figure 4.4.3a with its coupled branch representation. In Figure 4.4.3b there is shown a new MTP network N_2 and its coupled branch representation. This network has been formed from N_1 by defining new terminal pairs as indicated in this figure. The terminal pair voltages for N_2 are labeled $\tilde{e}_1$ and $\tilde{e}_2$. In Figure 4.4.4 the branch-loop currents of N_2 are shown circulating upon the coupled branch representation of N_1. By inspection, the tie set matrix β is

$$\beta = \begin{bmatrix} -1 & 0 \\ -1 & 1 \end{bmatrix} \qquad (4.4.7)$$

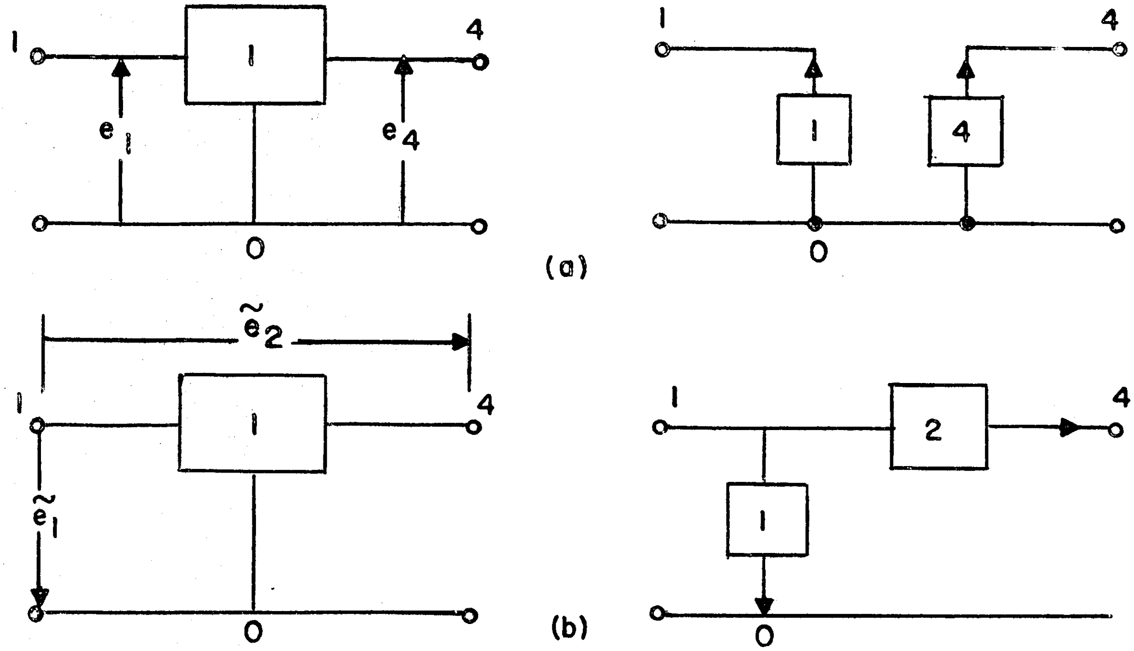


Figure 4.4.3. Formation Of MTP Network By Selection Of New Terminal Pairs

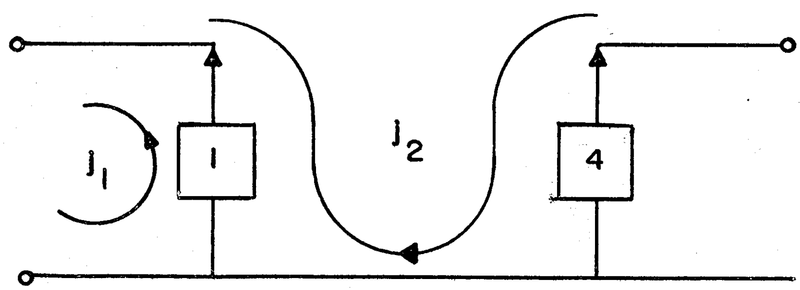


Figure 4.4.4. Network Pertinent To Obtaining The Tie Set Matrix β

The o.c. impedance matrix of N_2 is obtained by use of Equation 4.4.1 as follows

$$Z_2 = \begin{bmatrix} -1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \hat{z}_{11} & x\hat{z}_{14} \\ x^{-1}\hat{z}_{14} & \hat{z}_{44} \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \hat{z}_{11} & \hat{z}_{11} - x\hat{z}_{14} \\ \hat{z}_{11} - x^{-1}\hat{z}_{14} & \hat{z}_{11} + \hat{z}_{44} - (x + x^{-1})\hat{z}_{14} \end{bmatrix}$$

$$= \begin{bmatrix} \tilde{z}_{11} & \tilde{z}_{12} \\ \tilde{z}_{21} & \tilde{z}_{22} \end{bmatrix} \tag{4.4.8}$$

Of particular importance is the expression for $\tilde{z}_{22}$, the driving point impedance at terminal pair 2 of N_2,

$$\tilde{z}_{22} = \hat{z}_{11} + \hat{z}_{44} - (x + x^{-1})\hat{z}_{14} \tag{4.4.9}$$

This impedance may have zeroes anywhere provided $x = 1$. This is most easily understood by recognizing that

$$\hat{z} = \hat{z}_{11} + \hat{z}_{44} - 2\hat{z}_{14} \tag{4.4.10}$$

is a p.r. driving impedance of an RC-LLFPB network. With the impedance of Equation 4.4.10 we may express $\tilde{z}_{22}$ as

$$\tilde{z}_{22} = \hat{z} - (x + x^{-1} - 2)\hat{z}_{14} \tag{4.4.11}$$

It is not difficult to demonstrate that $\hat{z}_{22}$ may have zeroes placed anywhere in the complex frequency plane provided x is sufficiently greater than 1. When $x = 1$ then $\hat{z}_{22} = \hat{z}$ and its zeroes lie on the negative real axis. This might have been expected since it should be recognized that $x = 1$ is a condition that makes N_2 an

RC-LLFPB network. If we short terminal pair 2 of N_2 and leave terminal pair 1 open the resulting network has natural frequencies located at the zeroes of $\tilde{z}_{22}$ and these are complex. Consequently the voltage transfer ratio from terminal pair 2 to terminal pair 1 should be a transfer function whose poles and zeroes may be quite generally located. This transfer function is given by

$$\tilde{a}_{12} = \frac{\tilde{z}_{12}}{\tilde{z}_{22}} = \frac{\hat{z}_{11} - x\hat{z}_{14}}{\hat{z}_{11} + \hat{z}_{44} - (x + x^{-1})\hat{z}_{14}} \tag{4.4.12}$$

and does represent a transfer function which is potentially capable of exhibiting poles and zeroes of a rather general character. However, as discussed in Chapter 1, Section 1.5.2, three additional steps must be completed before one can specify a set of poles and zeroes for $\tilde{a}_{12}$ and then obtain the RC-LLF:R network which exhibits this transfer function. Unfortunately we are unable to complete the first step since the expression for $\tilde{a}_{12}$ involves a specification of the complete s.c. admittance matrix of the reference RC-LLFPB network with o.c. matrix $\hat{Z}$. One possible course of action that might be followed to complete step 1 for the particular RC-LLFPB network under discussion here, Figure 4.4.1, is to specialize the network as indicated in Figure 4.4.5.

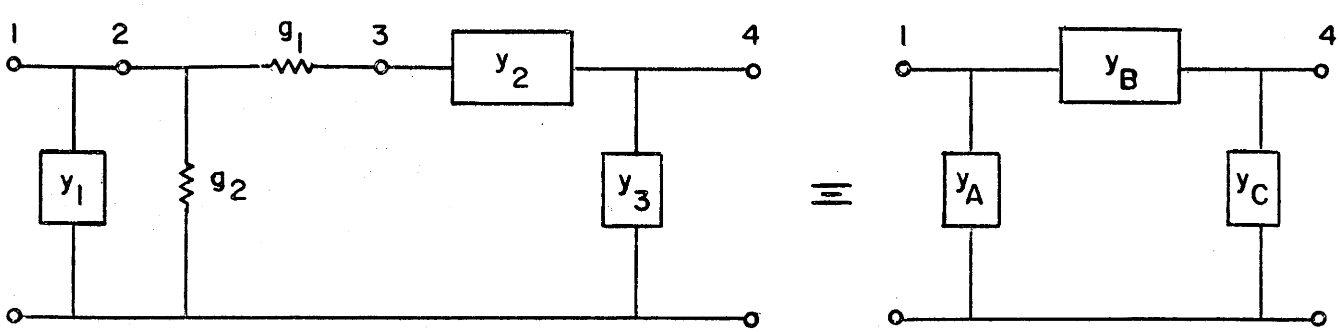


Figure 4.4.5. A Specialized Version Of The Network Of Figure 4.4.1

The conductances g_1 and g_2 are absorbed into y_2 and y_1 in the fashion indicated such that the final network assumes a π configuration. To find an expression for a_{12} in terms of y_A, y_B, and y_C we note that

$$\hat{z}_{11} = \frac{\hat{y}_{44}}{\Delta y} \; ; \; \hat{z}_{14} = \frac{-\hat{y}_{14}}{\Delta y} \; ; \; \hat{z}_{44} = \frac{\hat{y}_{11}}{\Delta y} \tag{4.4.13}$$

so that $\tilde{a}_{12}$ can be written in the equivalent form

$$\tilde{a}_{12} = - \frac{\hat{y}_{44} + x\hat{y}_{14}}{\hat{y}_{11} + \hat{y}_{44} + (x + x^{-1})\hat{y}_{14}} \tag{4.4.14}$$

But from Figure 4.4.4

$$\hat{y}_{11} = y_A + y_B$$

$$\hat{y}_{14} = -y_B$$

$$\hat{y}_{44} = y_C + y_B \tag{4.4.15}$$

so that

$$\tilde{a}_{12} = - \frac{y_C - (x - 1)y_B}{y_A + y_C - (x + x^{-1} - 2)y_B} \tag{4.4.16}$$

From Figure 4.4.5 we deduce that

$$y_A = y_1 + g_2$$

$$y_B = \frac{g_1 y_2}{g_1 + y_2}$$

$$y_C = y_3 \tag{4.4.17}$$

Examination of the denominator of the expression for $\tilde{a}_{12}$ given in Equation 4.4.16 show that the poles of $\tilde{a}_{12}$ are determined by the zeroes of the difference between two driving point admittances, $(y_A + y_C)$ and $(x + x^{-1} - 2)y_B$. In order to obtain realizable RC admittances y_1 and y_2 and positive values for g_1 and g_2 it is necessary and sufficient that y_A and y_B satisfy the following inequalities

$$y_A(0) > g_2$$

$$y_B(\infty) < g_1 \qquad (4.4.18)$$

in addition to being p.r. and RC admittances. One may always select values for g_1 and g_2 to satisfy Equation 4.4.18 and thus the poles of a_{12} may be placed arbitrarily. However it is not difficult to see that the zeroes of a_{12} may not be placed arbitrarily since the numerator expression involves the same admittance as the denominator expression in Equation 4.4.16. One may increase the freedom obtainable in locating the zeroes by placing another network in parallel with the LLF:R network.

Before we discuss this possibility it will help to clarify the above discussion if a more detailed picture is given of the network N_2 that results after the congruent transformation. The definition of terminal pairs for N_2 is shown in Figure 4.4.3b and the network N_1 is shown in Figure 4.4.2. With the help of these figures one may draw N_2 as indicated in Figure 4.4.6. The reorientation of terminal pairs "tips" the R-LLF device sideways. The R-LLF device shown in Figure 4.4.6 has the same associated MP network as the one in Figure 4.4.2 however new terminal-pairs have been defined to "right" the device. It is clear by comparison of Figures 4.4.6 and 4.4.2 that the R-LLF MTP elements of these figures differ only in the selection of datum node - 0 for Figure 4.4.2 and 1 for Figure 4.4.6. If the reference LLFPB network is the specialized one of Figure 4.4.5 the network N_2 takes the form shown in Figure 4.4.7.

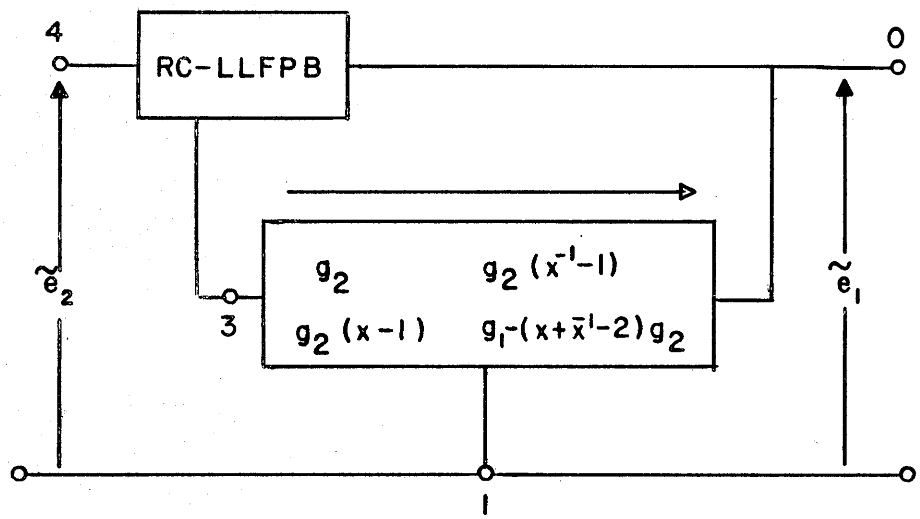


Figure 4.4.6. LLF:R Network After Congruent Transformation

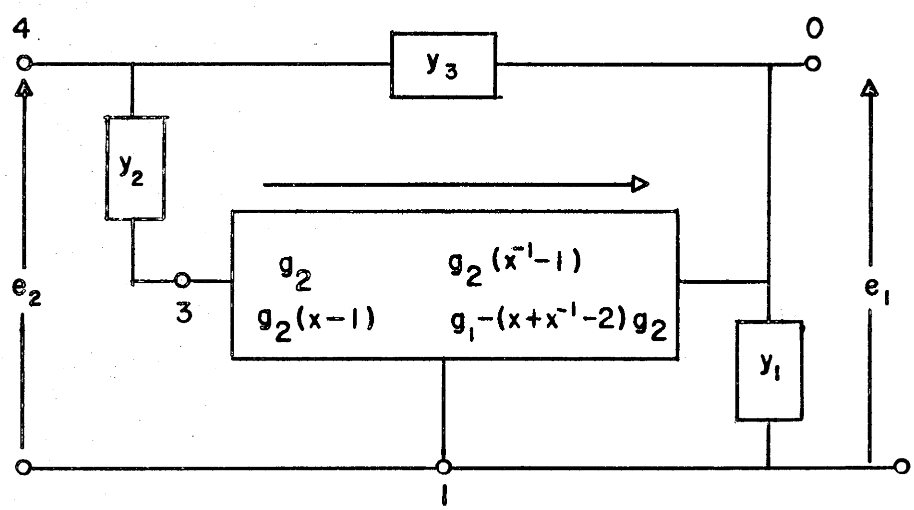


Figure 4.4.7. LLF:R Network of Figure 4.4.5 With Reference Network Of Figure 4.4.4

We have shown that the poles of the voltage transfer function $\tilde{a}_{12}$ for the network of Figure 4.4.7 may be located arbitrarily in the complex frequency plane (of course complex poles occur in complex conjugate pairs). However the zeroes are restricted. We may increase the generality of location of the zeroes of $\tilde{a}_{12}$ by placing a grounded two terminal-pair RC-LLFPB network in parallel with that of Figure 4.4.7 as shown in Figure 4.4.8a. It is clear from inspection of this latter figure that y_1 and y_2 may be absorbed into the paralleled network without loss of generality. When this is done the network takes the form shown in Figure 4.4.8b. The voltage transfer ratio A_{12} of the Network of Figure 4.4.8b from terminal pair 2 to 1 is readily found from the relationship

$$A_{12} = \frac{-Y_{12}}{Y_{11}} \qquad (4.4.19)$$

where Y_{12}, Y_{11} are s.c. transfer and driving point admittances for the same network. One may determine that

$$A_{12} = - \frac{\overline{y}_{12} - (x - 1)y_B}{\overline{y}_{11} + g_1 - (x + x^{-1} - 2)y_B} \qquad (4.4.20)$$

where $\overline{y}_{12}$, $\overline{y}_{11}$ are s.c. transfer and driving point admittances for the paralleled network. Inspection of Equation 4.4.20 indicates not only that the zeroes of A_{12} may be placed quite generally but that step 1 of Section 1.5.2 has been completed, i.e., only "constructible" specifications are involved in A_{12}. The next step of the synthesis procedure as outlined in Section 1.5.2 is the formulation of an Algorithm whereby one may go from a specified pole-zero pattern for

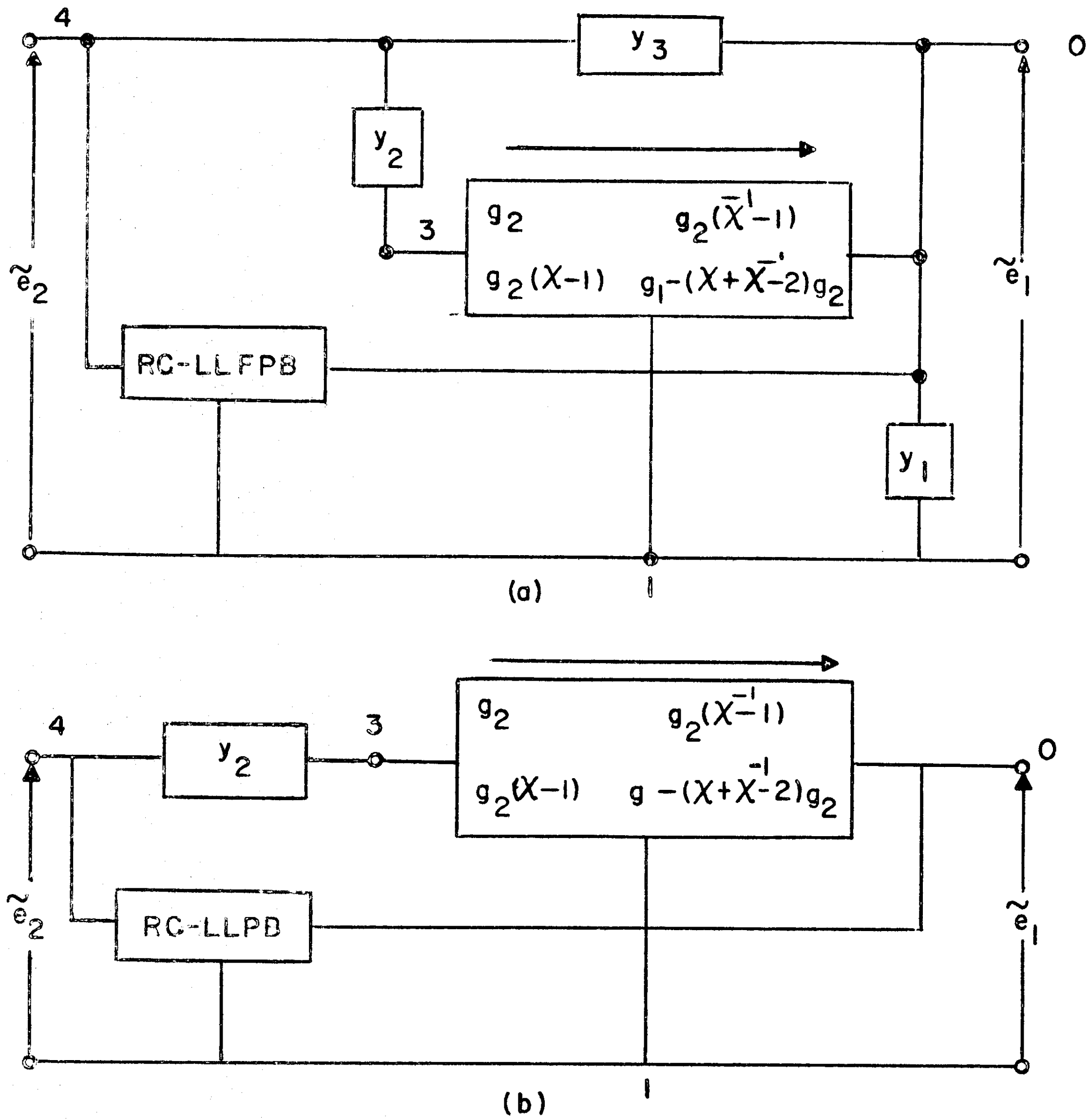


Figure 4.4.8. Network Of Figure 4.4.6 With Paralleled RC Network

A_{12} to physically realizable functions $\bar{y}_{12}$, $\bar{y}_{11}$, y_B. The author has been unable to successfully complete this second step. No discussion will be given of the difficulties involved but it will be stated that the source of these difficulties stems from the fact that $\bar{y}_{12}$ can only be synthesized to within a constant multiplier when $\bar{y}_{11}$ is completely specified.

CHAPTER 5

COMPLEX NATURAL FREQUENCIES OF AN RC-LLF:R NETWORK

5.1 Introduction

In Chapter 3 some particular techniques of LLF:R network analysis through the use of linear transformations were presented. The techniques involved both real and complex linear transformations. In Chapter 4 the analysis techniques involving real transformations were studied with the idea of inverting the analysis procedure and forming synthesis procedures. Chapter 6 will consider the use of the complex linear transformation analysis techniques of Chapter 3 as an aid in synthesizing transfer functions of RC-LLF:R networks. As groundwork for the material of Chapter 6, Chapter 5 will investigate the complex natural frequencies caused by the introduction of an R-LLF three terminal device into an RC-LLFPB network. It is shown in Section 5.2 that the zeroes of a certain Characteristic Determinant are the complex poles of the network. This determinant involves the parameters of the R-LLF device and the RC-LLFPB network in a relatively simple fashion. Attention is given in Section 5.3 to conditions on the R-LLF device and the RC-LLFPB network such that the characteristic determinant involves RC-LLFPB network functions that have "constructible" specifications. This is done as an aid in developing potentially acceptable transfer functions (i.e. those having the possibility of general pole-zero locations) which involve only "constructible" specifications. The approach used is general from the point of view that R-LLF devices may be handled that do not have a description on an impedance or admittance basis. In Section 5.4 the following question is investigated for some specific R-LLF devices. Can an RC-LLFPB network be found such that when the R-LLF device is embedded in the RC-LLFPB network, the resulting RC-LLF:R network will have a prescribed set of complex natural frequencies? A number of R-LLF devices are found to allow an arbitrary assignment of complex natural frequencies. It is shown that while the natural frequencies introduced by a gyrator may be in the complex plane they may not be generally assigned. Specifically the complex natural frequencies introduced by a gyrator are constrained to be the short circuit natural frequencies of the series combination of an RC and an RL impedance.

In Section 5.6 a general expression is given for the driving point impedance of an LLFPB network containing an embedded R-LLF device.

5.2 Characteristic Determinant

5.2.1 Impedance and Admittance Matrix Formulations

In Section 3.5 of Chapter 3 it was demonstrated that the natural frequencies caused by the introduction of an s terminal-pair R-LLF device into an LLFPB network are zeroes of the determinant

$$\Delta = \det\left[U_s + \hat{S}_{ss} e_{ss}\right] \qquad (5.2.1)$$

The determinant in Eq. 5.2 will be called the Characteristic Determinant. The combination of the LLFPB network and the R-LLF device are assumed to form an Additive network as discussed in

Section 3.5 U_s is an s x s unit matrix, $\hat{S}_{ss}$ is the s x s solution matrix of network seen from the s terminal pairs connected to R-LLF device, and e_{ss} is the s x s branch parameter matrix of R-LLF device that is used in formulating equilibrium equations for the LLF:R network. In this section we will consider the form that Eq. 5.2.1 takes when $\hat{S}_{ss}$ is an impedance or admittance solution matrix.

First consider the case in which S_{ss} is a Z x Z open circuit impedance matrix and e_{ss} is a Z x Z branch admittance matrix as indicated below

$$\left[\hat{S}_{ss}\right]_z = \begin{bmatrix} \hat{z}_{11} \hat{z}_{12} \\ \hat{z}_{12} \hat{z}_{22} \end{bmatrix}$$

$$\left[e_{ss}\right]_g = \begin{bmatrix} g_{11} g_{12} \\ g_{21} g_{22} \end{bmatrix} \equiv G \tag{5.2.2}$$

where the subscripts on $\hat{S}_{ss}$ and e_{ss} denote that $\hat{S}_{ss}$ is given by an o.c. impedance matrix and e_{ss} is characterized by admittance parameters. It is clear that we are discussing the case in which a three terminal R-LLF device described by admittance parameters is embedded in an LLFPB network. Figure 5.2.1 indicates the definition of terminal pairs involved such that the LLF:R network is Additive and the admittance equilibrium matrix is the sum of the branch admittance parameter matrices of the LLFPB MTP element and the R-LLF MTP element. The center terminal of the R-LLF device is labeled with a g to denote that the parameters indicated on the circuit

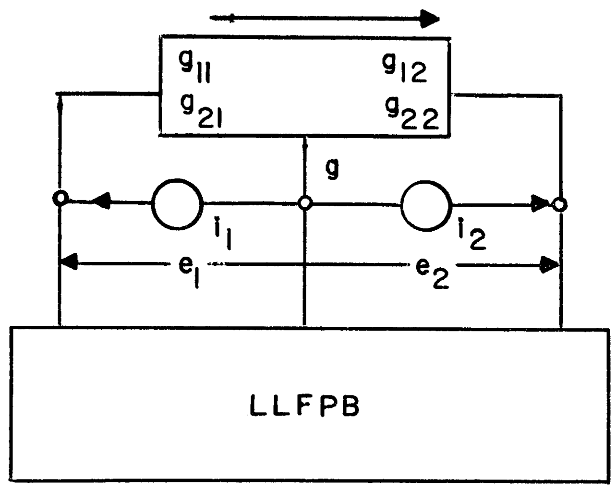


Figure 5.2.1. Definition Of Terminal Pairs For LLF:R Network With Additive Admittance Matrices

Symbol apply to a branch admittance description of the R-LLF device. Note that the LLFPB device is shown with only two terminal pairs. For the work of this chapter it is not necessary to evidence the other terminal pairs involved. They may be considered to be contained within the box labeled LLFPB in Fig. 5.2.1. The voltages e_1 and e_2 are terminal pair response voltages to current sources i_1, i_2 applied across the terminal pairs 1 and 2, respectively. When $\left[e_{ss}\right]_g$ becomes null, i.e. $g_{11} = g_{12} = g_{21} = g_{22} = 0$, one obtains the reference LLFPB network. Letting $\left[e_{ss}\right]_g$ become null effectively removes the R-LLF network and places the current sources i_1 and i_2 across the LLFPB network as indicated in Fig. 5.2.2.

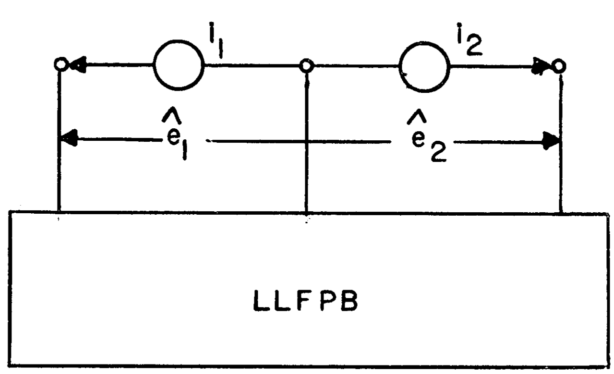


Figure 5.2.2. Network of Figure 5.2.1 With e_{ss_g} Null

The matrix $\left[\hat{S}_{ss}\right]_z$ then relates the response voltages $\hat{e}_1$, $\hat{e}_2$ to the current source excitations i_1, i_2. It should be noted that $\left[\hat{S}_{ss}\right]_z$ is the o.c. impedance matrix of a grounded two terminal-pair LLFPB network.

If we use the definitions of Eq. 5.2.2 then Eq. 5.2.1 becomes

$$\det\left\{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} \hat{z}_{11} & \hat{z}_{12} \\ \hat{z}_{12} & \hat{z}_{22} \end{bmatrix}\begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix}\right\} = \Delta^g \tag{5.2.3}$$

where the superscript g is used to denote that the R-LLF device is described by g parameters.

If the matrix operations are carried out in Eq. 5.2.3 and the determinant is evaluated one arrives at the equation

$$1 + \Delta g \Delta\hat{z} + g_{11}\hat{z}_{11} + g_{22}\hat{z}_{22} + \left[g_{12} + g_{21}\right]\hat{z}_{12} = \Delta^g \tag{5.2.4}$$

We will call Δ^g a g-type Characteristic Determinant

The complex frequencies which are zeroes of Δ^g are natural frequencies of the LLF:R network with o.c. constraints across terminal pairs 1 and 2. In Eq. 5.2.4 we have used the definitions

$$\Delta z = \hat{z}_{11}\hat{z}_{22} - \hat{z}_{12}^{\,2}$$

$$\Delta g = g_{11}g_{22} - g_{12}^{\,2} \tag{5.2.5}$$

To obtain the Characteristic Determinant in the case wherein the R-LLF device is described by a branch impedance parameter matrix we let $\hat{S}_{ss}$ and e_{ss} take the forms

$$\left[\hat{S}_{ss}\right]_y = \begin{bmatrix} \hat{y}_{11} & \hat{y}_{12} \\ \hat{y}_{12} & \hat{y}_{22} \end{bmatrix}$$

$$\left[e_{ss}\right]_r = \begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix} = R \tag{5.2.6}$$

where the subscripts on $\hat{S}_{ss}$ and e_{ss} denote that $\hat{S}_{ss}$ is given by a s.c. admittance matrix and e_{ss} is characterized by impedance parameters.

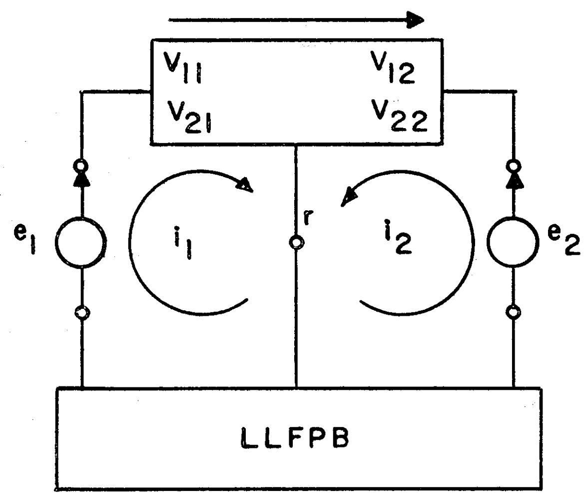


Figure 5.2.3

Figure 5.2.3 indicates the definition of terminal pairs involved such that the LLF:R network is Additive and the impedance equilibrium matrix is the sum of the branch impedance parameter matrices of the LLFPB MTP element and the R-LLF MTP element. The center terminal of the R-LLF device is labeled with an r to denote that the parameters indicated on the circuit symbol apply to a branch impedance description of the R-LLF device. The currents i_1, i_2 are response loop-currents to source voltages e_1 and e_2, respectively. When $\left[e_{ss}\right]_r$ becomes null, i.e., $r_{11} = r_{12} = r_{21} = r_{22} = 0$ one obtains the reference LLFPB network. Letting $\left[e_{ss}\right]_r$ become null effectively removes the R-LLF network and places the voltage sources e_1 and e_2 across terminal pairs of the LLFPB network as indicated in Fig. 5.2.4. The matrix $\left[\hat{S}_{ss}\right]_y$ then relates the response currents i_1, i_2 to voltage sources e_1, e_2.

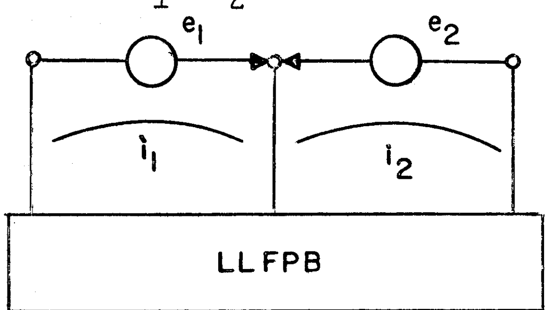


Figure 5.2.4. Network of Figure 5.2.3 With e_{ss} $_r$ Null

If we use the definitions of Eq. 5.2.6 then Eq. 5.2.1 becomes

$$\det\left\{\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} + \begin{bmatrix}\hat{y}_{11}\hat{y}_{12}\\ y_{12}y_{22}\end{bmatrix}\begin{bmatrix}r_{11}r_{12}\\ r_{21}r_{22}\end{bmatrix}\right\} = \Delta^r \tag{5.2.7}$$

where the superscript r denotes the fact that the R-LLF device is described by r parameters. If the matrix operations are carried out in Eq. 5.2.3 and the determinant is evaluated one arrives at the equation

$$1 + \Delta r\Delta\hat{y} + r_{11}\hat{y}_{11} + r_{22}\hat{y}_{22} + (r_{12} + r_{21})\hat{y}_{12} = \Delta^r \tag{5.2.8}$$

this determinant will be called an r-type Characteristic Determinant. It applies to the case wherein the R-LLF device is described by a branch impedance parameter matrix. Equation 5.2.8 is dual to Equation 5.2.4 since they pertain to dual situations. Thus Equation

5.2.8 could have been written by inspection. In Equation 5.2.8 we have used the definitions

$$\Delta\hat{y} = \hat{y}_{11}\hat{y}_{22} - \hat{y}_{12}^{2}$$

$$\Delta r = r_{11}r_{22} - r_{12}^{2} \tag{5.2.9}$$

It is not difficult to see from the above discussion that terminal pairs have been defined in such a way that $\left[S_{ss}\right]_y$ and $\left[S_{ss}\right]_z$ are inverses, i.e.,

$$\begin{bmatrix} \hat{z}_{11} & \hat{z}_{12} \\ \hat{z}_{12} & \hat{z}_{22} \end{bmatrix} = \begin{bmatrix} \hat{y}_{11} & \hat{y}_{12} \\ \hat{y}_{12} & \hat{y}_{22} \end{bmatrix}^{-1} \tag{5.2.10}$$

In addition, if $\left[e_{ss}\right]_g$ is not singular (and thus $\left[e_{ss}\right]_r$ is not singular)

$$\begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix}^{-1} \tag{5.2.11}$$

However Equation 5.2.4 applies whether $\Delta g = 0$ or not and Equation 5.2.8 applies whether $\Delta r = 0$ or not. Thus the pair of Equations 5.2.4 and 5.2.8 are able to handle the situation in which the R-LLF device can only be described either on an impedance or an admittance basis but not both. However there are situations in which the R-LLF device cannot be described on either an admittance or an impedance basis. In such situations a mixed basis description suffices. Thus the following sections discusses the formulation of the Characteristic Determinant for a mixed solution matrix S_{ss} and mixed parameter matrix e_{ss}.

5.2.2 Mixed Matrices

A description of a multiterminal-pair network on a mixed basis is one in which some voltages and some currents are dependent or response quantities rather than all voltages or all currents. We will not consider the mixed cases in which both the voltage and current at a terminal pair are regarded as dependent or independent quantities since such cases have no physical correspondence as far as the formulation of equilibrium equations is concerned.

Figures 5.2.5a and b show a two terminal-pair grounded network with the two possible types of mixed excitation, i.e., a voltage source at one terminal pair and a current source at the other terminal pair. In Fig. 5.2.5b the response variables are the current at

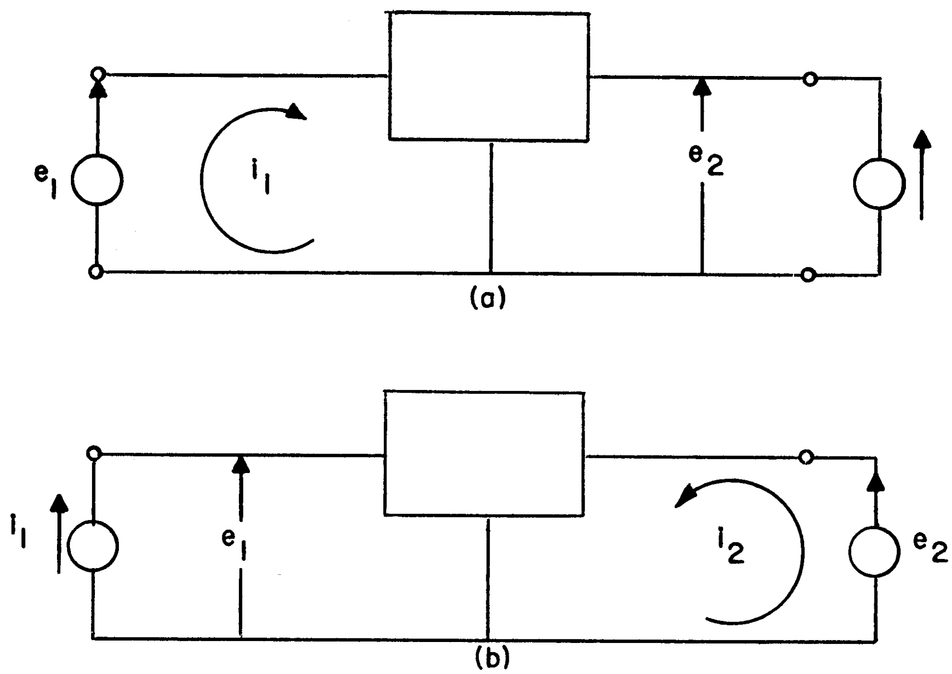


Figure 5.2.5. Mixed Excitations For A Two Terminal-Pair Network

terminal pair 2 and the voltage at terminal pair 1. The equilibrium equations for this case read as follows

$$
\begin{aligned}
i_1 &= u_{11}e_1 + b_{12}i_2 \\
e_2 &= a_{21}e_1 + u_{22}i_2
\end{aligned}
\tag{5.2.12}
$$

where i_1 and e_2 are source variables and

$$
U = \begin{bmatrix} u_{11} b_{12} \\ \\ a_{21} u_{22} \end{bmatrix}
\tag{5.2.13}
$$

is the equilibrium matrix. The first equation in 5.2.12 is an application of Kirchoff's Current Law and the second equation is an application of Kirchoff's Voltage Law. In Fig. 5.2.5a the response variables are the voltage at terminal pair 2 and the current at terminal pair 1. The corresponding equilibrium equations are given by

$$
\begin{aligned}
e_1 &= v_{11}i_1 + a_{12}e_2 \\
i_2 &= b_{21}i_1 + v_{22}e_2
\end{aligned}
\tag{5.2.14}
$$

where e_1 and i_2 are source quantities and

$$
V = \begin{bmatrix} v_{11} a_{12} \\ \\ b_{21} v_{22} \end{bmatrix}
\tag{5.2.15}
$$

is the solution matrix. If U and V are not singular then it is clear that

$$
U = V^{-1}
\tag{5.2.16}
$$

since the inverse of the equilibrium matrix is the solution matrix. We may regard the solution matrix for one mixed basis description as the equilibrium matrix of the other mixed basis description. The elements of these mixed matrices are related to the open circuit impedances and the short circuit admittances. Let

$$\dot{Y} = \begin{bmatrix} y_{11} y_{12} \\ y_{21} y_{22} \end{bmatrix} \quad Z = \begin{bmatrix} z_{11} z_{12} \\ z_{21} z_{22} \end{bmatrix} \tag{5.2.17}$$

be the s.c. admittance matrix and the o.c. impedance matrix as conventionally defined. Then

$$a_{jk} = \frac{z_{jk}}{z_{kk}} = -\frac{y_{jk}}{y_{jj}} \quad ; \quad b_{jk} = -\frac{z_{jk}}{z_{jj}} = \frac{y_{jk}}{y_{kk}}$$

$$u_{11} = \frac{1}{z_{11}} \quad , \quad u_{22} = \frac{1}{y_{22}} \quad ; \quad v_{11} = \frac{1}{y_{11}} \quad , \quad v_{22} = \frac{1}{z_{22}} \tag{5.2.18}$$

Since the mixed matrices of a two terminal pair R-LLF device are real, special symbols will be used for the elements of these matrices just as with impedance and admittance matrices. Figures 5.2.6a and b illustrate the circuit symbols for a two terminal-pair R-LLF device when mixed matrix descriptions are used. The equilibrium equations for the network of Fig. 5.2.6b are given by

$$\begin{aligned} i_1 &= k_{11} e_1 + k_{12} i_2 \\ e_2 &= k_{21} e_1 + k_{22} e_2 \end{aligned} \tag{5.2.19}$$

where i_1 and e_2 are excitations and

$$K = \begin{bmatrix} k_{11} k_{12} \\ k_{21} k_{22} \end{bmatrix} \tag{5.2.20}$$

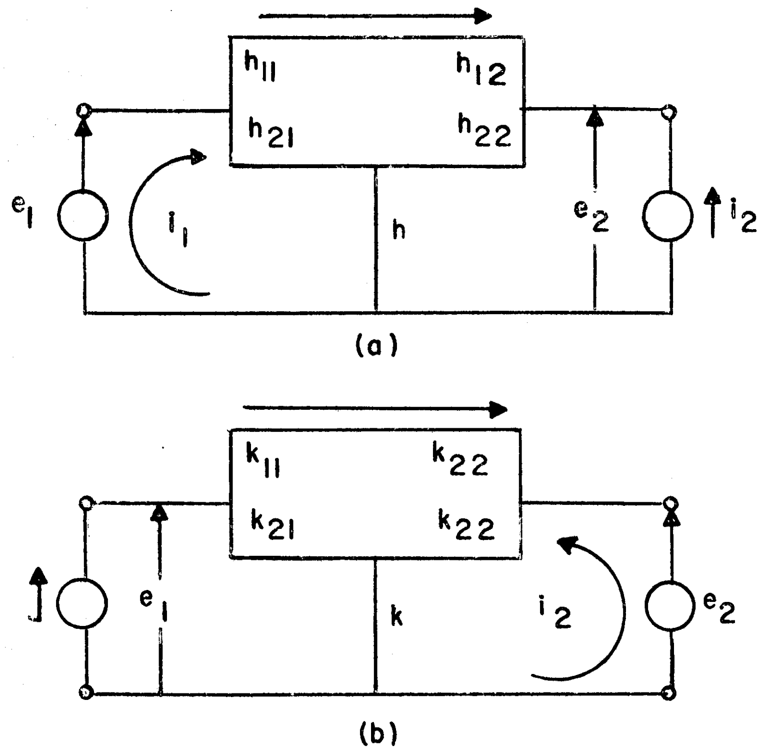


Figure 5.2.6. Mixed Excitations and Circuit Symbols For Two Terminal-Pair R-LLF Device

is the equilibrium matrix. The equilibrium equations for the network of Fig. 5.2.6b are given by

$$
\begin{aligned}
e_1 &= h_{11}i_1 + h_{12}e_2 \\
i_2 &= h_{21}i_1 + h_{22}e_2
\end{aligned}
\tag{5.2.21}
$$

where e_1 and i_2 are excitations and

$$
H = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix}
\tag{5.2.22}
$$

is the equilibrium matrix. Since H and K are real versions of U and V respectively one may construct the set of equations analogous to Eq's. 5.2.16 to 5.2.18.

5.2.3 Mixed Matrix Formulations

In this section we will obtain characteristic determinants for the cases where the R-LLF device is described by mixed equilibrium matrices. Consider first the case in which the R-LLF device is described by a K type equilibrium matrix. Figure 5.2.7 indicates

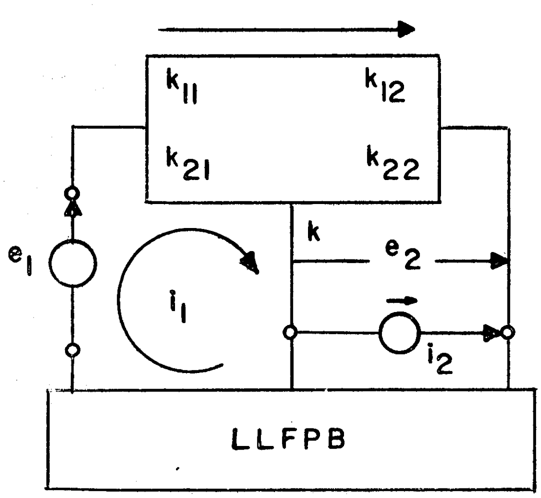


Figure 5.2.7. Definition Of Terminal Pairs For LLF:R Network With Additive Mixed Matrices

the definition of terminal pairs involved such that the LLF:R network is Additive and the mixed equilibrium matrix is the sum of the mixed equilibrium matrix of the R-LLF MTP element and that of the LLFPB MTP element. In this case we choose $\hat{S}_{ss}$ and e_{ss} as indicated below

$$\left[\hat{S}_{ss}\right]_u = \begin{bmatrix} \hat{u}_{11} & -\hat{b}_{12} \\ -\hat{a}_{21} & \hat{u}_{22} \end{bmatrix}$$

$$\left[e_{ss}\right]_k = \begin{bmatrix} k_{11} k_{12} \\ k_{21} k_{22} \end{bmatrix} = K \tag{5.2.23}$$

When $\left[e_{ss}\right]_k$ becomes null, i.e., $k_{11} = k_{12} = k_{21} = k_{22} = 0$ one obtains the reference LLFPB network. Letting $\left[e_{ss}\right]_k$ become null effectively removes the R-LLF device and places the voltage source e_1 and current source i_2 across terminal pairs of the LLFPB network as indicated in Fig. 5.2.8. The matrix $\left[\hat{S}_{ss}\right]_u$ then relates the response

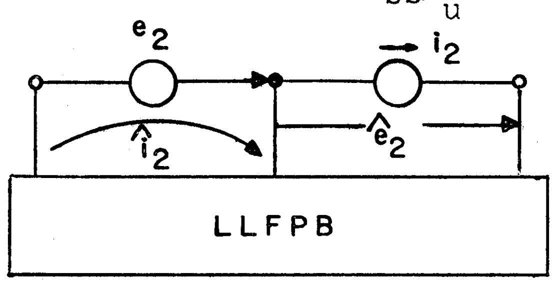


Figure 5.2.8. Network Of Figure 5.2.7 $e_{ss\,k}$ Null

quantities $\hat{i}_1$, $\hat{e}_2$ to the sources e_1, i_2. The minus signs are used in the definition of the elements of $\left[\hat{S}_{ss}\right]_u$ so that Eq.'s 5.2.18 may be used to relate parameters of $\left[\hat{S}_{ss}\right]_u$ with those of $\left[\hat{S}_{ss}\right]_y$ and $\left[\hat{S}_{ss}\right]_z$. The minus signs are needed due to the different positive reference directions assigned for the voltages and currents of the network of

Fig. 5.2.8 and those of Fig's. 5.2.2 and 5.2.4. For an LLFPB network

$$\hat{b}_{12} = -\hat{a}_{21} \tag{5.2.24}$$

i.e., the voltage transfer ratio in one direction is the negative of the current transfer ratio in the opposite direction. When we use the definitions of Eq. 5.2.23 and Eq. 5.2.24, Eq. 5.2.1 becomes

$$\det\left\{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} \hat{u}_{11} & \hat{a}_{21} \\ -\hat{a}_{21} & \hat{u}_{22} \end{bmatrix} \begin{bmatrix} \hat{k}_{11} & \hat{k}_{12} \\ \hat{k}_{21} & \hat{k}_{22} \end{bmatrix}\right\} = \Delta^k \tag{5.2.25}$$

If the matrix operations are carried out in Eq. 5.2.25 and the determinant evaluated we obtain the Characteristic Determinant

$$1 + \Delta k \Delta u + k_{11}u_{11} + k_{22}u_{22} + (k_{21} - k_{12})a_{21} = \Delta^k \tag{5.2.26}$$

where the following definitions have been used

$$\begin{aligned} \Delta k &= k_{11}k_{22} - k_{12}^{\;2} \\ \hat{\Delta u} &= \hat{u}_{11}\hat{u}_{22} + \hat{a}_{21}^{\;2} \end{aligned} \tag{5.2.27}$$

The determinant in Eq. 5.2.26 will be called a k-type Characteristic Determinant.

We will consider now the case in which the R-LLF device is described by an H type equilibrium matrix. Figure 5.2.9 indicates the definition of terminal pairs involved such that the LLF:R network is Additive and the mixed equilibrium matrix is the sum of the mixed equilibrium matrix of the R-LLF MTP element and that of the LLFPB MTP element. In this case we choose $\hat{S}_{ss}$ and e_{ss} as indicated below

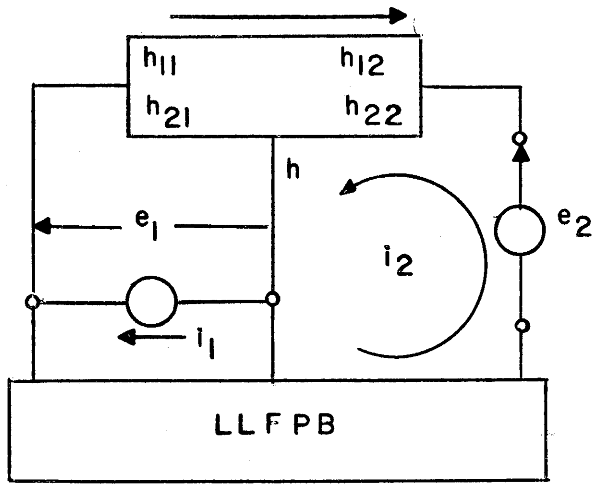


Figure 5.2.9

$$\left[\hat{S}_{ss}\right]_v = \begin{bmatrix} \hat{v}_{11} & -\hat{a}_{12} \\ -\hat{b}_{21} & \hat{v}_{22} \end{bmatrix}$$

$$\left[e_{ss}\right]_h = \begin{bmatrix} h_{11} h_{12} \\ h_{21} h_{22} \end{bmatrix} = H \tag{5.2.28}$$

When $\left[e_{ss}\right]_h$ becomes null, i.e., $h_{11} = h_{12} = h_{21} = h_{22} = 0$, one obtains the reference LLFPB network. Letting $\left[e_{ss}\right]_h$ become null effectively removes the R-LLF device and places the current source i_1 and the voltage source e_2 across the LLFPB network as shown in Fig. 5.2.10.

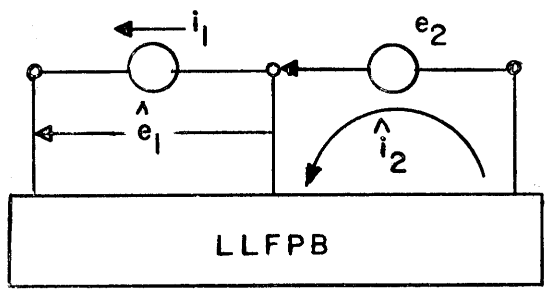


Figure 5.2.10. Network Of Figure 5.2.9 With $e_{ss\,h}$ Null

The matrix $\left[\hat{S}_{ss}\right]_v$ relates the response quantities $\hat{e}_1$ and $\hat{i}_2$ to the sources i_1 and e_2. The reason for the minus sign in $\left[\hat{S}_{ss}\right]_v$ is the same as for $\left[\hat{S}_{ss}\right]_u$. For an LLFPB network

$$\hat{b}_{21} = -\hat{a}_{12} \tag{5.2.29}$$

Thus Eq. 5.2.1 becomes

$$\det\left\{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} \hat{v}_{11} & -\hat{a}_{12} \\ \hat{a}_{12} & \hat{v}_{22} \end{bmatrix}\begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix}\right\} = \Delta^h \tag{5.2.30}$$

which leads to the characteristic equation

$$1 + \Delta h \Delta v + h_{11}v_{11} + h_{22}v_{22} + (h_{12} - h_{21})a_{12} = \Delta^h \tag{5.2.31}$$

The following definitions have been used

$$\Delta v = \hat{v}_{11}\hat{v}_{22} + \hat{a}_{12}^2$$

$$\Delta h = \hat{h}_{11}\hat{h}_{22} - \hat{h}_{12}^2 \tag{5.2.32}$$

The determinant in Eq. 5.2.31 will be called an h-type Characteristic Determinant.

5.3 Constructible Specifications on Complex Pole Locations

5.3.1 g and r - Type Characteristic Determinants

In this section we consider ways in which the g-type and r-type Characteristic Determinants may be specialized such that only "constructible" specifications are made upon the LLFPB network which is now considered to be RC. This is a preliminary step to finding potentially acceptable driving point and transfer functions (i.e. those having the possibility of general pole-zero locations) which involve only constructible specifications. We will discuss the g-type Characteristic Determinant first. This is rewritten below for convenience

$$1 + \Delta g \Delta \hat{z} + g_{11}\hat{z}_{11} + g_{22}\hat{z}_{22} + \left[g_{12} + g_{21}\right] \hat{z}_{12} = \Delta^{g} \qquad (5.3.1)$$

Examination of this determinant indicates 5 situations in which constructible specifications are involved upon the RC-LLFPB network. These are listed in table form in Fig. 5.3.1 with the corresponding specialized Characteristic Determinants and constructible specifications. Cases (3) and (4) have obvious variants of identical form by

	Specialized Parameters	Characteristic Determinant	Specifications on RC-LLFPB Network
1	$\Delta g = 0;\ g_{12} + g_{21} = 0$	$1 + g_{11}\hat{z}_{11} + g_{22}\hat{z}_{22} = \Delta^{g}$	$\hat{z}_{11}, \hat{z}_{22}$
2	$\Delta g = 0;\ \hat{z}_{12} = 0$	$1 + g_{11}\hat{z}_{11} + g_{22}\hat{z}_{22} = \Delta^{g}$	$\hat{z}_{11}, \hat{z}_{22};\ \hat{z}_{12} = 0$
3	$g_{12} = g_{11} = g_{22} = 0$	$1 + g_{21}\hat{z}_{12} = \Delta^{g}$	$\hat{z}_{12}$
4	$g_{11} = g_{12} = 0$	$1 + g_{22}\hat{z}_{22} + g_{21}\hat{z}_{12} = \Delta^{g}$	$\hat{z}_{22}, \hat{z}_{12}$
5	$\Delta g \neq 0;\ \hat{z}_{12} = 0$	$1 + \Delta g \hat{z}_{11}\hat{z}_{22} + g_{11}\hat{z}_{11} + g_{22}\hat{z}_{22} = \Delta^{g}$	$\hat{z}_{11}, \hat{z}_{22};\ \hat{z}_{12} = 0$

Figure 5.3.1. Special Situations Leading To Constructible Specifications: g-Type Characteristic Determinant

redefining terminal pairs. It should be noted that in cases (1) through (4) the G matrix is singular, i.e., $\Delta g = 0$. Thus the corresponding R-LLF devices have no representation in terms of R matrices although some may be represented by H or K matrices. The forms of the G matrices for cases (1) through (5) are listed below

$$(1) \quad G_1 = \begin{bmatrix} g_{11} & g \\ -g & g_{22} \end{bmatrix} \; ; \; g_{11}g_{22} = -\,g^2 \tag{5.3.2}$$

$$(2) \quad G_2 = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \; ; \; g_{11}g_{22} = g_{12}g_{21} \tag{5.3.3}$$

$$(3) \quad G_3 = \begin{bmatrix} 0 & 0 \\ g_{21} & 0 \end{bmatrix} \tag{5.3.4}$$

$$(4) \quad G_4 = \begin{bmatrix} 0 & 0 \\ g_{21} & g_{22} \end{bmatrix} \tag{5.3.5}$$

$$(5) \quad G_5 = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \; ; \; g_{11}g_{22} \neq g_{12}g_{21} \tag{5.3.6}$$

Note that in cases (2) and (5) z_{12} is required to be zero. In such a case the RC-LLFPB network must decomposable into two isolated sub-networks as shown in Fig. 5.3.2.

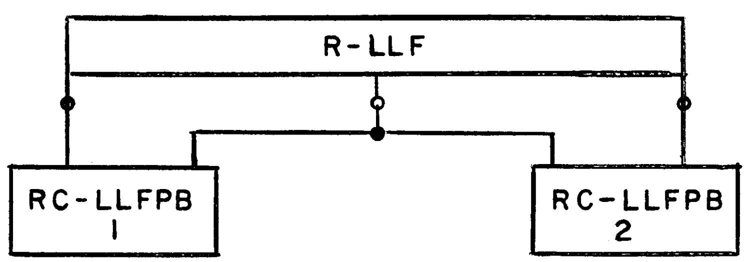


Figure 5.3.2. Illustration Of The Condition $\hat{z}_{12} = 0$

Since the r-type Characteristic Determinant is dual to the g-type we obtain the dual table shown in Fig. 5.3.3. Note in cases (1) through (4) that the R matrix is singular. Thus the corresponding

	Specialized Parameters	Characteristic Determinant	Specifications on RC-LLFPB Network
1	$\Delta r = 0;\ r_{12} + r_{21} = 0$	$1 + r_{11}\hat{y}_{11} + r_{22}\hat{y}_{22} = \Delta^r$	$\hat{y}_{11},\ \hat{y}_{22}$
2	$\Delta r = 0;\ \hat{y}_{12} = 0$	$1 + r_{11}\hat{y}_{11} + r_{22}\hat{y}_{22} = \Delta^r$	$\hat{y}_{11},\ \hat{y}_{22};\ \hat{y}_{12} = 0$
3	$r_{12} = r_{11} = r_{22} = 0$	$1 + r_{21}\hat{y}_{12} = \Delta^r$	$\hat{y}_{12}$
4	$r_{11} = r_{12} = 0$	$1 + r_{22}\hat{y}_{22} + r_{21}\hat{y}_{12} = \Delta^r$	$\hat{y}_{22},\ \hat{y}_{12}$
5	$\Delta r \neq 0;\ \hat{y}_{12} = 0$	$1 + \Delta r\hat{y}_{11}\hat{y}_{22} + r_{11}\hat{y}_{11} + r_{22}\hat{y}_{22} = \Delta^r$	$\hat{y}_{11},\ \hat{y}_{22};\ \hat{y}_{12} = 0$

Figure 5.3.3. Special Situations Leading To Constructible Specifications: r-Type Characteristic Determinant

R-LLF devices have no representation in terms of G matrices although some may be represented by H or K matrices. It is readily seen that case (5) of Fig. 5.3.3 is identical to case (5) of Fig. 5.3.1. Consequently Fig. 5.3.3 lists only four new possibilities for constructible specifications rather than 5. The R matrices for these four cases are listed below

$$(1)\quad R_1 = \begin{bmatrix} r_{11} & r \\ -r & r_{22} \end{bmatrix} \quad ; \quad r_{11}r_{22} = -r^2 \qquad (5.3.7)$$

$$(2)\quad R_2 = \begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix} \quad ; \quad r_{11}r_{22} = r_{12}^{\ 2} \qquad (5.3.8)$$

$$(3)\quad R_3 = \begin{bmatrix} 0 & 0 \\ r_{21} & 0 \end{bmatrix} \qquad (5.3.9)$$

$$(4)\quad R_4 = \begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix} \; ; \quad r_{11}r_{22} \neq r_{12}r_{21} \tag{5.3.10}$$

The condition $\hat{y}_{12} = 0$ has the same consequences as for $\hat{z}_{12} = 0$ since

$$y_{12} = -\frac{\hat{z}_{12}}{\Delta\hat{z}}$$

5.3.2 k And h-type Characteristic Determinants

In this section we consider ways in which k-type and h-type Characteristic Determinants may be specialized such that only "constructible" specifications are made upon the RC-LLFPB network. We will consider the k-type Characteristic Determinant first. This is rewritten below

$$1 + \Delta k\Delta\hat{u} + k_{11}\hat{u}_{11} + k_{22}\hat{u}_{22} + (k_{21} - k_{12})\hat{a}_{21} = \Delta k$$

The functions $\hat{u}_{11}$, $\hat{u}_{22}$, and $\hat{a}_{21}$ may be expressed in terms of the $\hat{y}$ and $\hat{z}$ functions with the aid of Eq.'s 5.2.18. In terms of these functions

$$\hat{u}_{11} = \frac{1}{\hat{z}_{11}} \; ; \; \hat{u}_{22} = \frac{1}{\hat{y}_{22}} \; ; \; \hat{a}_{21} = \frac{\hat{z}_{21}}{\hat{z}_{11}} = -\frac{\hat{y}_{21}}{\hat{y}_{22}}$$

Also one may readily determine that

$$\Delta\hat{u} = \frac{\hat{y}_{11}}{\hat{y}_{22}} = \frac{\hat{z}_{22}}{\hat{z}_{11}} \tag{5.3.11}$$

The R-LLF devices with the G matrices of cases (1), (2), (4), and (5) and in Fig. 5.3.1 the R matrices of cases (1), (2), and (5) in

Fig. 5.3. are such that K matrix description exist. Consequently they provide a number of situations in which the parameters of the K type Characteristic Determinant may be specialized to attain constructible specifications. Since they do not add anything new there is no point in bringing them up in the present discussions. Rather we will only mention those specializations of the parameters of the k-type Characteristic Determinant which lead to R-LLF devices with no R or G matrix description. The conditions for the non-existence of R or G matrix descriptions are that

$$k_{11} = 0$$

$$k_{22} = 0 \tag{5.3.12}$$

In such a case the k-type Characteristic Determinant becomes

$$1 - k_{12}k_{21}\Delta\hat{u} + (k_{21} - k_{12})\hat{a}_{21} = \Delta^k \tag{5.3.13}$$

This determinant may be rewritten in terms of z parameters as below

$$1 - k_{12}k_{21}\frac{\hat{z}_{22}}{\hat{z}_{11}} + (k_{21} - k_{12})\frac{\hat{z}_{12}}{\hat{z}_{11}} = \Delta^k \tag{5.3.14}$$

or

$$\hat{z}_{11} - k_{12}k_{21}\hat{z}_{22} + (k_{21} - k_{12})\hat{z}_{12} = \Delta^k z_{11} \tag{5.3.15}$$

Examination of Eq's. 5.3.13 to 5.3.15 indicates that constructible specifications will be obtained in the following three situations indicated in Fig. 5.3.4.

	Specialized Parameters	Characteristic Determinant	Specifications On LLFPB Network
1	$k_{11} = k_{22} = 0$; $z_{12} = 0$	$1 - k_{12}k_{21}\dfrac{\hat{z}_{22}}{\hat{z}_{11}} = \Delta^k$	$\hat{z}_{11}, \hat{z}_{22}; \hat{z}_{12} = 0$
2	$k_{11} = k_{22} = 0$; $k_{21} = k_{12}$	$1 - k_{12}k_{21}\dfrac{\hat{z}_{22}}{\hat{z}_{11}} = \Delta^k$	$\hat{z}_{11}, \hat{z}_{22}$
3	$k_{12} = 0$	$1 + k_{21}\hat{a}_{21} = \Delta^k$	$\hat{z}_{12}, \hat{z}_{11}$

Fig. 5.3.4 Some Conditions Leading To Constructible Specifications: k-Type Characteristic Determinant

The forms for the K matrices of the R-LLF devices for cases (1) through (3) are listed below

$$(1) \quad K_1 = \begin{bmatrix} 0 & k_{12} \\ k_{21} & 0 \end{bmatrix} \tag{5.3.16}$$

$$(2) \quad K_2 = \begin{bmatrix} 0 & k \\ k & 0 \end{bmatrix} \tag{5.3.17}$$

$$(3) \quad K_3 = \begin{bmatrix} 0 & 0 \\ k_{21} & 0 \end{bmatrix} \tag{5.3.18}$$

By direct analogy with the k-type characteristic equation we may construct the table of Fig. 5.3.5 for the h-type Characteristic Determinant.

It is readily seen that cases (1) and (2) of Fig. 5.3.5 lead to the same R-LLF device as cases (1) and (2) of Fig. 5.3.4. Thus only case (3) represents a new result. The H matrix for this case is

$$H_1 = \begin{bmatrix} 0 & 0 \\ h_{21} & 0 \end{bmatrix} \tag{5.3.19}$$

	Specialized Parameters	Characteristic Determinant	Specifications On LLFPB Network
1	$h_{11} = h_{22} = 0 \; ; \; z_{12} = 0$	$1 - h_{12}h_{21} \dfrac{\hat{z}_{11}}{\hat{z}_{22}} = \Delta^h$	$\hat{z}_{11}, \hat{z}_{22}; \hat{z}_{12} = 0$
2	$h_{11} = h_{22} = 0 \; ; \; h_{21} = h_{12}$	$1 - h_{12}h_{21} \dfrac{\hat{z}_{11}}{\hat{z}_{22}} = \Delta^h$	$\hat{z}_{11}, \hat{z}_{22}$
3	$h_{12} = 0$	$1 + h_{21}\hat{a}_{12} = \Delta^h$	$\hat{z}_{12}, \hat{z}_{22}$

Fig. 5.3.5 Some Conditions Leading To Constructible Specifications: h-Type Characteristic Determinant

5.4 Discussion Of Specific Three Terminal R-LLF Devices

In this section the following question is investigated for some specific R-LLF devices that involve Characteristic Determinants with constructible specifications. Can an RC-LLFPB network be found such that when the R-LLF device is embedded in the RC-LLFPB network the resulting RC-LLF:R network will have a prescribed set of complex natural frequencies? To answer this question we form the Characteristic Determinant pertinent to the particular description of the R-LLF device and then see what restrictions there are on its zeroes.

We will consider first the R-LLF devices and LLFPB networks that are pertinent to cases (1) through (5) of Fig. 5.3.1 and then we will discuss the cases of Figures 5.3.3, 5.3.4, and 5.3.5.

5.4.1 Cases (1) and (2); G Matrices

The Characteristic Determinant for cases (1) and (2) is shown below equated to a rational fraction in (s).

$$\Delta^g = 1 + g_{11}\hat{z}_{11} + g_{22}\hat{z}_{22} = \frac{P(s)}{Q(s)} \tag{5.4.1}$$

P(s) and Q(s) are polynomials in s with real coefficients. Since we are assuming that the LLFPB network is RC, then $\hat{z}_{11}$ and $\hat{z}_{22}$ have their poles constrained to the negative real axis. It follows that the poles of Δ^g which are the zeroes of Q(s) must lie on the negative real axis. Thus let

$$Q(s) = (s + \sigma_1)(s + \sigma_2)\cdots(s + \sigma_n) \tag{5.4.2}$$

where the

$$\sigma_j > 0; \; j = 1,2\cdots n \;\; ; \; \sigma_j \neq \sigma_k \tag{5.4.3}$$

It is assumed that P(s) is a polynomial in s of the n^{th} degree. We note first of all that if both g_{11} and g_{22} are positive then the zeroes of P(s) which are the zeroes of Δ^g, must lie on the negative real axis. This follows from the obvious fact that if g_{11} and g_{22} are > 0 then Δ^g is, according to Eq. 5.4.1, equal to an RC-LLFPB driving point impedance.

If g_1 and g_2 are both negative then Δ^g is expressible in the form

$$\Delta^g = 1 - Z(s) = \frac{P(s)}{Q(s)} \tag{5.4.4}$$

where Z(s) is an RC-LLFPB driving point impedance. It then follows that P(s) is the difference between the denominator and numerator polynomials of an RC-LLFPB driving point impedance. One may readily demonstrate that such a P(s) not only has all its zeroes on the σ axis but may have at most one zero on the positive real axis.

The remaining possibility is that g_{11} and g_{22} be of opposite algebraic sign. It will now be demonstrated that for this situation P(s) may have zeroes located arbitrarily in the complex plane. To prove this fact we will assign a P(s) with arbitrary zeroes and then find an algorithm whereby one may find physically realizable impedances $\hat{z}_{11}$ and $\hat{z}_{22}$. Let

$$P(s) = s^n + a_{n-1}s^{n-1} + \cdots a_1 s + a_0 \tag{5.4.5}$$

where the a's are restricted only in that they are real. Expand P/Q in partial fractions. This expansion takes the form

$$\frac{P(s)}{Q(s)} = 1 + \sum_1^n \frac{k_j}{s + \sigma_j} = 1 + \sum \frac{k_j^+}{s + \sigma_j^+} - \sum \frac{k_j^-}{s + \sigma_j^-} \tag{5.4.6}$$

Some of the residues will be positive and some will be negative. The last equality in Eq. 5.4.6, where k_j^+ and k_j^- are positive, evidences this fact in an obvious fashion. If we compare Eq. 5.4.6 with Δ^g in Eq. 5.4.1 and make the identifications

$$\hat{z}_{11} = \frac{1}{g_{11}} \sum \frac{k_j^+}{s + \sigma_j^+}$$

$$\hat{z}_{22} = -\frac{1}{g_{22}} \sum \frac{k_j^-}{s + \sigma_j^{-1}} \tag{5.4.7}$$

then $\hat{z}_{11}$ and $\hat{z}_{22}$ will be p.r. and RC if

$$g_{11} > 0, \; g_{22} < 0 \tag{5.4.8}$$

An interesting point here is that any set of g's which satisfy Eq. 5.4.8 will lead to p.r. and RC, $\hat{z}_{11}$ and $\hat{z}_{22}$. Thus for case (1) we conclude that if $\hat{z}_{12} = 0$ and the R-LLF device has a G matrix of the form

$$G_1 = \begin{bmatrix} a & b \\ -c & -d \end{bmatrix} \; ; \; ad = bc \tag{5.4.9}$$

then the resulting RC-LLF: device may be synthesized to have natural frequencies arbitrarily placed in the complex plane. Since $\hat{z}_{12} = 0$, the RC-LLF:R network must take the configuration indicated in Fig. 5.4.

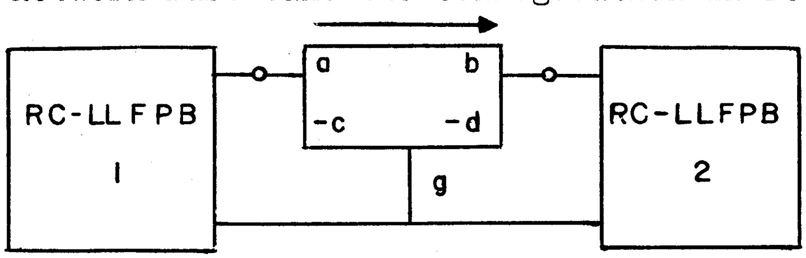


Figure 5.4.1. RC-LLF:R Network Permitting Arbitrary Assignment Of Natural Frequencies: G_1 Type Matrix

For case (2) we do not have to assume that $z_{12} = 0$ but then we have the restriction

$$b = c \tag{5.4.10}$$

leading to

$$G_2 = \begin{bmatrix} a & b \\ -b & -d \end{bmatrix} \; ; \; ad = b^2 \tag{5.4.11}$$

The RC-LLF:R network in this case is not restricted to the configuration of Fig. 5.4.1 but to the more general configuration of Fig. 5.4.2. While the above procedures allow complex natural frequencies to be arbitrarily assigned no attention has been given to the problem of

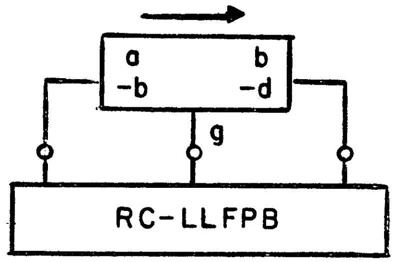


Figure 5.4.2. RC-LLF:R Network Permitting Arbitrary Assignment Of Natural Frequencies: G_2 Type Matrix

synthesizing RC-LLF:R transfer functions for specified pole-zero locations. This problem is discussed in Chapter 6 where terminal pairs are brought out from the RC-LLFPB networks. To maintain the natural frequencies as designed in this section one must observe the usual precaution of inserting voltage sources by a plier-type entry and current sources by a soldering iron-type entry into the RC-LLFPB networks.

5.4.2 Case (3); G Matrices

The Characteristic Determinant for case (3) is

$$\Delta^g = 1 + g_{21}\hat{z}_{12} = \frac{P(s)}{Q(s)} \tag{5.4.12}$$

If we solve for $\hat{z}_{12}$ we obtain

$$\hat{z}_{12} = \frac{1}{g_{21}} \left[\frac{P(s) - Q(s)}{Q(s)}\right] \tag{5.4.13}$$

We note that $\hat{z}_{12}$ is the transfer impedance of a grounded two-terminal pair RC-LLFPB network. The necessary and sufficient conditions for realizability of $\hat{z}_{12}$ are

(1) that it have simple negative real poles with real residues

(2) that the numerator polynomial have all positive coefficients and degree $\leq$ degree of denominator polynomial.

Condition (1) is automatically satisfied by selecting Q(s) as below

$$Q(s) = \alpha(s + \sigma_1)(s + \sigma_2) \cdots (s + \sigma_n) \tag{5.4.14}$$

where the σ's are positive and α is a positive constant. Let us assume first that g_{21} is positive. Then (2) can only be satisfied if P(s) has all positive coefficients. This is easily seen from the fact that -Q(s) has all negative coefficients. Consequently if P(s) has negative coefficients then P(s) - Q(s) will at least have negative coefficients in the same locations. Thus for g_{21} positive the assignment of natural frequencies must be the zeroes of a polynomial in s with positive coefficients.

If we let g_{21} be negative then it is readily seen that P(s) becomes unrestricted. This is best seen by rewriting $\hat{z}_{12}$ for this case as follows

$$\hat{z}_{12} = \frac{1}{|g_{21}|}\left[\frac{Q - P}{Q}\right] \tag{5.4.15}$$

Since Q has positive coefficients we may always adjust α in Eq. 5.4.14 such that Q - P has positive coefficients, whether or not P has positive coefficients. From a practical point of view, the case where $g_{21} > 0$ is not less useful from $z_{21} < 0$ since a stable transfer function always has a denominator polynomial which is Hurwitz and a Hurwitz polynomial has positive coefficients. It will be recognized that the G matrix for case (3) is that of an ideal vacuum tube when $g_{21} > 0$.

In practice when one synthesizes z_{12} one makes an initial synthesis synthesis which yields z_{12} to within a constant multiplier. Then impedance leveling adjusts the transfer impedance to the desired constant multiplier. It may be seen then, that regardless of the value of g_{21} one may obtain a physically realizable $\hat{z}_{21}$.

5.4.3 Case (4); G Matrices

The characteristic equation for this case is

$$\Delta^g = 1 + g_{22}\hat{z}_{22} + g_{21}\hat{z}_{12}$$

We are interested in the generality of location of the complex roots of

$$\Delta^g = 0 = 1 + g_{22}\hat{z}_{22} + g_{21}\hat{z}_{12} \tag{5.4.16}$$

But these are also the complex roots of the equation

$$1 + \frac{g_{21}\hat{z}_{12}}{1 + g_{22}\hat{z}_{22}} = 0 \tag{5.4.17}$$

obtained from Eq. 5.4.6 through dividing by $1 + g_{22}\hat{z}_{22}$. No complex roots are removed by this operation since $1 + g_{22}\hat{z}_{22}$ has only real axis zeroes whether or not g_{22} is negative. We may place

$$1 + \frac{\frac{g_{21}}{g_{22}}\hat{z}_{12}}{\frac{1}{g_{22}} + \hat{z}_{22}} = \frac{P(s)}{Q(s)} \tag{5.4.18}$$

Now the expression

$$z_{12} = \frac{\frac{1}{g_{22}}\hat{z}_{12}}{\frac{1}{g_{22}} + \hat{z}_{22}} \tag{5.4.19}$$

will be recognized as the transfer impedance of the network of Fig. 5.4.3.

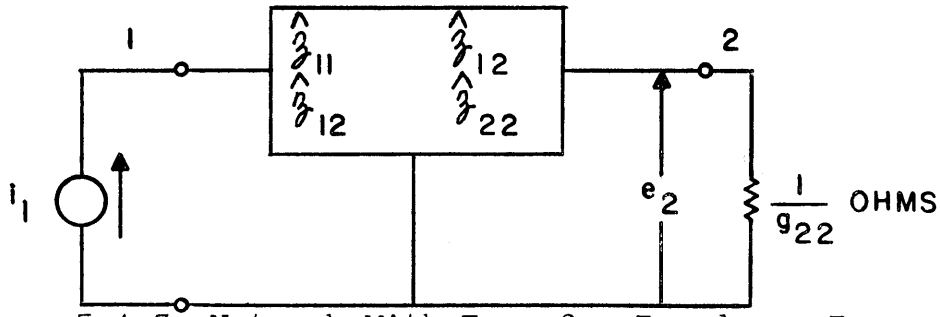


Figure 5.4.3. Network With Transfer Impedance Z_{12}

With this definition of Z_{12} Eq. 5.4.18 assumes the form

$$1 + g_{21}Z_{12} = \frac{P(s)}{Q(s)} \tag{5.4.20}$$

Note that this equation is identical in form Eq. 5.4.12 for case (3). Thus solving for Z_{12} we obtain

$$Z_{12} = \frac{1}{g_{21}}\left[\frac{P - Q}{Q}\right] \tag{5.4.21}$$

If we assume that g_{22} is positive then we can adjust the impedance levels of the passive network, without loss in generality, by letting

$$\hat{z}_{22} \rightarrow \frac{\hat{z}_{22}}{g_{22}}$$

$$z_{12} \rightarrow \frac{\hat{z}_{12}}{g_{22}}$$

$$Z_{12} \rightarrow \hat{Z}_{12} = \frac{\hat{z}_{12}}{1 + z_{22}} \quad \text{(one ohm termination)} \tag{5.4.22}$$

We define

$$u = \frac{g_{21}}{g_{22}} \tag{5.4.23}$$

as the amplification factor of the R-LLF device since when g_{21} and g_{22} are positive the R-LLF device has a G matrix identical in form to that of a vacuum tube at low frequencies (incremental linear operation) in which $1/g_{22}$ is the plate resistance and g_{21} is the transconductance. For g_{21} positive and normalization as indicated in Eq. 5.4.22, Eq. 5.4.21 becomes

$$\hat{Z}_{12} = u\left[\frac{P - Q}{Q}\right] \tag{5.4.24}$$

Since $\hat{Z}_{12}$ is an RC-LLFPB transfer function, the implications of Eq. 5.4.24 are almost identical to those of Eq. 5.4.13. Thus if $u > 0$, P must have positive coefficients while if $u < 0$, P may be generally assigned. There is however one important difference between the situation of Eq. 5.4.24 and Eq. 5.4.13. This is that while any value of g_{21} in Eq. 5.4.13 will lead to acceptable $\hat{z}_{12}$ functions, any value of u in Eq. 5.4.24 will not. To understand this fact note that $\hat{Z}_{12}$ is constrained to be the transfer impedance of an RC-LLFPB network terminated in 1 ohm while $\hat{z}_{12}$ is not restricted in this way. Thus while incorrect constant multipliers resulting from a synthesis of $\hat{z}_{12}$ can be compensated for by a change in impedance level, this is not generally the case with $\hat{Z}_{12}$. Synthesis of $\hat{Z}_{12}$ requires the simultaneous synthesis of $\hat{z}_{22}$ and $\hat{z}_{12}$. Synthesis procedures presently available allow $\hat{z}_{12}$ to be synthesized to within some maximum constant multiplier when $\hat{z}_{22}$ is also specified(13). A change in impedance level of $\hat{Z}_{12}$ is accomplished only by a change in level of $\hat{z}_{22}$, $\hat{z}_{12}$, and the one ohm termination by the same amount.

It will be demonstrated that if g_{22} is negative and g_{21} is positive then P can be chosen arbitrarily subject again to the condition that u may not be fixed a priori. In this case the Characteristic Determinant has the form

$$\Delta^{g} = 1 + g_{21}z_{12} - \left|g_{22}\right| z_{22} = \frac{P(s)}{Q(s)} \tag{5.4.25}$$

Expand P/Q, $\hat{z}_{12}$, and $\hat{z}_{22}$ in partial fractions (degree of P and Q the same)

$$\frac{P(s)}{Q(s)} = \sum \frac{k^{(j)}}{s + \sigma_j} + 1$$

$$\hat{z}_{12}(s) = \sum \frac{k_{12}^{(j)}}{s + \sigma_j}$$

$$\hat{z}_{22}(s) = \sum \frac{k_{22}^{(j)}}{s + \sigma_j} \tag{5.4.26}$$

If we equate residues we find that

$$g_{21}k_{12}^{(j)} - \left|g_{22}\right| k_{22}^{(j)} = k^{(j)} \tag{5.4.27}$$

It is clear that positive values of $k_{12}^{(j)}$ and $k_{22}^{(j)}$ may be found to satisfy Eq. 5.4.27 whether or not $k^{(j)}$ is positive. Since any poles that $\hat{z}_{12}$ has must also be contained in $\hat{z}_{22}$ we must always have $k_{22}^{(j)} > 0$ when $k_{12}^{(j)} \neq 0$. For a specified $z_{22}(s)$ one can synthesize $z_{12}(s)$ only to within a maximum constant multiplier. Thus the set of residues $k_{12}^{(j)}$ may all be off by a constant multiplier. It may be seen that if we are free to choose the ratio $g_{21}/\left|g_{22}\right|$ as large as we wish we can always compensate for any constant multiplier.

If g_{12} and g_{22} are both negative one may show that in general P(s) must have positive coefficients.

5.4.4 Case (5); G Matrices

We will only consider here the situation in which

$$g_{11} = 0$$

$$g_{22} = 0$$

$$\hat{z}_{12} = 0 \tag{5.4.28}$$

The R-LLF device then has the G-matrix

$$G_5 = \begin{bmatrix} 0 & g_{12} \\ g_{21} & 0 \end{bmatrix} \tag{5.4.29}$$

and the Characteristic Determinant is

$$\Delta^g = 1 - g_{12}g_{21}\hat{z}_{11}\hat{z}_{22} \tag{5.4.30}$$

The complex zeroes of Δ^g are also those for the function

$$\frac{1}{\hat{z}_{11}} - g_{12}g_{21}\hat{z}_{22} = \frac{P(s)}{Q(s)} \tag{5.4.31}$$

Let us first assume that g_{12} and g_{21} are of the same sign. In this case Eq. 5.4.31 takes the form

$$\frac{1}{\hat{z}_{11}} - \left|g_{12}g_{21}\right|\hat{z}_{22} = \frac{P(s)}{Q(s)} \tag{5.4.32}$$

It will now be shown that P(s) in Eq. 5.4.32 has its zeroes constrained to the σ axis. When $\hat{z}_{11}$ has no zero at ∞, $\frac{1}{\hat{z}_{11}}$ has the partial fraction expansion.

$$\frac{1}{z_{11}} = k_{11}^{0} - \sum_{1}^{n} \frac{k_{11}^{j}}{s + \alpha_j} = \hat{y}_{11} \tag{5.4.33}$$

The poles of $1/z_{11}$ located at $s = -\alpha_j$; $j = 1,2, \cdots n$ where α_j is positive. The quantities k_{11}^{0}, k_{11}^{j} are positive. Now the sum

$$Z = \sum_{1}^{n} \frac{k_{11}^{j}}{s + \alpha_j} \tag{5.4.34}$$

satisfies all the requirements for a p.r. RC impedance. Thus the RC admittance

$$y_{11} = k_{11}^{0} - Z \tag{5.4.35}$$

is the difference between a constant and an RC impedance. It follows that

$$\frac{P(s)}{Q(s)} = k_{11}^{0} - \left[Z + \left| g_{12} g_{21} \right| \hat{z}_{22} \right] \tag{5.4.36}$$

Thus P(s) is expressible as the difference between the numerator and denominator polynomial of an RC impedance. Such a difference has its zeroes on the σ axis with at most one positive σ axis zero.

When $1/z_{11}$ has a zero at infinity one may still demonstrate that P(s) has zeroes on the real axis although the number of positive real axis zeroes has not been determined at present. The proof offered here is somewhat indirect and makes use of a property of LLF networks demonstrated in Chapter 7. This property is that an RC-LLF:R network containing only active bilateral devices has its natural frequencies constrained to the real axis. To apply it to our present situation we note that if $g_{12} = g_{21}$ the form of Eq. 5.4.32 remains unchanged. Thus P(s) is no less general when $g_{12} = g_{21}$. But when $g_{12} = g_{21}$ the R-LLF device becomes bilateral. Thus applying the theorem we conclude that P(s) has only negative real axis zeroes whenever g_{12} and g_{21} are of the same sign.

Let us now assume that g_{12} and g_{21} are of opposite sign. Then Eq. 5.4.31 may be rewritten in the form

$$\hat{y}_{11} + \left| g_{12}g_{21} \right| \hat{z}_{22} = \frac{P(s)}{Q(s)} \tag{5.4.37}$$

An RC admittance behaves exactly like an RL impedance when considered as a function of s. Thus the zeroes of P(s) are constrained by Eq. 5.4.37 to be located no more generally than the zeroes of the sum of an RC and an RL impedance. Another way of saying this is that the zeroes of P(s) are the short circuit natural frequencies of the series combination of an RC and an RL impedance. It is difficult to state in a precise quantitative way how such zeroes are restricted. It is readily seen that j axis zeroes are forbidden. In a qualitative way one can see that a number of zeroes in the same area making small angles with the jw axis will, in general, be forbidden. The reasoning behind this statement is the fact that in the jw axis vicinity of such zeroes the phase of P(s) increases very rapidly by many radians. Since P/Q is a p.r. function its phase may not go beyond $\pm\, \pi/2$ radians. Thus the increase in phase of P must be matched by a corresponding increase in phase of Q(s). But Q(s) has only negative real poles which are located distantly from that portion of the jw axis under discussion. Thus the phase contribution from Q(s) will vary too slowly to compensate for the phase change of P(s). One may conjecture, however, that if we augment P/Q as follows to form P'/Q'

$$\frac{P'}{Q'} = \frac{PQ_2}{QQ_1}$$

where Q_2 and Q_1 have negative real zeroes, that the phase change of P(s) may be accommodated. The idea is that with enough poles and zeroes on the negative real axis we can approximate rather general phase characteristics.

When $g_{21}g_{12} = -g^2$ and $|g_{21}| = |g_{12}|$ the R-LLF device becomes a gyrator with G matrix

$$G = \begin{bmatrix} 0 & g \\ -g & 0 \end{bmatrix} \qquad (5.4.38)$$

Thus we have demonstrated that a gyrator in a network configuration like Fig. 5.3.2 produces a resulting RC-LLF:R network whose natural frequencies are no more general than the short circuit natural frequencies of the series combination of an RC and an RL impedance. We will now demonstrate that a gyrator quite generally embedded in an RC-LLFPB network leads to an RC-LLF:R network with the same restrictions. The characteristic equation for a generally embedded gyrator ($\hat{z}_{12} \neq 0$) is

$$\Delta^g = 1 + g^2\Delta\hat{z} = 1 + g^2\,\frac{\hat{z}_{22}}{y_{11}} \qquad (5.4.39)$$

where we have used the identity

$$\Delta^{\hat{z}} = \frac{\hat{z}_{22}}{\hat{y}_{11}} \tag{5.4.40}$$

It is clear that the complex zeroes of Δ^g of Eq. 5.4.39 are also zeroes of

$$\hat{y}_{11} + g^2\hat{z}_{22} = \frac{P(s)}{Q(s)} \tag{5.4.41}$$

Since Equation 5.4.41 is identical in form to 5.4.37 we have clearly obtained the desired result.

5.4.5 Other Cases; R, K, and H Matrices

Cases (1) through (5) of Fig. 5.3.3 wherein R matrices are used are dual to those of Cases (1) through (5) of Fig. 5.3.1 where G matrices are used. Consequently the discussion of Sections 5.4.1 through 5.4.4 may be carried over in dual form in order to discuss the locations of the natural frequencies of an RC-LLF:R network with embedded R-LLF devices described by the R matrices of Fig. 5.3.3. This process of dualizing the statements of the previous sections is quite straightforward and will not be carried through.

We will now consider the R-LLF devices and LLFPB networks that are pertinent to Cases (1) through (3) of Fig. 5.3.4. For Case (1) the Characteristic Determinant is given by

$$\Delta^k = 1 - k_{21}k_{12}\frac{\hat{z}_{22}}{\hat{z}_{11}} \tag{5.4.42}$$

The complex zeroes of Δ^k are the same as those of P(s) in the following equation

$$\hat{z}_{11} - k_{21}k_{12}\hat{z}_{22} = \frac{P(s)}{Q(s)} \tag{5.4.43}$$

Clearly if k_{21} and k_{12} are of opposite sign then the zeroes of $P(s)$ lie on the negative real axis since P/Q is a p.r. RC impedance in this case. When k_{21} and k_{12} are of the same sign then Eq. 5.4.43 becomes

$$\hat{z}_{11} - |k_{21}k_{12}|\hat{z}_{22} = \frac{P(s)}{Q(s)} \tag{5.4.44}$$

It is readily shown that the zeroes of $P(s)$ may now be assigned without restriction (providing of course as we have been assuming throughout that complex zeroes occur in conjugate pairs). To this end expand P/Q in partial fractions as indicated in Eq. 5.4.6 and then identify

$$\hat{z}_{11} = 1 + \sum \frac{k_j^+}{s + \sigma_j^+}$$

$$\hat{z}_{22} = \frac{1}{|k_{21}k_{12}|} \sum \frac{k_j^-}{s + \sigma_j^-} \tag{5.4.45}$$

The impedances $\hat{z}_{11}$ and z_{22} thus constructed are p.r. RC impedances. It is then clear that a network of the configuration shown in Fig. 5.4.4 may have an arbitrary set of complex natural frequencies specified. Case (2) yields a Characteristic Determinant with complex zeroes as generally located as zeroes of $P(s)$ of Eq. 5.4.44. Thus the network of Fig. 5.4.5 may have its natural frequencies arbitrarily assigned. When $b = c = 1$ or $k = 1$ the R-LLF devices in

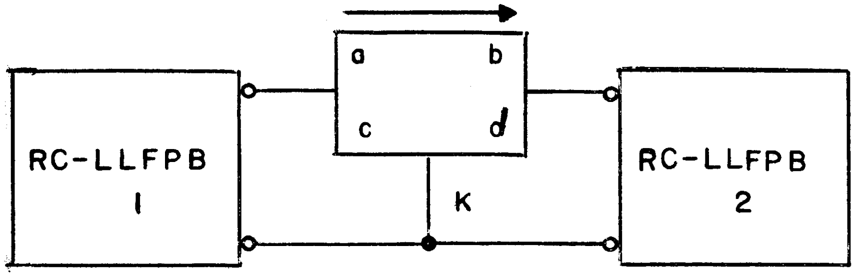


Figure 5.4.4. RC-LLF:R Network Permitting Arbitrary Assignment Of Natural Frequencies: K_1 Type Matrix

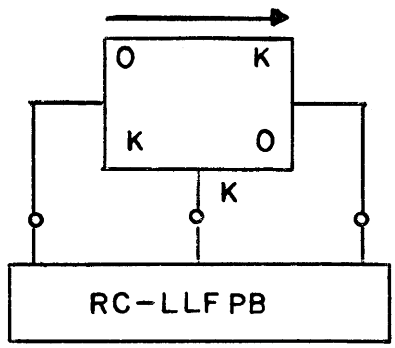


Figure 5.4.5
K_2 Type Matrix

Figures 5.4.4 and 5.4.5 become Negative Impedance Converters. A Negative Impedance Converter has the K matrix

$$K = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{5.4.46}$$

Linvill(4) has already demonstrated that a Negative Impedance Converter in the network configuration of Fig. 5.4.4 allows a general assignment of natural frequencies.

The Characteristic Determinant for Case (3) of Fig. 5.3.4 is

$$\Delta^k = 1 + k_{21}\hat{a}_{21} = \frac{P(s)}{Q(s)} \tag{5.4.47}$$

If we solve for the voltage transfer ratio $\hat{a}_{21}$ we find

$$a_{21} = \frac{1}{k_{21}}\left[\frac{P - Q}{Q}\right] \tag{5.4.48}$$

This case is entirely analogous to Cases (5) of Fig's. 5.3.1 and 5.3.3. We may state by analogy that if k_{21} is positive then P is restricted to have positive coefficients while if k_{21} is negative P is unrestricted. There is an important difference between Case (3) of Fig. 5.3.4 and Cases (5) of Fig's. 5.3.1 and 5.3.3. This is that while $\hat{z}_{12}$ and $\hat{y}_{12}$ may be synthesized exactly, $\hat{a}_{21}$ can only be synthesized to within a maximum constant multiplier. Thus while any g_{21} or r_{21} would suffice to obtain the desired P(s) in Cases (5) of Fig's. 5.3.1 and 5.3.3 only values of $|k_{21}|$ sufficiently large will be suitable for Case (3) of Fig. 5.3.4.

Of the situations shown in Cases (1) through (3) of Fig. 5.3.5 only Case (3) presents a new R-LLF device. The discussion of the possibilities of the R-LLF device of Case (3) of Fig. 5.3.5 is identical to that just completed for Case (3) of Fig. 5.3.5.

5.5 General Expression For Driving Point Impedance

In the previous sections we have been concerned with the complex natural frequencies caused by the introduction of an R-LLF device into an LLFPB network. Only those terminal pairs of the LLFPB network were evidenced that were connected to the R-LLF device. We now

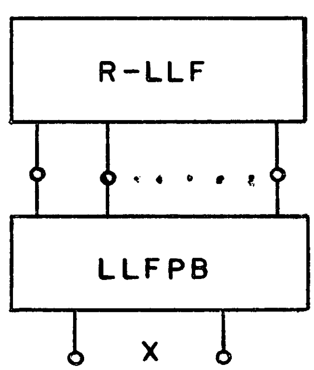


Figure 5.5.1. LLF:R Network Relevant To Determining General Expression For Impedance

extract a terminal pair x from the LLFPB network as shown in Fig. 5.5.1 and inquire as to the nature of the impedance, z, seen looking into this terminal pair. The poles of this impedance are the natural frequencies of the LLF:R network with x open while the zeroes are the natural frequencies of the LLF:R network with x shorted. We have previously determined that any complex natural frequencies of the LLF:R network are zeroes of the Characteristic Determinant

$$\Delta = \det \left\{ U_{ss} + \hat{S}_{ss} e_{ss} \right\}$$

When we examine the impedance at x we are concerned with two sets of complex natural frequencies. Those which exist for x open and those which exist for x shorted. Thus we define two Characteristic Determinants

$$\Delta_0 = \det \left\{ U_{ss} + \hat{S}^{0}_{ss} e_{ss} \right\} \tag{5.5.1}$$

and

$$\Delta_\infty = \det \left\{ U_{ss} + \hat{S}^{\infty}_{ss} e_{ss} \right\} \tag{5.5.2}$$

in which $\hat{S}^{0}_{ss}$ is the solution matrix of the LLFPB network seen from the s terminal pairs connected to the R-LLF device when x is shorted. Correspondingly $\hat{S}^{\infty}_{ss}$ is the solution matrix with x open.

It is then clear that the impedance seen at x may be expressed in the form

$$z(s) = F(s) \frac{\Delta_0}{\Delta_\infty} = F(s) \frac{\det \left\{ U_{ss} + \hat{S}^{0}_{ss} e_{ss} \right\}}{\det \left\{ U_{ss} + \hat{S}^{\infty}_{ss} e_{ss} \right\}} \tag{5.5.3}$$

The determinant Δ_0 contains the complex zeroes and Δ_∞ contains the complex poles. We may determine $F(s)$ in the following fashion. First we note that

$$\left\{\Delta_0\right\}_{e_{ss}=0} = \det\left\{U_{ss}\right\} = 1$$

$$\left\{\Delta_\infty\right\}_{e_{ss}=0} = \det\left\{U_{ss}\right\} = 1 \tag{5.5.4}$$

Thus

$$F(s) = \left[z(s)\right]_{\left[e_{ss}\right]=0} \tag{5.5.5}$$

i.e. $F(s)$ is the impedance seen at terminal pair x when the equilibrium matrix of the R-LLF device becomes null. As discussed previously letting $\left[e_{ss}\right]$ become null effectively removes the R-LLF device and places either open circuit or short-circuit constraints upon the set of terminal pairs of the LLFPB network that were connected to the R-LLF device. If $\left[e_{ss}\right]$ were described by a G matrix then open-circuit constraints would be applicable. If $\left[e_{ss}\right]$ were described by an R matrix then short-circuit constraints would be applicable. If $\left[e_{ss}\right]$ were described on a mixed basis some terminal pairs would be shorted and some opened.

To illustrate the above ideas we will consider a three terminal R-LLF device. There are then four types of Characteristic Determinants depending upon whether the device is described by a G, R, H, or K matrix. The impedance z(s) takes the four possible forms indicated below

$$z(s) = \hat{z}_g \frac{\Delta_0^g}{\Delta_\infty^g} = \hat{z}_r \frac{\Delta_0^r}{\Delta_\infty^r} = \hat{z}_k \frac{\Delta_0^k}{\Delta_\infty^k} = \hat{z}_h \frac{\Delta_0^h}{\Delta_\infty^h} \tag{5.5.6}$$

The appropriate R:LLF networks are shown in Fig. 5.2.1, Fig. 5.2.3, Fig. 5.2.7, and Fig. 5.2.9 with the terminal pair x still contained within the LLFPB network. One may readily define the impedances $\hat{z}_g$, $\hat{z}_r$, $\hat{z}_k$, and $\hat{z}_h$ with the aid of Fig's. 5.2.2, 5.2.4, 5.2.8, and 5.2.10. Extract the terminal pair x from the LLFPB network and then let the source quantities in these figures vanish. By this means we find that $\hat{z}_g$, $\hat{z}_r$, $\hat{z}_k$, and $\hat{z}_h$ are defined as shown in Fig's. 5.5.2a, b, c and d respectively.

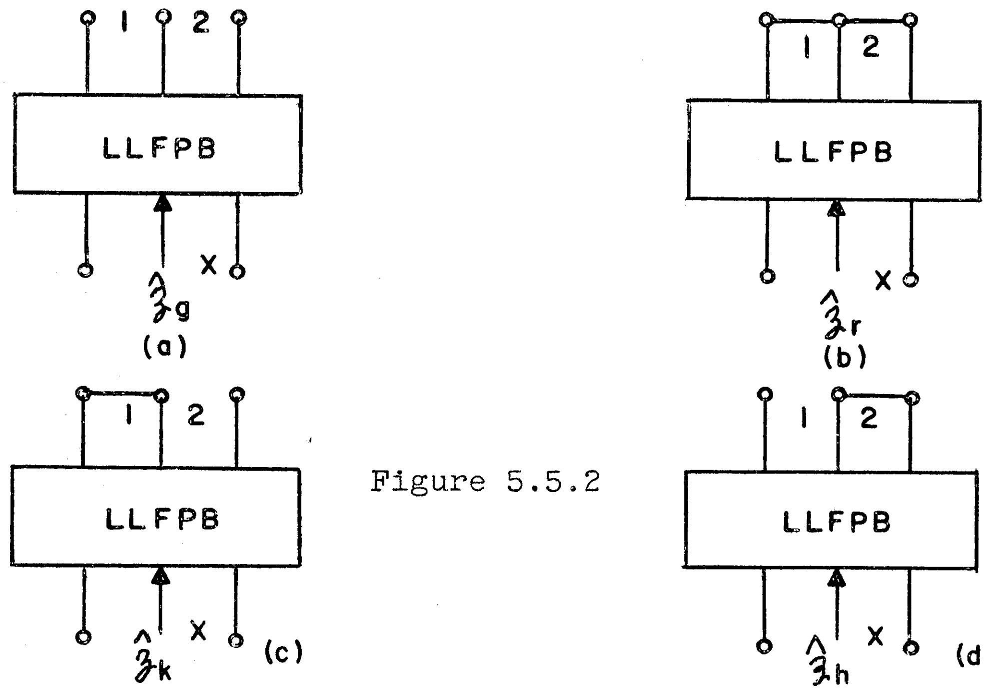


Figure 5.5.2

The general expression for z(s) will be written out below for the case in which the R-LLF device is described by a G matrix

$$z(s) = \hat{z}_g \frac{1 + \Delta g \Delta \hat{z}^0 + g_{11}\hat{z}_{11}^0 + g_{22}\hat{z}_{22}^0 + (g_{12} + g_{21})\hat{z}_{12}^0}{1 + \Delta g \Delta \hat{z}^\infty + g_{11}\hat{z}_{11}^\infty + g_{22}\hat{z}_{22}^\infty + (g_{12} + g_{21})\hat{z}_{12}^\infty} \tag{5.5.7}$$

where $\hat{z}_{jk}^0$ is an impedance of the LLFPB network with x shorted while z_{jk}^∞ is the corresponding impedance with x opened. It is of interest to consider some specific cases. Let

$$G_2 = \begin{bmatrix} g & g \\ -g & -g \end{bmatrix} \tag{5.5.8}$$

$$G_3 = \begin{bmatrix} 0 & 0 \\ g & 0 \end{bmatrix} \tag{5.5.9}$$

$$G_4 = \begin{bmatrix} 0 & 0 \\ g & g \end{bmatrix} \tag{5.5.10}$$

$$G_5 = \begin{bmatrix} 0 & g \\ -g & 0 \end{bmatrix} \tag{5.5.11}$$

It will be recognized that G_2 through G_5 are special versions of Cases (2) through (5) of Fig. 5.3.1. The device with matrix G_2 of Eq. 5.5.8 will be called an Activated Gyrator. The devices corresponding to G matrices of Eq's. 5.5.9 through 5.5.11 will be recognized as an Ideal Vacuum Tube, Vacuum Tube, and Gyrator, respectively. The corresponding expressions for driving point impedances are

$$z_2(s) = \hat{z}_g \frac{1 + g\left[\hat{z}_{11}^{0} - \hat{z}_{22}^{0}\right]}{1 + g\left[\hat{z}_{11}^{\infty} - \hat{z}_{22}^{\infty}\right]} \tag{5.5.12}$$

$$z_3(s) = \hat{z}_g \frac{1 + g\hat{z}_{12}^{0}}{1 + g\hat{z}_{12}^{\infty}} \tag{5.5.13}$$

$$z_4(s) = \hat{z}_g \frac{1 + g\left[\hat{z}_{22}^{0} + \hat{z}_{12}^{0}\right]}{1 + g\left[\hat{z}_{22}^{\infty} + \hat{z}_{12}^{\infty}\right]} \tag{5.5.14}$$

$$z_5(s) = \hat{z}_g \frac{1 + g^2 \Delta \hat{z}^0}{1 + g^2 \Delta \hat{z}^\infty} \qquad (5.5.15)$$

The expression 5.5.13 has already been derived by De Claris[9] although in an entirely different fashion.

CHAPTER 6

APPLICATIONS OF COMPLEX LINEAR TRANSFORMATIONS TO THE SYNTHESIS OF TRANSFER FUNCTIONS OF RC-LLF:R NETWORKS

6.1 Introduction

In Chapter 3 some particular techniques of LLF:R network analysis through the use of linear transformations were presented. The techniques involved both real and complex linear transformations. In Chapter 4 the analysis techniques involving real transformations were studied with the idea of inverting the analysis procedure and forming synthesis procedures. The inversion process was not entirely successful since the "constructible" specifications requirement was not met in general. In Chapter 6 the analysis techniques involving complex linear transformations are inverted successfully. Three general transfer function synthesis procedures are presented that involve 2 two-terminal-pair RC-LLFPB networks and one three terminal R-LLR device described by a G matrix. Two of these methods will synthesize any stable transfer function to within a constant multiplier. The starting point for developing these synthesis procedures is an analysis of RC-LLF:R networks of the type described in Chapter 5 which lead to constructible specifications on locations of complex natural frequencies. Specifically, network Configurations and R-LLF devices suggested by Section 5.4.1, 5.4.3, and 5.4.4 are used. The three synthesis techniques are presented in Sections 6.2, 6.3, and 6.4. Section 6.5 presents an illustrative example of transfer function synthesis for each of the three synthesis techniques.

6.2 Synthesis Technique No. 1

6.2.1 Transfer Impedance

In this section we will first analyze the LLF:R network of Figure 6.2.1 with the aid of complex linear transformations in the manner discussed in Section 3.5 of Chapter 3, as a preliminary to the synthesis procedure.

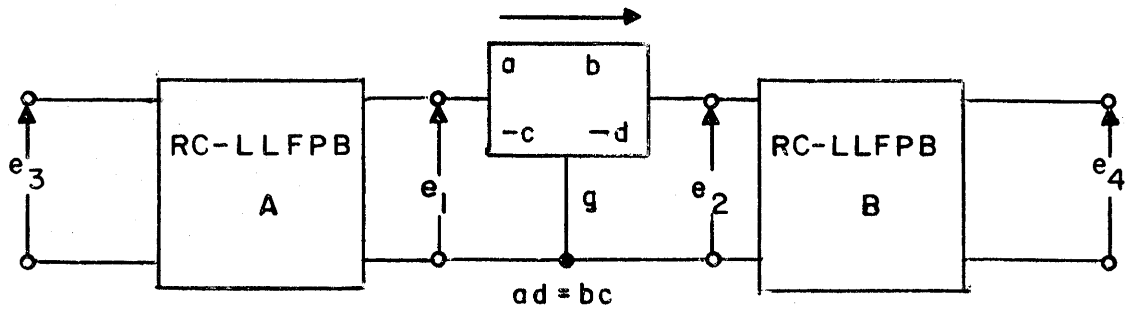


Figure 6.2.1 LLF:R Network Applicable To Synthesis Technique No. 1

It will be recognized that this network falls under case (2) of Figure 5.3.1 and has been discussed with regard to generality of location of complex natural frequencies in Section 5.4.1 of Chapter 5. In particular the network of Figure 6.2.1 is obtained from that of Figure 5.4.1 by extracting a terminal pair from each RC-LLFPB network, leaving an RC-LLF:R network with 4 terminal pairs. Let us assume that current sources are placed across terminal pairs so that the response quantities are terminal-pair voltages. We may now apply the results of Section 3.5 to express the o.c. impedance matrix Z of this RC-LLF:R network in the form

$$Z = \tau\hat{Z} \tag{6.2.1}$$

where Z is the o.c. impedance matrix of the network with the R-LLF device removed (leaving open circuit constraints at terminal pairs 1 and 2) and τ is a complex transformation matrix. In order to give a desired partitioned expression for τ it will be necessary to express Z and $\hat{Z}$ in partitioned form.

$$Z = \left[\begin{array}{cc|cc} z_{11} & z_{12} & z_{13} & z_{14} \\ z_{21} & z_{22} & z_{23} & z_{24} \\ \hline z_{31} & z_{32} & z_{33} & z_{34} \\ z_{41} & z_{42} & z_{43} & z_{44} \end{array}\right] = \left[\begin{array}{c|c} Z_{ss} & Z_{sr} \\ \hline Z_{rs} & Z_{rr} \end{array}\right] \tag{6.2.2}$$

$$\hat{Z} = \left[\begin{array}{cc|cc} \hat{z}_{11} & \hat{z}_{12} & \hat{z}_{13} & \hat{z}_{14} \\ \hat{z}_{21} & \hat{z}_{22} & \hat{z}_{23} & \hat{z}_{24} \\ \hline \hat{z}_{31} & \hat{z}_{32} & \hat{z}_{33} & \hat{z}_{34} \\ \hat{z}_{41} & \hat{z}_{42} & \hat{z}_{43} & \hat{z}_{44} \end{array}\right] = \left[\begin{array}{c|c} Z_{ss} & Z_{sr} \\ \hline Z_{rs} & Z_{ss} \end{array}\right] \tag{6.2.3}$$

Of course, for $\hat{Z}$, $\hat{z}_{jk} = \hat{z}_{kj}$.

The branch admittance matrix of the R-LLF device is

$$g_{ss} = \begin{bmatrix} a & b \\ -c & -d \end{bmatrix} \; ; \quad ad = bc \tag{6.2.4}$$

In terms of the matrices defined in Equations 6.2.2, 6.2.3 and 6.2.4 we may express the complex transformation matrix τ as (See Equation 3.5.14),

$$\tau = \left[\begin{array}{c|c} T & 0 \\ \hline -\hat{Z}_{rs} g_{ss} T & U_r \end{array}\right] \qquad (6.2.5)$$

where

$$T = U_s + \hat{Z}_{ss} g_{ss}^{-1} \qquad (6.2.6)$$

Use of τ as expressed by Equation 6.2.5 in Equation 6.2.1 yields the following partitioned expression for Z,

$$Z = \left[\begin{array}{c|c} Z_{ss} & Z_{sr} \\ \hline Z_{rs} & Z_{rr} \end{array}\right] = \left[\begin{array}{c|c} T\hat{Z}_{ss} & T\hat{Z}_{sr} \\ \hline \hat{Z}_{rs} - \hat{Z}_{rs} g_{ss} T \hat{Z}_{ss} & \hat{Z}_{rr} - \hat{Z}_{rs} g_{ss} T \hat{Z}_{sr} \end{array}\right] \qquad (6.2.7)$$

We will be interested in this section in the transfer impedances between terminal pairs 3 and 4 of Figure 6.2.1, z_{34}, z_{43}. These are elements of the submatrix Z_{rr} given by

$$Z_{rr} = \hat{Z}_{rr} - \hat{Z}_{rs} g_{ss} T \hat{Z}_{sr} \qquad (6.2.8)$$

To determine Z_{rr} as given by Equation 6.2.8 we need to evaluate the LLFPB open circuit impedance matrix $\hat{Z}$. Figure 6.2.2 shows the RC-LLFPB network with o.c. impedance matrix $\hat{Z}$.

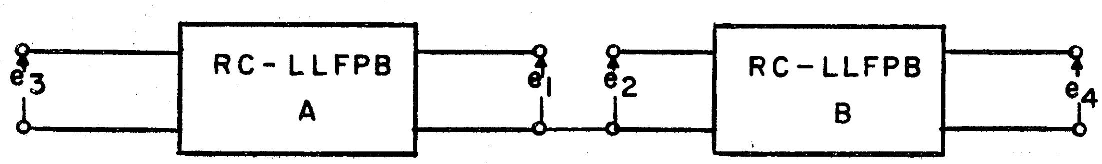


Figure 6.2.2 RC-LLFPB Network With O.C. Impedance Matrix $\hat{Z}$

The networks A and B are isolated so that transfer relations are zero between the terminal pairs (1,3) and (2,4). Thus $\hat{Z}$ takes the form

$$Z = \left[\begin{array}{cc|cc} \hat{z}_{11} & 0 & \hat{z}_{13} & 0 \\ 0 & \hat{z}_{22} & 0 & z_{24} \\ \hline \hat{z}_{31} & 0 & \hat{z}_{33} & 0 \\ 0 & \hat{z}_{42} & 0 & \hat{z}_{34} \end{array}\right] \tag{6.2.9}$$

and the submatrices

$$\begin{aligned} \hat{Z}_{ss} &= \begin{bmatrix} \hat{z}_{11} & 0 \\ 0 & \hat{z}_{22} \end{bmatrix} \quad \hat{Z}_{32} = \begin{bmatrix} \hat{z}_{13} & 0 \\ 0 & \hat{z}_{24} \end{bmatrix} \\ \hat{Z}_{rs} &= \begin{bmatrix} \hat{z}_{31} & 0 \\ 0 & \hat{z}_{42} \end{bmatrix} \quad \hat{Z}_{rr} = \begin{bmatrix} \hat{z}_{33} & 0 \\ 0 & \hat{z}_{34} \end{bmatrix} \end{aligned} \tag{6.2.10}$$

The remaining matrix needed to evaluate Z_{rr} is T. This is given by

$$\begin{aligned} T &= \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} \hat{z}_{11} & 0 \\ 0 & \hat{z}_{22} \end{bmatrix} \begin{bmatrix} a & b \\ -c & -d \end{bmatrix} \right\}^{-1} \\ &= \begin{bmatrix} 1+a\hat{z}_{11} & b\hat{z}_{11} \\ -c\hat{z}_{22} & 1-d\hat{z}_{22} \end{bmatrix} \\ &= \frac{1}{\Delta^g} \begin{bmatrix} 1-d\hat{z}_{22} & -b\hat{z}_{11} \\ c\hat{z}_{22} & 1+a\hat{z}_{11} \end{bmatrix} \end{aligned} \tag{6.2.11}$$

where Δ^g, the Characteristic Determinant, is given by

$$\Delta^g = 1 + a\hat{z}_{11} - \hat{d}z_{22} \tag{6.2.12}$$

Continuing the evaluation of $\hat{z}_{43}$ and $\hat{z}_{34}$ we note that

$$Z_{rs}g_{ss}T\,\hat{Z}_{sr} = -\begin{bmatrix} \hat{z}_{31} & o \\ o & \hat{z}_{42} \end{bmatrix}\begin{bmatrix} a & b \\ -c & -d \end{bmatrix}\begin{bmatrix} 1-d\hat{z}_{22} & -b\hat{z}_{11} \\ c\hat{z}_{22} & 1+a\hat{z}_{11} \end{bmatrix}\begin{bmatrix} \hat{z}_{13} & o \\ o & \hat{z}_{24} \end{bmatrix}\frac{1}{\Delta^g}$$

(6.2.13)

$$= -\begin{bmatrix} a\hat{z}_{31} & b\hat{z}_{31} \\ -c\hat{z}_{42} & -d\hat{z}_{42} \end{bmatrix}\begin{bmatrix} (1-d\hat{z}_{22})\hat{z}_{13} & -b\hat{z}_{11}\hat{z}_{24} \\ c\hat{z}_{22}\hat{z}_{13} & (1+a\hat{z}_{11})\hat{z}_{24} \end{bmatrix}\frac{1}{\Delta^g}$$

Since $\hat{Z}_{rr}$ contains no off diagonal terms, the off diagonal terms of $\hat{Z}_{rs}g_{ss}T\,\hat{Z}_{sr}$ yield z_{34} and z_{43} directly. By inspection of Equation 6.2.13 these are given by

$$z_{34} = \frac{ab\hat{z}_{31}\hat{z}_{11}z_{24} - b\hat{z}_{31}\hat{z}_{24}(1 + a\hat{z}_{11})}{\Delta^g} = -\,b\,\frac{\hat{z}_{31}\hat{z}_{24}}{\Delta^g}$$

(6.2.14)

$$z_{43} = \frac{cd\hat{z}_{42}\hat{z}_{22}\hat{z}_{13} + c\hat{z}_{42}\hat{z}_{13}(1 - d\hat{z}_{22})}{\Delta^g} = c\,\frac{z_{13}z_{42}}{\Delta^g}$$

Since z_{43} and z_{34} differ by a constant multiplier we need only examine z_{43}, the transfer impedance from terminal pair 3 to terminal pair 4 of the RC-LLF:R network of Figure 6.2.2. This is given by

$$z_{43} = \frac{d\hat{z}_{13}\hat{z}_{42}}{1 + a\hat{z}_{11} - d\hat{z}_{22}} \tag{6.2.15}$$

It is clear that z_{43} is a potentially satisfactory RC-LLF:R transfer function from the point of view of generality of location of poles and zeroes. Moreover, constructible specifications are clearly involved since $\hat{z}_{11}$ and $\hat{z}_{13}$ are a driving point and transfer function of network A and $\hat{z}_{22}$, $\hat{z}_{42}$ are a driving point and transfer function of network B. Thus we have completed the first of three steps which must be completed before z_{43} may be synthesized to specifications. (See discussion in Section 1.5.2 of Chapter 1). The next step is to form an Algorithm whereby physically realizable $\hat{z}_{11}$, $\hat{z}_{13}$, $\hat{z}_{22}$, and $\hat{z}_{42}$ may be determined from a given pole-zero specification of z_{43}. This is readily done as follows. Let

$$z_{43} = \frac{N(s)}{D(s)} \quad \frac{\frac{N(s)}{Q(s)}}{\frac{D(s)}{Q(s)}} \tag{6.2.16}$$

where N(s) and D(s) are polynomials in s with real coefficients. The polynomial Q(s) is of the same degree as D(s) and has negative real zeroes,

$$Q(s) = (s + \sigma_1)(s + \sigma_2) \cdots\cdots (s + \sigma_n)$$

$$D(s) = s^n + d_{n-1}s^{n-1} + \cdots\cdots d_1 s + d_o \tag{6.2.17}$$

A partial fraction expansion of D/Q then has the form

$$\frac{D(s)}{Q(s)} = 1 + \sum \frac{k_j^+}{s + \sigma_j^+} - \sum \frac{k_j^-}{s + \sigma_j^-} \tag{6.2.18}$$

where

$$k_j^+ > o$$
$$k_j^- > o \tag{6.2.19}$$

If we now identify

$$z_{11} = \frac{1}{a} \sum \frac{k_j^+}{s + \sigma_j^+} = \frac{P_{11}(s)}{Q_{11}(s)}$$
$$z_{22} = \frac{1}{d} \sum \frac{k_j^-}{s + \sigma_j^-} = \frac{P_{22}(s)}{Q_{22}(s)} \tag{6.2.20}$$

where

$$Q(s) = Q_{11}(s)Q_{22}(s) \tag{6.2.21}$$

then (assuming $a, d > o$) $\hat{z}_{11}$ and $\hat{z}_{22}$ will be p.r., RC driving point impedances. It remains to determine $\hat{z}_{13}$ and $\hat{z}_{42}$. This is readily done by factoring N/Q in the form

$$\frac{N(s)}{Q(s)} = \frac{N_{13}(s)N_{42}(s)}{Q_{11}(s)Q_{22}(s)} \tag{6.2.22}$$

where the factors N_{13} and N_{24} of N(s) are chosen so that

$$z_{13} = \frac{N_{13}(s)}{Q_{11}(s)} \quad z_{42} = \frac{N_{42}(s)}{Q_{22}(s)} \tag{6.2.23}$$

are RC-LLFPB transfer impedances. Since Q_{11} and Q_{22} already have negative real zeroes we need only require that the degree of N_{13} (N_{42}) is not greater than the degree of Q_{11} (Q_{22}). Otherwise $\hat{z}_{13}$ ($\hat{z}_{42}$) would have a pole at infinity, which is forbidden for an

RC-LLFPB transfer impedance. There is one special situation in which this requirement cannot be met. This is the special case in which the following three conditions exist at the same time

(1) $N(s)$ has all complex zeroes

(2) the degree of N is the same as the degree of D

(3) the degrees of Q_{11} and Q_{22} are odd.

Then it will be found that either $\hat{z}_{13}$ or $\hat{z}_{42}$ will have a pole at infinity when Equation 6.2.22 and 6.2.23 are used. This difficulty is easily remedied by augmenting N and D by a factor $(s + \alpha)$ as indicated below

$$\frac{N(s)}{D(s)} = \frac{N(s)(s + \alpha)}{D(s)(s + \alpha)} = \frac{\dfrac{N(s)(s + \alpha)}{Q(s)}}{\dfrac{D(s)(s + \alpha)}{Q(s)}} \tag{6.2.24}$$

where $(s + \alpha)$ is a factor not contained in $Q(s)$. Of course $Q(s)$ must also be provided with an additional factor to maintain the degree of $N(s)(s + \alpha)$ the same as that of $Q(s)$. It is clear from the above that an Algorithm has been established whereby one may go from a specified pole-zero pattern for z_{43} to physically realizable $\hat{z}_{11}$, $\hat{z}_{13}$, $\hat{z}_{22}$, and $\hat{z}_{42}$. Since $\hat{z}_{13}$ and $\hat{z}_{42}$ can be synthesized only to within a maximum constant multiplier, z_{43} will have a maximum constant multiplier. If we were free to choose c we could have any constant multiplier we might desire. The last step in synthesizing z_{43} is that of synthesizing the LLFPB and the R-LLF portion of the RC-LLF:R network. Since, as discussed in Section 1.5.2, our reference LLFPB network is identical to the LLFPB portion of the RC-LLF:R network, the establishment of the Algorithm in step 2 effectively solves half of step 3, the synthesis of the LLFPB portion of the RC-LLF:R network. The remaining half of the problem is the synthesis of an R-LLF device with G matrix

$$G = \begin{bmatrix} a & b \\ -c & -d \end{bmatrix} \qquad ad = bc \tag{6.2.25}$$

As far as the synthesis procedure is concerned, the numbers a, b, c, and d may be any real numbers that satisfy ad = bc. A particular case of such a device has already appeared in the literature. Horowitz* shows a device composed of two transistors which realizes a G matrix of the form of Equation 6.2.25 for the particular case in which

$$\begin{aligned} b &= -a \\ c &= -d \end{aligned} \tag{6.2.26}$$

As discussed in the introductory chapter the primary emphasis of this thesis is to present a new approach to the synthesis of LLF:R networks, the linear transformation theory approach. It is not within the scope of this thesis to be concerned with the practical design of R-LLF devices. Thus, no detailed discussion will be given of the practical realization of the R-LLF devices appearing in this chapter. It is felt, however, that the nature of the devices are such that practical realizations are possible.

6.2.2 Transfer Admittance, Voltage Ratio, Current Ratio

When current sources are placed at terminal pairs 3 and 4 in addition to 1 and 2 of Figure 6.2.1 the solution matrix is an open

*See page 40 of reference 7.

circuit impedance matrix Z. By using a complex transformation matrix the elements of Z are expressed in terms of those of an LLFPB open circuit impedance matrix $\hat{Z}$. The previous section has developed a synthesis technique for the transfer impedance z_{43}. If we apply voltage sources at terminal pairs 3 and 4 and still use current sources at 1 and 2 the solution matrix becomes a mixed matrix, say, M_1. By using a complex transformation matrix the elements of M_1 may be expressed in terms of those of an LLFPB mixed solution matrix $\hat{M}_1$. One may then develop a synthesis technique for the transfer admittance y_{43} in an entirely analogous fashion to that developed for z_{43} in the previous section. Similarly one may apply a current source at 4 and a voltage source at 3 and still use current sources at 1 and 2. The mixed solution matrix M_2 may also be expressed in terms of an LLFPB mixed matrix $\hat{M}_2$. A synthesis technique for the voltage transfer ratio a_{43} may then be developed. Finally if we apply a current source at 3 and a voltage source at 4 then one may deal with the synthesis of the transfer current ratio b_{13}. The details of the matrix manipulations used to find expressions for y_{43}, a_{43}, and b_{43} will not be given here. Since the procedure in their evaluation is entirely analogous to the step leading to the evaluation of z_{43} we may determine the correct expressions for y_{43}, a_{43}, and b_{43} by a comparison of the elements of $\hat{Z}$ with those of $\hat{M}_1$, $\hat{M}_2$, and $\hat{M}_3$. The matrices $\hat{M}_1$, $\hat{M}_2$, and $\hat{M}_3$ are given below. The solution matrix $\hat{M}_1$ relates the following quantities of the network of Figure 6.2.2.

$$\begin{bmatrix} e_1 \\ e_2 \\ i_3 \\ i_4 \end{bmatrix} = \hat{M}_1 \begin{bmatrix} i_1 \\ i_2 \\ e_3 \\ e_4 \end{bmatrix} \tag{6.2.27}$$

with $\hat{M}_1$ given by

$$\hat{M}_1 = \left[\begin{array}{cc|cc} \frac{1}{y_{11}} & 0 & \hat{a}_{13} & 0 \\ 0 & \frac{1}{\hat{y}_{22}} & 0 & \hat{a}_{24} \\ \hline \hat{b}_{31} & 0 & \frac{1}{\hat{z}_{33}} & 0 \\ 0 & \hat{b}_{42} & 0 & \frac{1}{\hat{z}_{44}} \end{array}\right] \tag{6.2.28}$$

The solution matrix $\hat{M}_2$ relates the quantities

$$\begin{bmatrix} e_1 \\ e_2 \\ i_3 \\ e_4 \end{bmatrix} = \hat{M}_2 \begin{bmatrix} i_1 \\ i_2 \\ e_3 \\ i_4 \end{bmatrix} \tag{6.2.29}$$

with $\hat{M}_2$ given by

$$\hat{M}_2 = \left[\begin{array}{cc|cc} \frac{1}{\hat{y}_{11}} & o & \hat{a}_{13} & o \\ o & \hat{z}_{22} & o & z_{24} \\ \hline \hat{b}_{31} & o & \frac{1}{\hat{z}_{33}} & o \\ o & \hat{z}_{42} & o & \hat{z}_{44} \end{array}\right] \tag{6.2.30}$$

The solution matrix $\hat{M}_3$ relates the quantities

$$\begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ i_4 \end{bmatrix} = \hat{M}_3 \begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ e_4 \end{bmatrix} \tag{6.2.31}$$

with $\hat{M}_3$ given by

$$\hat{M}_3 = \left[\begin{array}{cc|cc} \hat{z}_{11} & o & \hat{z}_{13} & o \\ o & \frac{1}{\hat{y}_{22}} & o & \hat{a}_{24} \\ \hline \hat{z}_{31} & o & \hat{z}_{33} & o \\ o & \hat{b}_{42} & o & \frac{1}{\hat{z}_{44}} \end{array}\right] \tag{6.2.32}$$

The voltage and transfer current ratios are related by

$$\hat{a}_{jk} = -\hat{b}_{kj} \tag{6.2.33}$$

due to the bilateral nature of LLFPB networks. It should be noted that all of these mixed matrices apply to the situation wherein terminal pairs 1 and 2 are excited by current sources. The reason for this is that only in this way may we be sure of describing the R-LLF device by g parameters, i.e., the matrix e_{ss} of Section 3.5 is, for this type of excitation, a branch admittance matrix.

A comparison of the solution matrix $\hat{Z}$ of Equation 6.2.9 with the mixed solution matrices of Equations 6.2.30, 6.2.31, and 6.2.32 allow us to write down expressions for y_{43}, a_{43} and b_{43} by inspection. To find y_{43} we replace quantities in the expression for z_{43} (Equation 6.2.15) as indicated below

$$\begin{aligned} z_{43} &\to y_{43} \\ \hat{z}_{13} &\to \hat{a}_{13} \\ \hat{z}_{42} &\to \hat{b}_{42} \\ \hat{z}_{11} &\to \frac{1}{\hat{y}_{11}} \\ \hat{z}_{22} &\to \frac{1}{\hat{y}_{22}} \end{aligned} \tag{6.2.34}$$

with the result

$$y_{43} = \frac{c\hat{a}_{13}\hat{b}_{42}}{1 + a\frac{1}{\hat{y}_{11}} - d\frac{1}{\hat{y}_{22}}} \tag{6.2.35}$$

To find a_{43} we make the replacements

$$\begin{aligned} z_{43} &\to a_{43} \\ \hat{z}_{13} &\to \hat{a}_{13} \\ \hat{z}_{11} &\to \frac{1}{\hat{y}_{11}} \end{aligned} \tag{6.2.36}$$

with the result

$$a_{43} = \frac{c\hat{a}_{13}\hat{z}_{24}}{1 + a\frac{1}{\hat{y}_{11}} - d\,\hat{z}_{22}} \tag{6.2.37}$$

To find b_{43} we make the replacements

$$\begin{aligned} z_{43} &\rightarrow b_{43} \\ \hat{z}_{42} &\rightarrow \hat{b}_{42} \\ \hat{z}_{22} &\rightarrow \frac{1}{\hat{y}_{22}} \end{aligned} \tag{6.2.38}$$

with the result

$$b_{43} = \frac{c\hat{z}_{13}\hat{b}_{42}}{1 + a\,\hat{z}_{11} - \frac{d}{\hat{y}_{22}}} \tag{6.2.39}$$

It is readily seen that only constructible specifications are involved in y_{43}, a_{43}, and b_{43}. This fact is perhaps more evident if the following relationships are recognized

$$\begin{aligned} \hat{a}_{13} &= -\frac{\hat{y}_{13}}{\hat{y}_{11}} \\ \hat{b}_{42} &= -\frac{\hat{y}_{42}}{\hat{y}_{22}} \end{aligned} \tag{6.2.40}$$

Thus synthesis of y_{43} involves a synthesis of networks A and B for a s.c. driving point admittance and transfer admittance; synthesis of $\hat{a}_{43}$ involves a synthesis of network A for driving point and transfer admittance and network B for driving point and transfer impedance; and synthesis of $\hat{b}_{42}$ requires a synthesis of network A for driving point and transfer impedance and network B for driving point and transfer admittance.

The formulation of Algorithms to determine the constructible LLFPB network specifications proceeds almost identically as with z_{43}. We will illustrate the procedure with the construction of the Algorithm relevant to the synthesis of a specified b_{43} (to within a constant multiplier). Thus suppose it is desired to synthesize

$$b_{43} = \frac{N(s)}{D(s)}$$

Perform the following steps.

1. Find the polynomials N_{13}, N_{42}, Q_{11}, Q_{22}, P_{11}, and P_{22} which are applicable to the synthesis of z_{43}.
2. Determine $\hat{y}_{42}$, $\hat{y}_{22}$, $\hat{z}_{13}$, $\hat{z}_{11}$ as follows

$$\hat{y}_{22} = \frac{Q_{22}(s)}{P_{22}(s)}$$

$$\hat{y}_{42} = \frac{N_{42}(s)}{P_{22}(s)}$$

$$\hat{z}_{11} = \frac{P_{11}(s)}{Q_{11}(s)}$$

$$\hat{z}_{13} = \frac{N_{13}(s)}{Q_{11}(s)}$$

The corresponding Algorithms for y_{43} and a_{43} are readily formulated

6.3 <u>Synthesis Technique No. 2.</u>

6.3.1 <u>Transfer Impedance</u>

As a preliminary step in the development of the synthesis procedure we will first analyze the LLF:R network of Figure 6.3.3 with the aid of complex linear transformations in the manner discussed in Section 3.5 of Chapter 3. It will be recognized that this network falls under case (3) of Figure 5.3.1 and has been discussed with regard to generality of location of complex natural frequencies in Section 5.4.2 of Chapter 5. The R-LLF device in Figure 6.3.1 is recognized to be an Ideal Vacuum Tube.

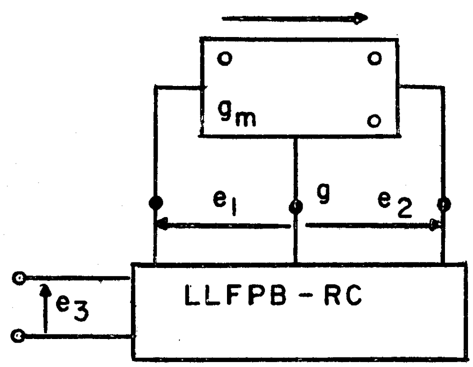


Figure 6.3.1 Network Configuration Applicable to Synthesis Technique No. 2

In contrast to the previous section the LLFPB network has only one rather than two terminal pairs in addition to the two already connected to the R-LLF device. The resulting RC-LLF network has three terminal pairs. Let us assume in this section that current sources are placed across terminal pairs so that the response quantities are terminal-pair voltages. We may now apply the results of Section 3.5 to express the o.c. impedance matrix Z of RC-LLF:R network in the form

$$Z = \tau \hat{Z}$$

where $\hat{Z}$ is the o.c. impedance matrix of the network with the R-LLF device removed (leaving open circuit constraints at terminal pairs

1 and 2) and τ is a complex transformation matrix. Partitioned expressions for Z and $\hat{Z}$ analogous to Equations 6.2.2 and 6.2.3 are shown below

$$Z = \left[\begin{array}{c|c} Z_{ss} & Z_{sr} \\ \hline Z_{rs} & Z_{rr} \end{array}\right] = \left[\begin{array}{cc|c} z_{11} & z_{12} & z_{13} \\ z_{21} & z_{22} & z_{23} \\ \hline z_{31} & z_{32} & z_{33} \end{array}\right] \tag{6.3.1}$$

$$\hat{Z} = \begin{bmatrix} \hat{Z}_{ss} & \hat{Z}_{sr} \\ \hat{Z}_{rs} & \hat{Z}_{rr} \end{bmatrix} = \left[\begin{array}{cc|c} \hat{z}_{11} & \hat{z}_{12} & \hat{z}_{13} \\ \hat{z}_{21} & \hat{z}_{22} & \hat{z}_{23} \\ \hline \hat{z}_{31} & \hat{z}_{32} & \hat{z}_{33} \end{array}\right] \tag{6.3.2}$$

in which $\hat{z}_{jk} = \hat{z}_{kj}$.

The branch admittance matrix of the R-LLF device is

$$g_{ss} = \begin{bmatrix} o & o \\ g_m & o \end{bmatrix} \quad ; \; g_m > o \tag{6.3.3}$$

The partitioned expression in Equation 6.2.5 for the transformation matrix τ and that in Equation 6.2.7 for Z are valid here also. We will be interested in the transfer impedance z_{13}. Inspection of Equation 6.3.1 shows that this is an element of Z_{sr}. According to Equation 6.2.7, Z_{sr} is given by

$$Z_{sr} = T \hat{Z}_{sr} \tag{6.3.4}$$

where T has the general expression given by Equation 6.2.6. In our case

$$T = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{z}_{11} & \hat{z}_{12} \\ \hat{z}_{21} & \hat{z}_{22} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ g_m & 0 \end{bmatrix} \right\}^{-1}$$

$$= \begin{bmatrix} 1+g_m\hat{z}_{12} & 0 \\ g_m\hat{z}_{22} & 1 \end{bmatrix}^{-1} \tag{6.3.6}$$

$$= \frac{1}{\Delta^g} \begin{bmatrix} 1 & 0 \\ -g_m\hat{z}_{22} & 1+g_m\hat{z}_{12} \end{bmatrix}$$

where Δ^g is the appropriate Characteristic Determinant,

$$\Delta^g = 1 + g_m\hat{z}_{12} \tag{6.3.7}$$

Applying Equation 6.3.4 to evaluate Z_{sr} we find

$$Z_{sr} = \begin{bmatrix} z_{13} \\ z_{23} \end{bmatrix} = \frac{1}{\Delta^g} \begin{bmatrix} 1 & 0 \\ -g_m\hat{z}_{22} & 1+g_m\hat{z}_{12} \end{bmatrix} \begin{bmatrix} \hat{z}_{13} \\ \hat{z}_{23} \end{bmatrix} \tag{6.3.8}$$

$$= \begin{bmatrix} \dfrac{\hat{z}_{13}}{1+g_m\hat{z}_{12}} \\ \hat{z}_{23} - g_m \dfrac{\hat{z}_{13}\hat{z}_{22}}{1+g_m\hat{z}_{12}} \end{bmatrix}$$

The expression for z_{13} is then

$$z_{13} = \frac{\hat{z}_{13}}{1+g_m\hat{z}_{12}} \tag{6.3.9}$$

It is clear that z_{13} is a potentially satisfactory RC-LLF:R transfer function from the point of view of generality of location of poles and zeroes. However, as discussed in Section 1.5.2 of Chapter 1, there are three steps that must be completed in succession before one may synthesize the RC-LLF:R network for prescribed z_{13}. The first step is the requirement that constructible specifications be involved in the expression for z_{13}. Examination of Equation 6.3.9 shows that z_{13} is a function of two transfer impedances of a three terminal-pair RC-LLFPB network. In order to obtain constructible specifications, the three terminal-pair LLFPB network must be restricted to a composition of two terminal-pair networks. The particular composition used is shown in Figure 6.3.2.

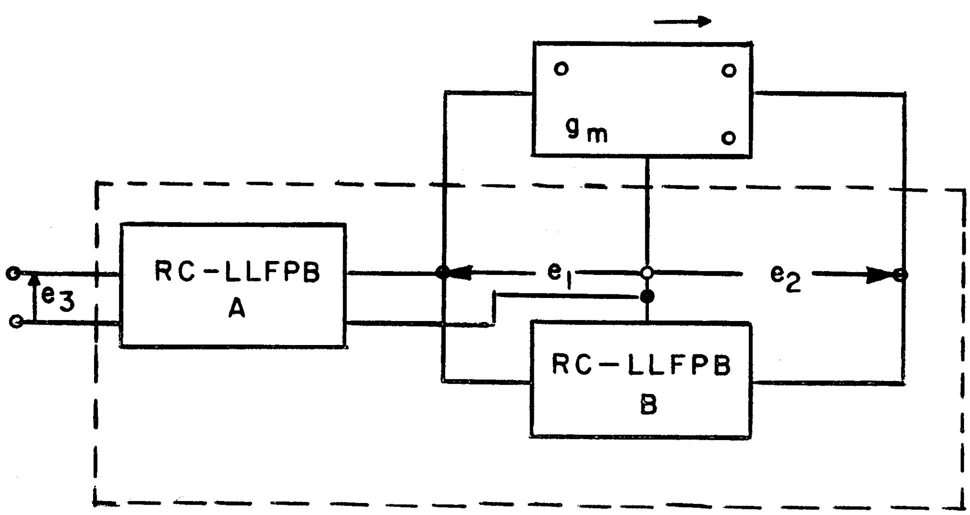


Figure 6.3.2 Network For Synthesis Technique No. 2

Expressions for $\hat{z}_{13}$ and $\hat{z}_{12}$ in terms of the driving point and transfer impedances of networks A and B will now be evaluated. First, the o.c. impedance matrix of network A, Z_A, and network B, Z_B are defined below,

$$Z_A = \begin{bmatrix} z_{11}^A & z_{13}^A \\ z_{31}^A & z_{33}^A \end{bmatrix} \quad Z_B = \begin{bmatrix} z_{11}^B & z_{12}^B \\ z_{21}^B & z_{22}^B \end{bmatrix} \tag{6.3.10}$$

The network with o.c. impedance matrix $\hat{Z}$ is shown in Figure 6.3.3.

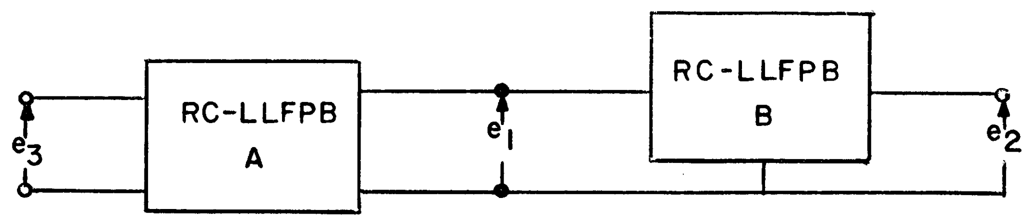


Figure 6.3.3 RC-LLFPB Reference Network

We may evaluate the transfer impedances $\hat{z}_{13}$ and $\hat{z}_{12}$ by inspection of Figure 6.3.3 if we make use of the following well known expression* for the transfer impedance Z_{12} of two networks in cascade.

$$Z_{12} = \frac{z_{12}^{(1)} z_{12}^{(2)}}{z_{22}^{(1)} + z_{11}^{(2)}} \tag{6.3.11}$$

where $z_{12}^{(1)}$ and $z_{12}^{(2)}$ are the transfer impedances of the first and second network respectively. The impedances $z_{11}^{(2)}$ and $z_{22}^{(1)}$ are the driving point impedances of the component networks that are applicable to the terminal pairs that join the two networks.

We immediately determine that

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* Reference 3, page 372

$$\hat{z}_{13} \quad \frac{z^A_{13} z^B_{11}}{z^A_{11} + z^B_{11}}$$

$$\hat{z}_{12} = \frac{z^B_{12} z^A_{11}}{z^A_{11} + z^B_{11}} \tag{6.3.12}$$

Using these expressions for $\hat{z}_{13}$ and $\hat{z}_{12}$ in Equation 6.3.8 we find that

$$z_{13} = \frac{z^A_{13} z^B_{11}}{z^A_{11} + z^B_{11} + g_m z^B_{12} z^A_{11}} \tag{6.3.13}$$

Examination of Equation 6.3.13 shows that we have completed step 1, i.e., only constructible specifications are made upon z_{13}. The next step is to find an Algorithm which will yield physically realizable z^A_{11}, z^A_{13} and z^B_{11}, z^B_{12} when a specified pole-zero pattern is given for z_{13}. Examination of Equation 6.3.13 shows that z_{13} will in general have zeroes when z^B_{11} has zeroes. This is undesirable because the zeroes of z^B_{11} must lie on the negative real axis of the complex frequency plane. We may kill two birds with one stone by letting

$$z^B_{11} = \beta\, z^A_{11} \tag{6.3.14}$$

where β is a positive constant, because then z_{13} takes the form,

$$z_{13} = \frac{\beta\, z^A_{13}}{\beta + 1 + g_m z^B_{12}} = \frac{\beta}{\beta + 1} \cdot \frac{z^A_{13}}{1 + \frac{g_m}{1+\beta} z^B_{12}} \tag{6.3.15}$$

The zeroes of z_{13} are those of z^A_{13}. Since network A may be an ungrounded two terminal-pair RC network, the zeroes of z^A_{13} and thus

z_{13} are unrestricted. The poles of z_{13} are the zeroes of the equation

$$1 + \frac{g_m}{1 + \beta} z_{12}^B = \frac{D(s)}{Q(s)} \tag{6.3.16}$$

where Q(s) is a factor that contains the poles of z_{12}^B. From the discussion in Section 5.4.2 we recognize that a physically realizable z_{12}^B can be found for any pos. g_m, β and any D(s) with positive coefficients. Since a Hurwitz polynomial has positive coefficients, it is clear that z_{13} is as general as any stable transfer function as far as pole-zero locations is concerned

The construction of the Algorithm proceeds as follows. Let

$$z_{13} = \frac{N(s)}{D(s)} \tag{6.3.17}$$

be an arbitrary stable transfer function. Then let

$$z_{13}^A = \frac{N(s)}{Q(s)} \tag{6.3.18}$$

and

$$z_{12}^B = \frac{1 + \beta}{g_m} \; \frac{D(s) - Q(s)}{Q(s)} \tag{6.3.19}$$

The polynomial Q is assumed to have the form

$$Q(s) = \alpha(s + \sigma_1)(s + \sigma_2) \cdots (s + \sigma_n); \; \sigma_j \neq \sigma_k \tag{6.3.20}$$

with α and the σ's positive and to be of the same degree, n, as D(s). For any selection of σ's we may always find an α for which D-Q has positive coefficients if D has positive coefficients. Of course, if we are dealing with a denominator polynomial D(s) which is Hurwitz, then D must have positive coefficients. There is one limiting case which can cause difficulty and that is when D(s) has all j axis zeroes. Then D is either an even or an odd polynomial and no non-zero value of α can be found for which D-Q has positive coefficients. This difficulty is easily remedied by augmenting N and D by the factor $(s + \gamma)$ where γ is a positive constant. Whereas D(s) may be either even or odd $(s + \gamma)$ D(s) will have no missing coefficients. It is assumed that $\gamma \neq \sigma_j$; $j = 1, 2 \cdots n$. In order for z_{12}^A to be a physically realizable RC-LLFPB transfer impedance z_{13}^A must not have a pole at infinite frequency. Thus the degree of N must not be greater than the degree of Q. Examination of Equation 6.3.19 shows that the degree of Q must be equal or less than the degree of D in order for z_{12}^B to have positive numerator coefficients (as is required for a grounded transfer function). Thus the specified stable transfer function z_{13} may not have a pole at infinity.

Once a realizable z_{12}^B has been found from Equation 6.3.19 a polynomial $P_{12}(s)$ is selected such that

$$z_{11}^B = \frac{P_{12}(s)}{Q(s)} \tag{6.3.21}$$

is a physically realizable RC-LLFPB impedance. An RC-LLFPB grounded two terminal-pair network is synthesized to have driving point and transfer impedances z_{11}^B, z_{12}^B. If α and g_m are selected <u>a priori</u> then z_{12}^B is specified completely, i.e., the constant multiplier is fixed by Equation 6.3.19. Since in practice we can either synthesize z_{11}^B, exactly and z_{12}^B to within a constant multiplier or else z_{11}^B to within a constant multiplier and z_{12}^B exactly we must accept whatever constant multiplier appears in z_{11}^{B}. Once z_{11}^{B} together with its constant multiplier has been determined by the synthesis procedures we determine z_{11}^{A} by Equation 6.3.14. Then z_{11}^{A} is synthesized exactly and z_{12}^{A} to within a maximum constant multiplier. The final result is a synthesis of z_{13} to within a constant multiplier. Since the R-LLF device used in the synthesis procedure may be represented as a vacuum tube with no interelectrode capacitance and infinite plate resistance, the RC-LLF:R network may be represented as shown in Figure 6.3.4 where the vacuum tube circuit symbol is used to represent the R-LLF device. If a sufficiently small shunt resistance can be extracted at terminal pair 2 of network B, this may be used as the plate resistance of the vacuum tube so that a physical vacuum tube may be used (neglecting interelectrode capacitances), in this case.

6.3.2 Transfer Voltage Ratio

When a current source is placed at terminal pair 3 in addition to current sources at terminal pairs 1 and 2 of Figure 6.3.2, the solution matrix is an open circuit impedance matrix Z. By means of a complex transformation matrix the elements of Z are expressed in terms of those of an LLFPB open circuit matrix $\hat{Z}$. The previous section has developed a synthesis technique for the transfer impedance z_{13}. If we apply a voltage source at terminal pair 3 and still use current sources at 1 and 2, the solution matrix becomes a mixed matrix, M. By using a complex transformation matrix the elements of M may be expressed in terms of those of an LLFPB mixed solution matrix $\hat{M}$. One may then develop a synthesis technique for the transfer voltage ratio a_{13} in an entirely andogous fashion to that developed for z_{13} in the previous section. The details of the matrix manipulations used to find the expression for a_{13} will not be given here. Since the procedure in its evaluation is entirely analogous to the steps leading to the evaluation of z_{13}, we may determine the correct expression for a_{13} by a comparison of the elements of $\hat{Z}$ with those of $\hat{M}$. The solution matrix $\hat{M}$ relates the following quantities as follows

$$\begin{bmatrix} e_1 \\ e_2 \\ i_3 \end{bmatrix} = \hat{M} \begin{bmatrix} i_1 \\ i_2 \\ e_3 \end{bmatrix} \tag{6.3.22}$$

where the matrix $\hat{M}$ has the form

$$\hat{M} = \left[\begin{array}{cc|c} \hat{v}_{11} & v_{12} & \hat{a}_{13} \\ v_{21} & \hat{v}_{22} & \hat{a}_{23} \\ \hline \hat{b}_{31} & \hat{b}_{32} & v_{33} \end{array}\right] = \left[\begin{array}{c|c} \hat{M}_{ss} & \hat{M}_{sr} \\ \hline \hat{M}_{rs} & \hat{M}_{rr} \end{array}\right] \tag{6.3.23}$$

To obtain a_{13} we replace various network functions in Equation 6.3.9 as follows

$$\begin{aligned} z_{13} &\to a_{13} \\ \hat{z}_{12} &\to \hat{v}_{12} \\ \hat{z}_{13} &\to \hat{a}_{13} \end{aligned} \tag{6.3.24}$$

with the result

$$a_{13} = \frac{\hat{a}_{13}}{1 + g_m \hat{v}_{12}} \tag{6.3.25}$$

From Equation 6.3.22 it is seen that v_{12} is the transfer impedance between terminal-pairs 1 and 2 of Figure 6.3.3 when terminal pair 3 is shorted, and that a_{13} is the transfer voltage ratio between terminal pairs 3 and 1 (voltage source at 3). One may readily determine that

$$\hat{a}_{13} = \frac{\hat{y}_{13}}{\hat{y}_{11}} = -\frac{y_{13}^A}{1 + y_{11}^A + \frac{1}{z_{11}^B}} = \frac{a_{13}^A}{1 + \frac{1}{y_{11}^A z_{11}^B}}$$

$$v_{12} = \frac{z_{12}^B \frac{1}{y_{11}^A}}{z_{11}^B + \frac{1}{y_{11}^A}} = \frac{z_{12}^B}{1 + y_{11}^A z_{11}^B} \tag{6.3.26}$$

where $a_{13}^A = - y_{13}^A/y_{11}^A$ is a transfer voltage ratio of network A. With these expressions for $\hat{a}_{13}$ and $\hat{v}_{12}$, the transfer voltage ratio a_{13} takes the form

$$a_{13} = \frac{a_{13}^A}{1 + \frac{1}{y_{11}^A z_{11}^B} + g_m z_{12}^B \frac{1 + \frac{1}{y_{11}^A z_{11}^B}}{1 + y_{11}^A z_{11}^B}} \tag{6.3.27}$$

If we let

$$y_{11}^A z_{11}^B = \beta \tag{6.3.28}$$

where β is a positive constant then

$$a_{13} = \frac{\beta}{\beta + 1} \frac{a_{13}^A}{1 + \frac{g_m}{1 + \beta} z_{12}^B} \tag{6.3.29}$$

The construction of an Algorithm appropriate to the determination of physically realizable $\hat{a}_{13}$ and $\hat{z}_{12}$ for a specified a_{13} is essentially identical to that for the determination of z_{13}^{A} and z_{12}^{B} in the previous section.

6.4 Synthesis Technique No. 3

In this section we will present a transfer function synthesis technique which does not allow a completely general assignment of pole locations. This synthesis technique is presented primarily for its collateral interest with a result of Chapter 7. This is in regard to the fact that nonbilaterality is the essential ingredient that allows the assignment of complex natural frequencies in an RC network.

The network to be considered is shown in Figure 6.4.1.

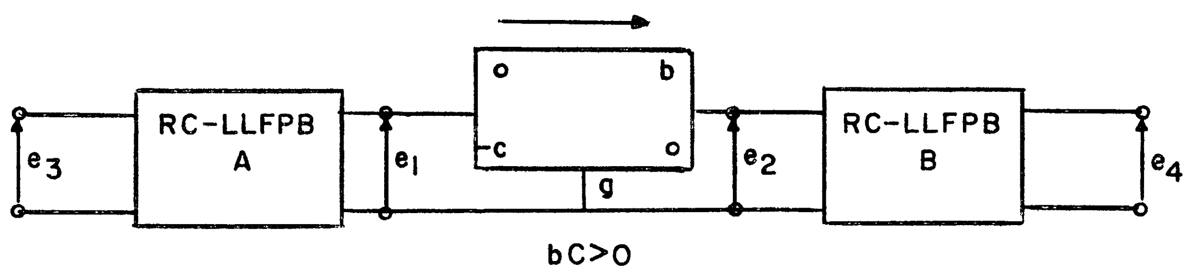


Figure 6.4.1 RC-LLF:R Network Applicable To Synthesis Technique No. 3

It will be noted that the R-LLF device becomes a Gyrator when b = c. Since a Gyrator is lossless, the RC-LLF:R network must in such a case become passive. The synthesis technique to be presented requires only that bc > o so that a gyrator is suitable if desired. It will be recognized that the type of network shown in Figure 6.4.1

falls into Case 5 of Figure 5.3.1 and has been discussed in Section 5.4.4. It was demonstrated in this latter section that the natural frequencies of such a network are constrained to be the zeroes of the series combination of an RC and an RL impedance.

Since the cascade network of Figure 6.4.1 is of the same form as that of Figure 6.2.1, the analysis by complex linear transformations follows an identical pattern. The matrix $\hat{Z}$ is the same in both cases. Only the matrices T and g_{ss} are different here. In our case

$$g_{ss} = \begin{bmatrix} o & b \\ -c & o \end{bmatrix} \quad ; \quad bc > o \tag{6.4.1}$$

and

$$\begin{aligned} T &= \left\{ \begin{bmatrix} 1 & o \\ o & 1 \end{bmatrix} + \begin{bmatrix} \hat{z}_{11} & o \\ o & \hat{z}_{22} \end{bmatrix} \begin{bmatrix} o & b \\ -c & o \end{bmatrix} \right\}^{-1} \\ &= \begin{bmatrix} 1 & b\hat{z}_{11} \\ -c\hat{z}_{22} & 1 \end{bmatrix}^{-1} = \frac{1}{\Delta^g} \begin{bmatrix} 1 & -b\hat{z}_{11} \\ c\hat{z}_{22} & 1 \end{bmatrix} \end{aligned} \tag{6.4.2}$$

where the Characteristic Determinant Δ^g is given by

$$\Delta^g = 1 + bc\,\hat{z}_{11}\hat{z}_{22} \tag{6.4.3}$$

We will concern ourselves only with the transfer impedance z_{43} although, just as with Synthesis Technique No. 1, we can derive corresponding synthesis procedures for y_{43}, a_{43}, and b_{43}. The transfer impedance z_{43} is contained within the matrix $Z_{\check{r}r}$ of Equation 6.2.7. Using Equation 6.2.8 and the expressions for g_{ss} and T applicable in this section we find that

$$z_{43} = \frac{c\,\hat{z}_{13}\hat{z}_{42}}{1 + bc\,\hat{z}_{11}\hat{z}_{22}} = \frac{c\,\hat{z}_{13}\hat{a}_{42}}{bc\,\hat{z}_{11} + \frac{1}{z_{22}}} \tag{6.4.4}$$

where we have used the relationship

$$a_{42} = \frac{\hat{z}_{42}}{\hat{z}_{22}} \tag{6.4.5}$$

Let us presume that it is desired to synthesize

$$z_{43} = \frac{N(s)}{D(s)} = \frac{\frac{N(s)}{Q(s)}}{\frac{D(s)}{Q(s)}} \tag{6.4.6}$$

where as usual the polynomial Q(s) is of the same degree as D(s) and has negative real zeroes. A partial fraction expansion of D/Q has the form

$$\frac{D}{Q} = \sum \frac{k_j^+}{s + \sigma_j^+} + 1 - \sum \frac{k_j^-}{s + \sigma_j^-} \tag{6.4.7}$$

where k_j^+ and k_j^- are positive. The admittance

$$\hat{y} = 1 - \sum \frac{k_j^-}{s + \sigma_j^-}$$

will be p.r. and RC if its zero frequency value is positive, i.e.,

$$\hat{y}(o) = 1 - \sum \frac{k_j^-}{\sigma_j^-} > 0 \tag{6.4.8}$$

We will confine our attention to only those denominator polynomials D(s) for which it is possible to find a Q(s) for which inequality 6.4.8 is satisfied. In this case we may identify

$$\begin{aligned} \frac{1}{z_{22}} = y = \frac{P_{22}(s)}{Q_{22}(s)} &= 1 - \sum \frac{k_j^-}{s + \sigma_j^-} \\ bc\,\hat{z}_{11} = \frac{P_{22}(s)}{Q_{11}(s)} &= \sum \frac{k_j^+}{s + \sigma_j^+} \end{aligned} \tag{6.4.9}$$

to obtain physically realizable $\hat{z}_{11}$ and $\hat{z}_{22}$. Then factor

$$\frac{N}{Q} = \frac{N_{13}(s)}{Q_{11}(s)} \frac{N_{42}(s)}{Q_{22}(s)} \tag{6.4.10}$$

where

$$\begin{aligned} Q &= Q_{11}Q_{22} \\ N &= N_{13}N_{42} \end{aligned} \tag{6.4.11}$$

We may then identify

$$\hat{z}_{13}(s) = \frac{N_{13}(s)}{Q_{11}(s)}$$

$$\hat{a}_{42}(s) = \frac{N_{42}(s)}{Q_{22}(s)} \; ; \; \hat{z}_{42}(s) = \hat{a}_{42}z_{22} = \frac{N_{42}(s)}{P_{22}(s)} \tag{6.4.12}$$

Since $\hat{z}_{11}$ as given in Equation 6.4.9 has a zero at infinity, then so must $\hat{z}_{13}$. Thus one must select $N_{13}(s)$ so that its degree is less than that of Q_{11}. From similar considerations we deduce that the degree of M_{42} may not exceed that of $Q_{22}(s)$. Thus the degree of $N(s)$ must be less than $D(s)$. If $y(o)$ is sufficiently large, then some of this d.c. value may be added to $\hat{z}_{11}$ relaxing the requirement that it have a zero at infinity. In such a case the degree of $M(s)$ may be equal to that of $D(s)$, but no greater.

For an attempt at a practical realization of a Gyrator in terms of vacuum tubes see Reference 23.

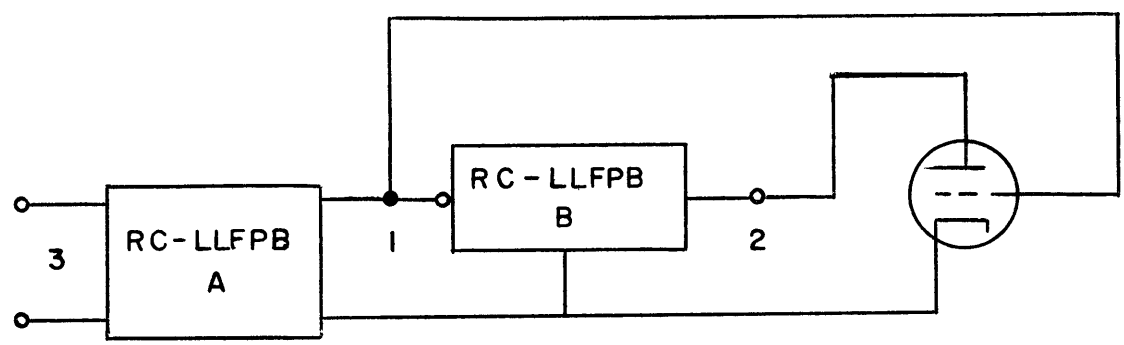


Figure 6.2.4. RC-LLF:R Network For Synthesis Technique No. 2

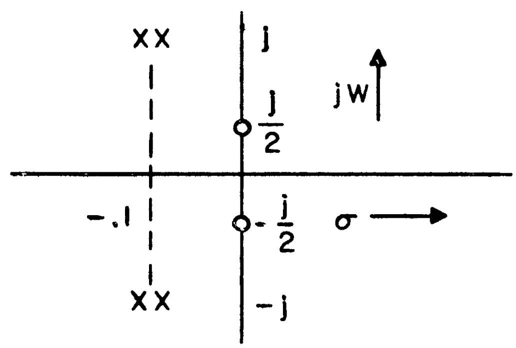
6.5 Illustrative Examples

6.5.1 Synthesis Technique No. 1

To illustrate synthesis technique No. 1 we shall consider the synthesis of the transfer current ratio

$$b_{43} = K \frac{\left[s^2 + 0.25\right]}{\left[s^2 + 0.2\ s + 1.01\right]^2} = K\,\frac{N(s)}{D(s)} \tag{6.5.1}$$

with pole-zero pattern indicated below



the quantity K is a constant that may be determined after the network has been synthesized.

For ease in presentation (and no loss in generality) the R-LLF device of Synthesis Technique No. 1 is chosen to have the G matrix

$$G_1 = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \tag{6.5.2}$$

which corresponds to

$$a = b = c = d = 1 \tag{6.5.3}$$

for the general R-LLF device applicable to this synthesis technique. For such a G matrix, the network of Fig. 6.2.1 has the transfer function

$$b_{43} = \frac{\hat{z}_{13}\hat{b}_{42}}{1 + \hat{z}_{11} - \frac{1}{\hat{y}_{22}}} \tag{6.5.4}$$

Proceeding as discussed in Section 6.2 we arbitrarily select

$$Q(s) = (s + .025)(s + 0.50)(s + 2.00)(s + 4) \tag{6.5.5}$$

and from the partial fraction expansion

$$\frac{D(s)}{Q(s)} = \frac{\left[s^2 + 0.25 + 1.0\right]^2}{(s + 0.25)(s + 0.50)(s + 2.00)(s + 4.00)}$$

$$= 1 + \frac{0.6635}{(s + 0.25)} + \frac{4.048}{(s + 2.00)} - \frac{1.0252}{(s + 0.50)} - \frac{14.0141}{(s + 4.00)} \tag{6.5.6}$$

We then make the identifications

$$\hat{z}_{11} = \frac{0.6635}{s + 0.25} + \frac{4.048}{s + 2.00} = \frac{4.7115s + 2.3389}{s^2 + 2.25s + 0.50} = \frac{P_{11}(s)}{Q_{11}(s)}$$

$$\hat{y}_{22} = \frac{1}{\frac{1.0252}{s + 0.50} + \frac{14.0141}{s + 4.00}} = \frac{s^2 + 4.5s + 2.00}{15.0392s + 11.1079} = \frac{Q_{22}(s)}{P_{22}(s)} \tag{6.5.7}$$

The zeroes of b_{43} may be associated with $\hat{z}_{13}$ or $\hat{b}_{42}$. We choose to associate them with $\hat{b}_{42}$. Then

$$\hat{z}_{13} = \frac{1}{s^2 + 2.25s + 0.50}$$

$$\hat{b}_{42} = \frac{s^2 + 0.25}{s^2 + 4.5s + 2.00} \tag{6.5.8}$$

From the equation

$$\hat{b}_{42} = \frac{\hat{y}_{42}}{\hat{y}_{22}}$$

we determine that

$$\hat{y}_{42} = \frac{s^2 + 0.25}{15.0392s + 11.1079} \qquad (6.5.9)$$

Network B is synthesized by the parallel ladder technique*. Since Network A has all transfer function zeroes at infinity, the realization of $\hat{z}_{11}$ as a Cauer Canonic form** with shunt capacitances and series resistances is appropriate. The final realization is shown in Fig. 6.5.1. Capacitance and resistance values are in farads and ohms, respectively.

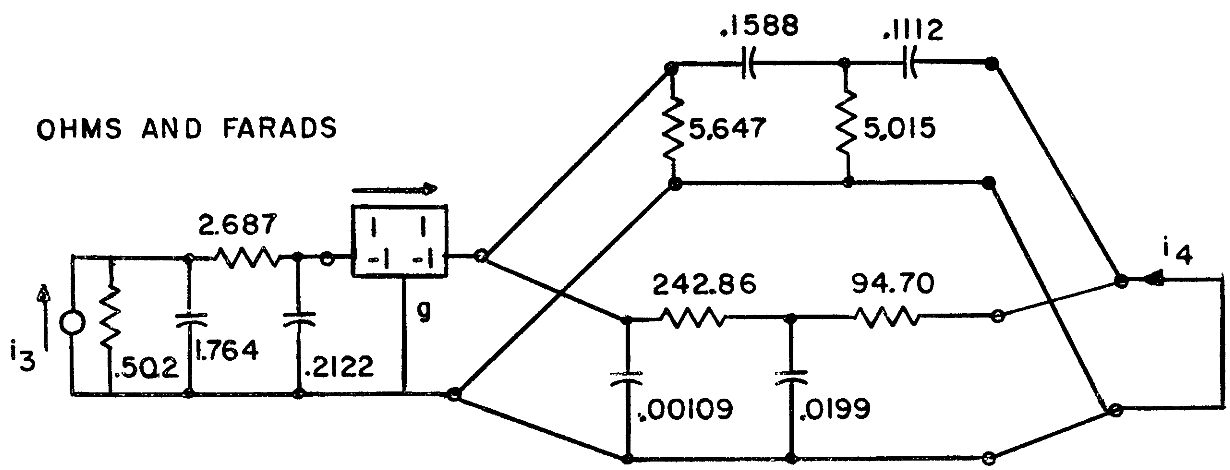


Figure 6.5.1. Network For Illustrated Example: Synthesis Technique No. 1

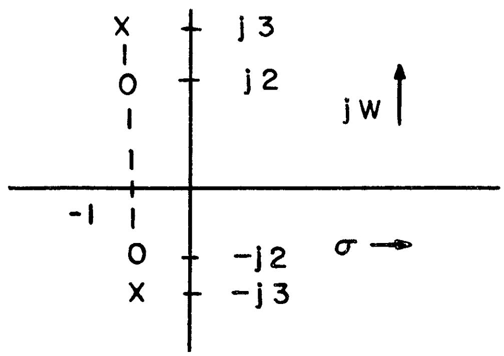
* Ref. (3) Page 555
** Ref. (3) Page 115

6.5.2 Synthesis Technique NO. 2

We will illustrate synthesis technique No. 2 with the transfer voltage ratio

$$a_{13} = \frac{s^2 + 2s + 5}{s^2 + 2s + 10} = K \frac{N(s)}{D(s)} \tag{6.5.10}$$

that has the pole-zero pattern indicated below



For ease in presentation (and no loss in generality) the ideal vacuum tube is assumed to have the normalized G matrix

$$G_4 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \tag{6.5.11}$$

which corresponds to a transconductance of 1 mho. In addition we will arbitrarily assume that $\beta = 1$. The voltage transfer function of the network of Fig. 6.3.4 then takes the form

$$a_{13} = \frac{1}{2} \frac{a_{13}^A}{1 + \frac{1}{2} z_{12}^B} \tag{6.5.12}$$

where it is assumed that

$$z_{11}^B = \frac{1}{y_{11}^A} \tag{6.5.13}$$

Proceeding as discussed in Section 6.3 we arbitrarily select

$$Q(s) = s^2 + 2s + 0.5 = (s + 1 + \frac{\sqrt{2}}{2})(s + 1 - \frac{\sqrt{2}}{2}) \qquad (6.5.14)$$

and determine z_{12}^B as

$$z_{12}^B = 2 \left[\frac{D - Q}{Q}\right] = \frac{19}{s^2 + 2s + 0.5} = \frac{P_{12}(s)}{Q(s)} \qquad (6.5.15)$$

We must then form the driving point impedance z_{11}^B. Only the numerator polynomial of z_{11}^B may be selected arbitrarily since the denominator polynomial is Q. The z_{11}^B used is

$$z_{11}^B = \left[\frac{s + 1}{s^2 + 2s + 0.5}\right] x \qquad (6.5.16)$$

The constant x is used to denote the fact that if z_{12}^B is to be synthesized exactly then z_{11}^B can be synthesized only to within a constant multiplier. This multiplier is determined by the synthesis procedure. In our case this multiplier turns out to be 38, i.e.,

$$x = 38 \qquad (6.5.17)$$

Applying Eq. 6.5.13

$$y_{11}^A = \frac{1}{38} \left[\frac{s^2 + 2s + 0.5}{s + 1}\right] \qquad (6.5.18)$$

The voltage transfer function for network A is then given by

$$a_{13}^A = \left[\frac{s^2 + 2s + 5}{s^2 + 2s + 0.5}\right] K_1 \qquad (6.5.19)$$

where K_1 is a constant multiplier. The s.c. transfer admittance of network A is

$$y_{13}^{A} = - y_{11}^{A} a_{13}^{A} = - \frac{K_1}{38} \left[\frac{s^2 + 2s + 5}{s + 1} \right] \tag{6.5.20}$$

Network B has its transfer function zeroes at $s = \infty$. The Cauer development of z_{11}^{β} with shunt capacitances will automatically ensure that z_{12}^{B} has its zeroes at $s = \infty$. Network A has complex zeroes of transmission and may be synthesized by the parallel ladder development of Guillemin. The final network is shown in Fig. 6.5.2.

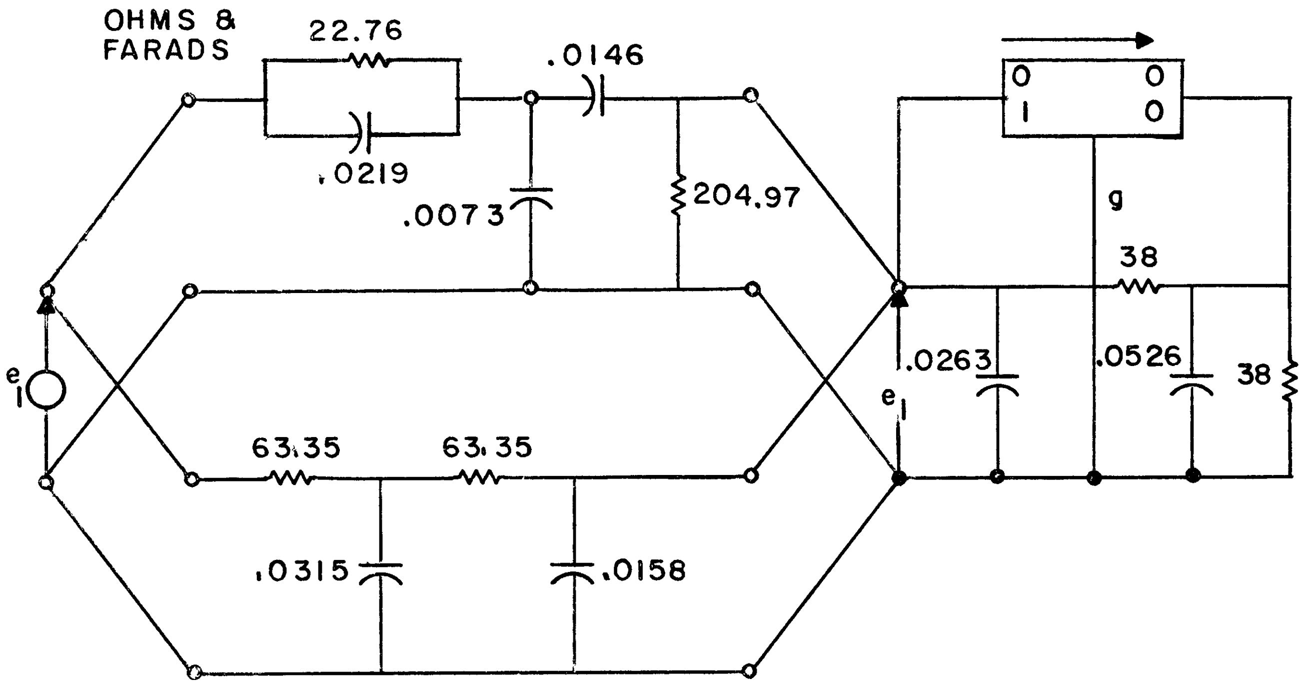


Figure 6.5.2. Network For Illustrative Example: Synthesis Technique No. 2

Note that the shunt 38 ohms at terminal-pair 2 of Network B may be used as the plate resistance of the normalized ideal vacuum tube. If the combination is to represent a physical vacuum tube (after impedance leveling) then the required amplification factor is

$$u = \frac{g_{21}}{g_{22}} = \frac{1}{1/38} = 38 \tag{6.5.21}$$

6.5.3 Synthesis Technique No. 3

The transfer impedance

$$z_{43} = K \frac{1}{(s + 1)(s^2 + s + 1)} = \frac{N(s)}{D(s)} \tag{6.5.22}$$

which is the third order Butterworth filter, will be synthesized with synthesis technique No. 3. Before this may be done we must find a polynomial Q(s) with negative real zeroes such that D/Q is expressible as the sum of an RC impedance and an RL impedance (or RC admittance). After some trial and error it is found that

$$Q(s) = (s + 0.25)(s + 1.25)(s + 4.00) \tag{6.5.23}$$

is suitable since

$$\frac{D(s)}{Q(s)} = \frac{(s + 1)(s^2 + s + 1)}{(s + 0.25)(s + 1.25)(s + 4.00)}$$

$$= \left[\frac{0.1625}{s + 0.25} + \frac{0.1193}{s + 1.25}\right] + \left[1 - \frac{3.782}{s + 4.00}\right] \tag{6.5.24}$$

where the first term in brackets is an RC impedance and the second term in brackets is an RC admittance. The R-LLF device for synthesis technique No. 2 will be specialized to the case

$$b = c = 1 \tag{6.5.25}$$

so that

$$G_5 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \tag{6.5.26}$$

It will be recognized that G_5 is the s.c. admittance matrix of a Gryator. With the Gyrator, the transfer impedance of the network of Fig.6.4.1 becomes

$$z_{43} = \frac{\hat{z}_{13}\hat{a}_{12}}{\hat{z}_{11} + \frac{1}{\hat{z}_{22}}} \tag{6.5.27}$$

We then identify

$$\hat{z}_{11} = \frac{0.1625}{s + 0.25} + \frac{0.1193}{s + 1.25} = \frac{0.2818s + 0.2329}{s^2 + 1.5s + 0.3125}$$

$$\hat{z}_{22} = \frac{1}{1 - \frac{3.782}{s + 4.00}} = \frac{s + 4.00}{s + 0.218} \tag{6.5.28}$$

Since z_{43} has its zeroes at $s = \infty$,

$$\hat{z}_{13} = \frac{1}{s^2 + 1.5s + 0.3125}$$

$$\hat{a}_{12} = \frac{1}{s + 4.0}$$

$$\hat{z}_{12} = \hat{a}_{12}z_{22} = \frac{1}{s + 0.218} \tag{6.5.29}$$

Since the transfer function zeroes of Networks A and B are at $s = \infty$, a Cauer development of $\hat{z}_{11}$ and $\hat{z}_{22}$ with shunt capacitances is appropriate. The final network is shown in Fig. 6.5.3.

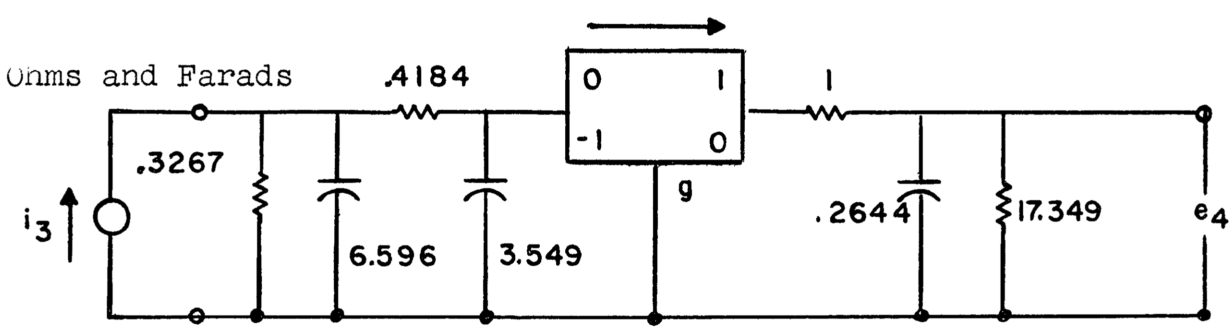


Figure 6.5.3. Network For Illustrative: Synthesis Technique No. 3

CHAPTER 7

SOME PROPERTIES OF DRIVING POINT AND TRANSFER IMPEDANCES OF LLF NETWORKS

7.1 Introduction

An effective analytic approach to the study of the fundamental properties of driving point and transfer functions of LLFPB networks is based upon expressing these network functions in terms of the energy functions associated with the network. This approach is extended to LLF networks in Chapter 7. As a result of this extension some new properties of LLF networks are found. A particularly interesting property is demonstrated for RC-LLF:R networks. Namely, that if the embedded R-LLF devices are active and bilateral the resulting network may not have complex natural frequencies, i.e., the natural frequencies are restricted to the real axis. Thus the repeated emphasis of the word active in the phrase Active RC Networks in many previous papers is misdirected. It is not the activity of the so-called "active" device involved but rather its nonbilaterality which allows the placement of poles in the complex plane. The most elementary lossless non-bilateral device is the gyrator. It is well known that an RLC driving point or transfer function can be realized with positive resistors, capacitors, and gyrators. But it has never been pointed out that not only are the gyrators sufficient but they are also necessary if generality of location of natural frequencies is to be obtained. Of collateral interest it is demonstrated in Section 6.5 of Chapter 6 that any transfer function (no pole at infinity) whose poles are equal to the S.C. natural frequencies of the series combination of an RC and an RL impedance may be realized by means of positive R's, C's, and one gyrator.

7.2 LLFPB Networks

In this section we will discuss briefly the energy function approach used to study the fundamental properties of driving point and transfer functions of LLFPB networks. Let us consider first the equilibrium equations formulated on the loop basis for an LLFPB network. These take the form

$$e = \left[\hat{R} + s\hat{L} + \frac{1}{s}\hat{S}\right] i \tag{7.2.1}$$

where $\hat{R}$, $\hat{L}$, and $\hat{S}$ are the resistance, inductance, and elastance loop parameter matrices, respectively. The column matrix e is that of source voltages in loop s while i is the column matrix of loop currents. We shall assume ℓ independent loops with source voltages in only the first p loops. Thus

$$e = \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_p \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad i = \begin{bmatrix} i_1 \\ i_2 \\ \vdots \\ i_p \\ i_{p+1} \\ \vdots \\ i_\ell \end{bmatrix} \tag{7.2.2}$$

If we enclose the network in a black box and bring out the p loop voltage sources (without disconnecting them) we form a p terminal-pair black box. The loop branch parameter matrix $\hat{Z}_{pp}$ of this MTP network relates the p voltages sources to the p currents traversing them. These currents are by definition $i_1, i_2 \ldots i_p$. Thus if we define the column vectors

$$E_p = \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_p \end{bmatrix} \quad I_p = \begin{bmatrix} i_1 \\ i_2 \\ \vdots \\ i_p \end{bmatrix} \tag{7.2.3}$$

then

$$E_p = \hat{Z}_{pp} I_p \tag{7.2.4}$$

Of course $\hat{Z}_{pp}$ is also the open circuit impedance matrix of the MTP network.

The expression

$$Q = i_t^* e = \sum_{m=1}^{\ell} e_m i_m^* = \sum_{m=1}^{p} e_m i_m^* = \left[I_p\right]_t^* E_p \tag{7.2.5}$$

will now be formulated in two ways - in terms of operations upon $\hat{R}, \hat{L}, \hat{S}$ and upon $\hat{Z}_{pp}$. The subscript t on a matrix denotes the transpose of the matrix. It is readily seen that Q takes the two forms

$$Q = i_t^* \hat{R} i_t + s i_t^* \hat{L} i + \frac{1}{s} i_t^* \hat{S} i = \left[I_p\right]_t^* \hat{Z}_{pp} I_p \tag{7.2.6}$$

If we define $\hat{R}, \hat{L}, \hat{S}$, and $\hat{Z}$ as follows

$$\hat{R} = \begin{bmatrix} \hat{R}_{11} & \hat{R}_{12} & \cdots & \hat{R}_{1\ell} \\ \hat{R}_{21} & \hat{R}_{22} & & \vdots \\ \vdots & & & \vdots \\ \hat{R}_{\ell 1} & & \cdots & \hat{R}_{\ell\ell} \end{bmatrix} ; \quad L = \begin{bmatrix} \hat{L}_{11} & \hat{L}_{12} & \cdots & \hat{L}_{1\ell} \\ \hat{L}_{21} & \hat{L}_{22} & & \vdots \\ \vdots & & & \vdots \\ \hat{L}_{\ell 1} & & \cdots & \hat{L}_{\ell\ell} \end{bmatrix} ;$$

$$S = \begin{bmatrix} \hat{S}_{11} & \hat{S}_{12} & \cdots & \hat{S}_{1\ell} \\ \hat{S}_{21} & \hat{S}_{22} & & \vdots \\ \vdots & & & \vdots \\ \hat{S}_{\ell 1} & & \cdots & \hat{S}_{\ell\ell} \end{bmatrix} ; \quad Z_{pp} = \begin{bmatrix} \hat{z}_{11} & \hat{z}_{12} & \cdots & \hat{z}_{1p} \\ \hat{z}_{21} & \hat{z}_{22} & & \vdots \\ \vdots & & & \vdots \\ \hat{z}_{p1} & & \cdots & \hat{z}_{pp} \end{bmatrix} \qquad (7.2.7)$$

then

$$\left[I_p^*\right]_t \hat{Z}_{pp} I_p = \sum\sum_{r,s=1}^{p} \hat{z}_{rs} i_r i_s^*$$

$$i_t^* \hat{R} i = \sum\sum_{r,s=1}^{\ell} \hat{R}_{rs} i_r i_s^* = F_o$$

$$i_t^* \hat{L} i = \sum\sum_{r,s=1}^{\ell} \hat{L}_{rs} i_r i_s^* = T_o \qquad (7.2.8)$$

$$i_t^* \hat{S} i = \sum\sum_{r,s=1}^{\ell} \hat{S}_{rs} i_r i_s^* = V_o$$

We may rewrite Equation 7.2.6 in the form

$$\sum_{r,s=1}^{p} \hat{z}_{rs} i_r i_s^* = F_o + sT_o + \frac{V_o}{s} \qquad (7.2.9)$$

The importance of this equation lies in the fact that the so-called energy functions F_o, T_o, and V_o are positive no matter how the network is excited provided $\hat{R}$, $\hat{L}$, and $\hat{S}$ are LLFPB loop parameter matrices. To demonstrate this property of F_o, T_o, and V_o let the typical current i_r be written in the form

$$i_r = a_r + jb_r \tag{7.2.10}$$

where a_r is the real part and b_r is the imaginary part of i_r. The typical product $i_r i_s^*$ in Equation 7.2.8 then may be written as

$$i_r i_s^* = \left[a_r a_s + b_r b_s\right] + j\left[a_s b_r - a_r b_s\right] \tag{7.2.11}$$

Substitution of this expression for $i_r i_s^*$ in the expression for F_o gives the equivalent expression

$$F_o = \sum\sum_{r,s=1}^{\ell} \hat{R}_{rs}\left[a_r a_s + b_r b_s\right] + j\sum\sum_{r,s=1}^{\ell} \hat{R}_{rs}\left[a_s b_r - a_r b_s\right] \tag{7.2.12}$$

Now due to the bilateral nature of LLFPB networks

$$\hat{R}_{rs} = \hat{R}_{sr}$$

It immediately follows that the second double sum in Equation 7.2.12 must vanish since the rs terms in the sum is the negative of the sr term. Thus

$$F_o = \sum\sum_{r,s=1}^{\ell} \hat{R}_{rs} a_r a_s + \sum\sum_{r,s=1}^{\ell} \hat{R}_{rs} b_r b_s \tag{7.2.13}$$

The two double sums in Equation 7.2.13 are known to be positive definite quadratic forms* and thus, as stated above, F_o is positive for all conditions of network excitation. In a parallel fashion one may demonstrate that T_o and V_o are the sums of positive definite quadratic forms and thus are positive for any manner of network excitation. It should be noted that an entirely dual discussion is applicable to the case in which we initially formulate equilibrium equations on the node basis.

A study of Equation 7.2.9 yields some fundamental properties of LLFPB networks. These arise from the "positive real" character of $F_o + sT_o + \frac{V_o}{s}$. For a discussion of these properties the reader is referred to the literature.*

7.3 LLF Networks

In this section we will consider the extension of the approach of the previous section to LLF networks. As defined in the introductory chapter an LLF network consists of branches which are the usual R's, L's, and C's, plus multiterminal-pair black boxes. These black boxes exhibit a resistive, inductive, or capacitive behavior. Let us consider the formulation of equilibrium equations on the loop basis for an LLF network. It will be assumed that the elements of the network are MTP elements of four types: LLFPB, R-LLF, L-LLF, C-LLF. Each MTP element is replaced by a set of mutually coupled branches. The loop equilibrium equations take the form

$$e = \left[Z_{\ell\ell} + R + sL + \frac{1}{s} S\right] i \qquad (7.3.1)$$

* Reference (3). Chapters 1 and 2.

where $\hat{Z}_{\ell\ell}$, R, L, and S are the loop parameter matrices of the LLFPB portion, the R-LLF portion, the L-LLF portion, and the C-LLF portion of the network, respectively. The matrix e is the column matrix of source voltages and the matrix i is the column vector of loop currents. Assuming that voltage sources are present in the first p loops and that there are ℓ loops, e and i take the forms

$$e = \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_p \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad i = \begin{bmatrix} i_1 \\ i_2 \\ \vdots \\ i_p \\ i_{p+1} \\ \vdots \\ i_\ell \end{bmatrix}$$

Just as in the development in the previous section we may define the column matrices E_p and I_p (Equation 7.2.3) and the open circuit impedance matrix Z_{pp} such that

$$E_p = Z_{pp} I_p \tag{7.3.2}$$

The Q function takes the two forms

$$Q = i_t^* \hat{Z}_{\ell\ell} i + i_t^* R i_t + s i_t^* L i + \frac{1}{s} i_t^* S i$$
$$Q = \left[I_p^*\right]_t Z_{pp} I_p \tag{7.3.3}$$

If we give $\hat{Z}_{\ell\ell}$, R, L, S, and Z_{pp} the following definitions

$$Z_{pp} = \begin{bmatrix} z_{11} & z_{12} & \cdots & z_{1p} \\ z_{21} & z_{22} & & \vdots \\ \vdots & & & \vdots \\ z_{p1} & & \cdots & z_{pp} \end{bmatrix}; \quad z_{\ell\ell} = \begin{bmatrix} \hat{z}_{11} & \hat{z}_{12} & \cdots & \hat{z}_{1\ell} \\ \hat{z}_{21} & \hat{z}_{22} & & \vdots \\ \vdots & & & \vdots \\ \hat{z}_{\ell 1} & & \cdots & \hat{z}_{\ell\ell} \end{bmatrix};$$

$$R = \begin{bmatrix} r_{11} & r_{12} & \cdots & r_{1\ell} \\ r_{21} & r_{22} & & \vdots \\ \vdots & & & \vdots \\ r_{\ell 1} & & \cdots & r_{\ell\ell} \end{bmatrix}; \quad L = \begin{bmatrix} \ell_{11} & \ell_{12} & \cdots & \ell_{1\ell} \\ \ell_{21} & \ell_{22} & & \vdots \\ \vdots & & & \vdots \\ \ell_{\ell 1} & & \cdots & \ell_{\ell\ell} \end{bmatrix} \qquad (7.3.4)$$

$$S = \begin{bmatrix} s_{11} & s_{12} & \cdots & s_{1\ell} \\ s_{21} & s_{22} & & \vdots \\ \vdots & & & \vdots \\ s_{\ell 1} & & \cdots & s_{\ell\ell} \end{bmatrix}$$

then

$$i_t^* \hat{Z}_{\ell\ell} i = \sum_{r,s=1}^{\ell}\sum \hat{z}_{rs} i_r i_s^*$$

$$i_t^* R i = \sum_{r,s=1}^{\ell}\sum r_{rs} i_r i_s^* = F$$

$$i_t^* L i = \sum_{r,s=1}^{\ell}\sum \ell_{rs} i_r i_s^* = T \qquad (7.3.5)$$

$$i_t^* S i = \sum_{r,s=1}^{\ell}\sum s_{rs} i_r i_s^* = V$$

$$\left[I_p^*\right]_t Z_{pp} I_p = \sum_{r,s=1}^{p}\sum z_{rs} i_r i_s^*$$

The first double sum in Equations 7.3.5 is of the same form as the left hand side of Equation 7.2.9. Thus we may express it in the form

$$\sum_{r,s=1}^{\ell}\sum \hat{z}_{rs} i_r i_s^* = F_o + sT_o + \frac{V_o}{s} \tag{7.3.6}$$

where F_o, T_o, and V_o are each expressible as the sum of two positive definite quadratic forms. With the functions F, T, and V defined in Equation 7.3.5 we may write

$$\sum_{r,s=1}^{p}\sum z_{rs} i_r i_s^* = F_o + F + s(T_o + T) + \frac{1}{s}(V_o + V) \tag{7.3.7}$$

The functions F, T, and V are not expressible as the sum of positive definite quadratic forms and in fact, are complex. We may separate them into real and imaginary parts in the manner that will now be described for F. First express the parameter matrix R as the sum of a symmetric matrix R_b and a skew-symmetric matrix R_n as follows

$$R = R_b + R_n \tag{7.3.8}$$

where

$$\begin{aligned} R_b &= \frac{1}{2}\left[R + R_t\right] \\ R_n &= \frac{1}{2}\left[R - R_t\right] \end{aligned} \tag{7.3.9}$$

One readily deduces from Equations 7.3.9 that

$$\begin{aligned} R_b &= \left[R_b\right]_t \\ R_n &= -\left[R_n\right]_t \end{aligned} \tag{7.3.10}$$

so that R_b is symmetric and R_n is skew-symmetric as desired. The elements of R_b and R_n are defined in terms of the elements of R as follows

$$
\begin{aligned}
r_{jk}^{(b)} &= \frac{r_{jk} + r_{kj}}{2} = r_{kj}^{(b)} \\
r_{jk}^{(n)} &= \frac{r_{jk} - r_{kj}}{2} = -r_{kj}^{(n)}
\end{aligned}
\tag{7.3.11}
$$

where $r_{jk}^{(b)}$ is a typical element of R_b and $r_{jk}^{(n)}$ is a typical element of R_n. The function F then takes the form

$$
F = \frac{1}{2}\sum_{r,s=1}^{\ell}\sum r_{rs}^{(b)} i_r i_s^* + \frac{1}{2}\sum_{r,s=1}^{\ell}\sum r_{rs}^{(n)} i_r i_s^* \tag{7.3.12}
$$

Let us now define

$$
\begin{aligned}
F_b &= \frac{1}{2}\sum_{r,s=1}^{\ell}\sum r_{rs}^{(b)} i_r i_s^* \\
jF_n &= \frac{1}{2}\sum_{r,s=1}^{\ell}\sum r_{rs}^{(n)} i_r i_s^*
\end{aligned}
\tag{7.3.13}
$$

so that

$$
F = F_b + jF_n \tag{7.3.14}
$$

It will now be demonstrated that F_n and F_b are real so that F_b is the real and F_n is the imaginary part of F. To demonstrate these facts we use in Equation 7.3.13 the expression for $i_r i_s^*$ given in Equation 7.2.11. The following expressions are then obtained for F_b and F_n

$$F_b = \frac{1}{2}\sum\sum r_{rs}^{(b)}\left[a_r a_s + b_r b_s\right] + j\,\frac{1}{2}\sum\sum r_{rs}^{(b)}\left[a_s b_r - a_r b_s\right]$$

$$F_n = \frac{1}{2}\sum\sum r_{rs}^{(n)}\left[a_s b_r - a_r b_s\right] - j\,\frac{1}{2}\sum\sum r_{rs}^{(n)}\left[a_r a_s + b_r b_s\right] \tag{7.3.15}$$

The second double sums in the expressions for F_b and F_n are readily seen to vanish because the rs and sr terms are of the same magnitude but of opposite sign. Thus

$$F_b = \frac{1}{2}\sum\sum r_{rs}^{(b)}\left[a_r a_s + b_r b_s\right] = \frac{1}{2}\sum\sum r_{rs}^{(b)}\ \mathrm{Re}\left[i_r i_s^*\right]$$

$$F_n = \frac{1}{2}\sum\sum r_{rs}^{(n)}\left[a_s b_r - a_r b_s\right] = \frac{1}{2}\sum\sum r_{rs}^{(n)}\ \mathrm{Im}\left[i_r i_s^*\right] \tag{7.3.16}$$

where $\mathrm{Re}\left[i_r i_s^*\right]$ stands for the real part and $\mathrm{Im}\left[i_r i_s^*\right]$ the imaginary part of $i_r i_s^*$.

The real functions F_b and F_n are implicit functions of the complex frequency variable s which satisfy the equations

$$F_b(s^*) = F_b(s) \tag{7.3.17}$$

$$F_n(s^*) = -F_n(s) \tag{7.3.18}$$

To demonstrate Equations 7.3.17 and 7.3.18 one need only note that i_r and i_s are real rational functions of the complex frequency variable s. Thus when s is replaced by s^* the real part of $i_r i_s^*$ stays the same but the imaginary part changes sign. Equation 7.3.18 implies that

$$F_n(\sigma) = -\,F_n(\sigma) \tag{7.3.19}$$

But this can only be true if

$$F_n(\sigma) = 0 \tag{7.3.20}$$

As a final point it should be noted that F_b is a function only of the symmetric part of R while F_n is a function only of the skew-symmetric part of R. Thus if R has no skew-symmetric portion $F_n \equiv 0$, while if R has no symmetric portion $F_b \equiv 0$. It is readily seen that R will have no skew-symmetric portion if the R-LLF devices have symmetric branch parameter matrices (O.C. impedance or S.C. admittance matrices) and that R will have no symmetric portion if the R-LLF devices have skew symmetric branch parameter matrices. It is well known that an R-LLF device with a skew-symmetric impedance (or admittance) matrix is a lossless device, i.e., it can neither dissipate nor generate power.[24] Thus a network containing lossless R-LLF devices must have $F_b \equiv 0$.

The above discussion may be carried along in parallel fashion for T and V to show that

$$\begin{aligned} T &= T_b + jT_n \\ V &= V_b + jV_n \end{aligned} \tag{7.3.21}$$

The real functions T_b, T_n, V_b, and V_n are given by

$$T_b = \frac{1}{2} \sum\sum \mathcal{l}_{rs}^{(b)} \left[a_r a_s + b_r b_s \right]$$

$$T_n = \frac{1}{2} \sum\sum \mathcal{l}_{rs}^{(n)} \left[a_s b_r - a_r b_s \right]$$

$$V_b = \frac{1}{2}\sum\sum s_{rs}^{(b)}\left[a_r a_s + b_r b_s\right]$$

$$V_n = \frac{1}{2}\sum\sum s_{rs}^{(n)}\left[a_s b_r - a_r b_s\right] \tag{7.3.22}$$

where $\ell_{rs}^{(b)}$, $\ell_{rs}^{(n)}$, $s_{rs}^{(b)}$ and $s_{rs}^{(n)}$ are defined analogously to $r_{rs}^{(b)}$ and $r_{rs}^{(n)}$.

It follows from Equation 7.3.22 that

$$\begin{aligned} T_b(s^*) &= T_b(s) \; ; \quad T_n(s^*) = -T_n(s) \; ; \quad T_n(\sigma) = 0 \\ V_b(s^*) &= V_b(s) \; ; \quad V_n(s^*) = -V_n(s) \; ; \quad V_n(\sigma) = 0 \end{aligned} \tag{7.3.23}$$

Thus Equation 7.3.7 becomes

$$\sum\sum_{r,s=1}^{p} z_{rs} i_r i_s^* = \left[F_o + F_b + jF_r\right] + s\left[T_o + T_b + jT_n\right] + \frac{1}{s}\left[V_o + V_b + jV_n\right] \tag{7.3.24a}$$

which is the desired extension of Equation 7.2.9 to LLF networks. By formulating equilibrium equations on the node basis one may derive the expression dual to Equation 7.3.24a as given below

$$\sum\sum_{r,s=1}^{p} y_{rs} e_r e_s^* = \left[\overline{F}_o + \overline{F}_b + j\overline{F}_n\right] + s\left[\overline{V}_o + V_b + j\overline{V}_n\right] + \frac{1}{s}\left[\overline{T}_o + \overline{T}_b + j\overline{T}_n\right] \tag{7.3.24b}$$

in which y_{rs}; $r,s = 1 \cdots p$ is the set of short circuit admittances for a network with p terminal pairs of access and e_r; $r = 1 \cdots p$ are terminal-pair voltages. To obtain expressions for the functions on the right hand side of Equation 7.3.24b one need only replace the corresponding expression on the right hand side of Equation 7.3.24a by its dual. Thus since

$$F_b = \frac{1}{2} \sum_{r,s=1}^{\ell}\sum r_{rs}^{(b)} \text{ Re } \left\{ i_r i_s^* \right\}$$

its dual has the expression

$$\overline{F}_b = \frac{1}{2} \sum_{r,s=1}^{n}\sum g_{rs}^{(b)} \text{ Re } \left\{ e_r e_s^* \right\}$$

where n is the total number of independent node pairs (the dual of ℓ, the number of independent loops) and $g_{rs}^{(b)}$ is an element of the symmetric portion of the branch admittance matrix of the R-LLF portion of the network. It is then only necessary to note which quantities in Equation 7.3.24a and Equation 7.3.24b are dual. The pertinent dual quantities are shown below

The dual of T_j is $\overline{V}_j$

The dual of F_j is $\overline{F}_j$

The dual of V_j is $\overline{T}_j$

where

$j = 0$, b, or n.

We may give a physical interpretation to a quantity of the form

$$Z(s) = \sum_{r,s=1}^{p}\sum z_{rs}x_r x_s^* \tag{7.3.25}$$

as the driving point impedance of an active-nonbilateral linear network just as the corresponding expression for LLFPB networks was given a physical interpretation by Brune.* The quantity Z(s) will be called a Brune form. In order to give a physical interpretation to the Brune form we have to introduce a new device which is the logical generalization of the ideal transformer to the case in which the turns ratio is a complex quantity. An ideal transformer is shown in Figure 7.3.1a.

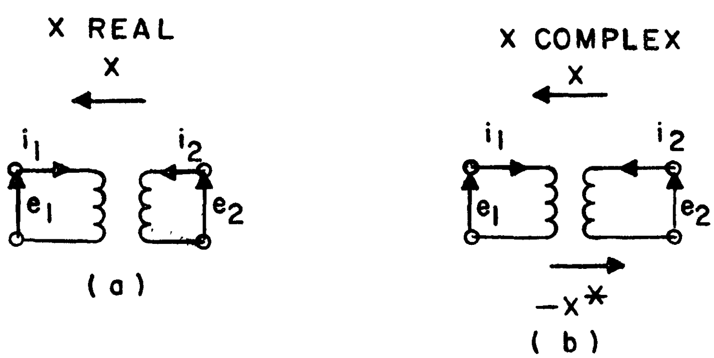


Figure 7.3.1. The Ideal And Generalized Ideal Transformer

The arrow and symbol x indicate that e_1 is stepped up by a factor of x relative to e_2. Thus the following mixed matrix equation characterizes the ideal transformer

$$\begin{bmatrix} e_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} 0 & x \\ -x & 0 \end{bmatrix} \begin{bmatrix} i_1 \\ e_2 \end{bmatrix} \tag{7.3.26}$$

*See Reference 3, page 8.

The net average power flowing into a two terminal pair device is given by

$$P_{AV} = \text{Re}\ \frac{1}{2}\left[e_1 i_1^* + e_2 i_2^*\right] = \text{RE}\ \frac{1}{2}\left[e_1^* i_1 + e_2^* i_2\right] \qquad (7.3.27)$$

For the ideal transformer

$$P_{AV} = \frac{1}{2}\ \text{Re}\left[xe_2\left[\frac{i_2^*}{-x}\right] + e_2 i_2^*\right] = 0 \qquad (7.3.28)$$

as is well known because the ideal transformer is lossless.

The device of Figure 7.3.1b which is the generalized ideal transformer is defined to have the following constraints between its voltages and currents.

$$\begin{bmatrix} e_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} 0 & x \\ -x^* & 0 \end{bmatrix} \begin{bmatrix} i_1 \\ e_2 \end{bmatrix} \qquad (7.3.29)$$

The net average power flowing into this generalized transformer is

$$P_{AV} = \frac{1}{2}\ \text{Re}\left[xe_2\left[\frac{i_2^*}{-x}\right] + e_2 i_2^*\right] = 0 \qquad (7.3.30)$$

i.e., the generalized transformer is a lossless device also. Note that when x is real the generalized ideal transformer becomes the conventional ideal transformer. Two arrows are used on the symbol for the generalized transformer to distinguish it from the conventional type. One may regard the generalized ideal transformer as a combination of an ideal transformer and phase shifters since the complex character of x merely indicates that i_1, e_1 are phase shifted relative to i_2, e_2.

A physical interpretation of $Z(s)$ is now given. Figure 7.3.2 shows schematically a p-terminal pair network N. The voltages and

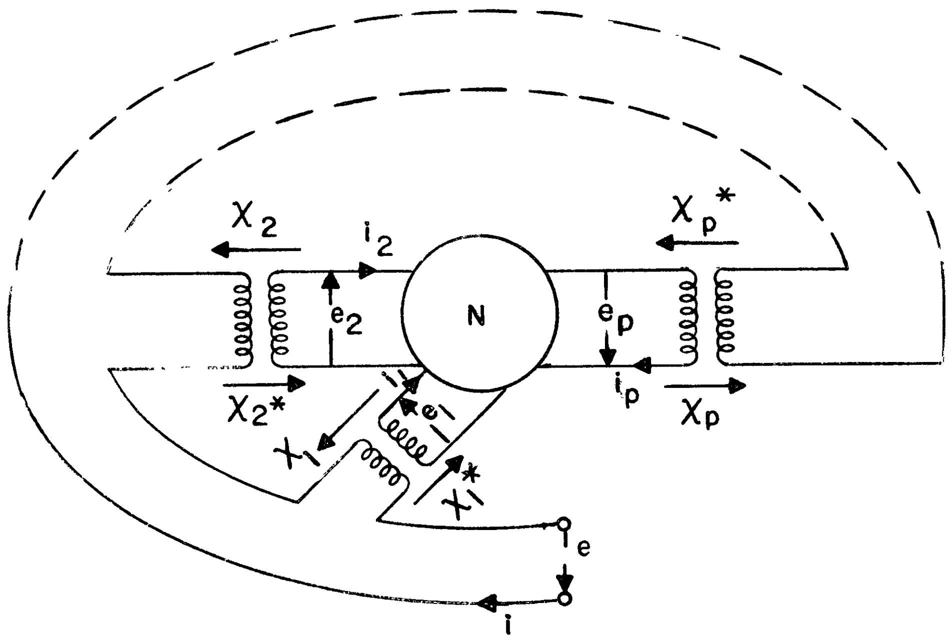


Figure 7.3.2. Relevant To The Physical Interpretation Of Equation 7.3.25

currents at the terminal pairs are denoted by e_1, i_1; e_2, i_2; $\cdots$ e_p; i_p. At each terminal pair the primary of a generalized ideal transformer is placed and all the secondaries are connected in series. Since the voltages on the secondaries are x_1e_1, x_2e_2, $\cdots$ x_pe_p, we see that the net voltage across their series combination is given by

$$e = x_1e_1 + x_2e_2 + \cdots x_pe_p \tag{7.3.31}$$

The common secondary current is i and is related to the various terminal pair currents by

$$i_1 = x_1^* i,\ i_2 = x_2^* i,\ \cdots\ i_p = x_p^* i \tag{7.3.32}$$

In matrix notation we may rewrite the last two equations in the form,

$$e = \begin{bmatrix} x_1 x_2 & \cdots & x_p \end{bmatrix} \cdot \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_p \end{bmatrix} \tag{7.3.33}$$

and

$$\begin{bmatrix} i_1 \\ i_2 \\ \vdots \\ i_p \end{bmatrix} = \begin{bmatrix} x_1^* \\ x_2^* \\ \vdots \\ x_p^* \end{bmatrix} i \tag{7.3.34}$$

The terminal pair voltages and currents of N are assumed to be related by the O.C. impedance matrix Z_{pp} of Equation 7.3.4. Thus

$$\begin{bmatrix} z_{11} & \cdots & z_{1p} \\ & & \\ z_{p1} & \cdots & z_{pp} \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ \vdots \\ i_p \end{bmatrix} = \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_p \end{bmatrix} \tag{7.3.35}$$

If we substitute for the column matrix of currents the equivalent expression in Equation 7.3.34 and then premultiply both sides by the row matrix

$$\begin{bmatrix} x_1 x_2 & \cdots & x_p \end{bmatrix}$$

we get, after noting Equation 7.3.33,

$$\begin{bmatrix} x_1 x_2 \cdots x_p \end{bmatrix} \begin{bmatrix} z_{11} & \cdots & z_{1p} \\ \vdots & & \vdots \\ z_{p1} & \cdots & z_{pp} \end{bmatrix} \begin{bmatrix} x_1^* \\ x_2^* \\ \vdots \\ x_p^* \end{bmatrix} = \frac{e}{i} = Z(s) \tag{7.3.36}$$

The left-hand side of this equation is recognized as being the matrix equivalent of the double sum in Equation 7.3.25 and the ratio e/i is the net impedance looking into the series combination of all the transformer secondaries in Figure 7.3.2. Thus the Brune form is given the simple physical interpretation of being the net impedance formed through series connection of the p terminal pairs of the network, each provided with a generalized ideal transformer having an independently controllable complex "turns" ratio.

7.4 Properties of LLF:R Network

The fact that the functions F, T, and V may be decomposed in the following way

$$F = F_b + jF_n$$

$$T = T_b + jT_n$$

$$V = V_b + jV_n$$

indicates that there is a fundamental physical reason for regarding the symmetric and skew-symmetric portions of R-LLF, L-LLF, or C-LLF branch parameter matrices as separate physical entities. Thus it is proper to investigate the forms that the Brune form takes when the R, L, and R-LLF devices have either symmetric or skew-symmetric parameter matrices. When

$$p = 1; \quad \left|i_1\right|^2 = 1 \tag{7.4.1}$$

Equation 7.3.24 yields the following general expression for the impedance of an LLF network

$$z_{11}(s) = F_o + F_b + jF_n + s\left[T_o + T_b + jT_n\right] + \frac{1}{s}\left[V_o + V_b + jV_n\right] \tag{7.4.2}$$

The expression $z_{11}(s)$ is not a p.r. function, although it is a rational function of s with real coefficients. This latter fact has already been discussed in Chapter 1 but it is instructive to prove this result from Equation 7.4.2. To demonstrate the "real" character of z_{11} (assuming rationality in s), it is only necessary, by definition, to show that z_{11} is real for s real. Forming $z_{11}(\sigma)$,

$$\begin{aligned} z_{11}(\sigma) &= F_o(\sigma) + F_b(\sigma) + jF_n(\sigma) + \sigma\left[T_o(\sigma) + T_b(\sigma) + jT_n(\sigma)\right] \\ &\quad + \frac{1}{\sigma}\left[V_o(\sigma) + V_b(\sigma) + jV_n(\sigma)\right] \end{aligned} \tag{7.4.3}$$

But it has been shown in the previous section that

$$F_n(\sigma) = T_n(\sigma) = V_n(\sigma) = 0 \tag{7.4.4}$$

Thus

$$z_{11}(\sigma) = F_o(\sigma) + F_b(\sigma) + \sigma\left[T_o(\sigma) + T_b(\sigma)\right] + \frac{1}{\sigma}\left[V_o(\sigma) + V_b(\sigma)\right] \tag{7.4.5}$$

which is obviously real.

We will now confine our attention to LLF:R networks, i.e., networks containing positive R's, L's, and C's in addition to R-LLF devices. For LLF:R networks

$$T = 0; \; V = 0 \tag{7.4.6}$$

and

$$Z(s) = \sum\sum_{r,s=1}^{p} z_{rs} i_s i_r^* = sT_o + \frac{1}{s} V_o + F_o + F_b + jF_n \tag{7.4.7}$$

The following two properties will now be demonstrated.

Property 1: An LLF:R network containing R-LLF devices with only skew-symmetric parameter matrices exhibits a positive real Brune form.

Property 2: An RC-LLF:R or RL-LLF:R network containing R-LLF devices with only symmetric parameter matrices must have its natural frequencies constrained to the σ axis.

The network of Property 1 might properly be called an LLFP:R network since the R-LLF devices involved are lossless. To demonstrate Property 1 we note that the presence of only skew-symmetric R-LLF parameter matrices implies that

$$F_b = 0$$

so that

$$Z(s) = \sum\sum_{r,s=1}^{p} z_{rs} i_r i_s^* = sT_o + \frac{1}{s} V_o + F_o + jF_n \tag{7.4.8}$$

The Brune form Z(s) (or any rational function of s) is defined to be positive real if

(a) It is real for real values of s

(b) It has a positive real part for values of s with a positive real part.

The real character of Z(s) has already been demonstrated. Since

$$\mathrm{Re}\left[Z(s)\right] = \mathrm{Re}\left[sT_o + \frac{1}{s} V_o + F_o\right] \tag{7.4.9}$$

it is clear that (b) is satisfied also since the quantity $sT_o + \frac{1}{s} V_o + F_o$ is a p.r. (positive real) function. Thus, Property 1 has been demonstrated.

The following properties may readily be deduced by application of Property 1.

<u>Property 1a</u>: The driving point functions of an LLFP:R network are positive real.

<u>Property 1b</u>: For an LLFP:R network any jw axis poles must be simple and the matrix of residues of driving point and transfer impedances at a jw axis pole must be a positive hermitian matrix. In particular, jw axis poles of driving point functions must be simple and have positive real residues.

<u>Property 1c</u>: The driving point functions of a network containing positive L's, C's, and skew-symmetric R-LLF devices are subject to the same restrictions with regard to s plane behaviour as those of a positive L,C network. In particular the natural frequencies of such a network are constrained to the jw axis.

Property 1a is obtained from 1 by letting p = 1 and $|i_1| = 1$ in Equation 7.4.8 for then the Brune form becomes equal to z_{11}. Since the reciprocal of a p.r. function is p.r. then $1/z_{11}$ is p.r. also.

Property 1b is obtained from the fact that if Z(s) is p.r. then it must have simple j axis poles with positive real residues. From the simplicity of the j axis poles of Z(s) we deduce that the $z_{rs}(s)$ must have simple j axis poles. If the residues of $z_{rs}(s)$ in their j axis poles are denoted by k_{rs} and those of Z(s) by k, then it is clear that

$$k = \sum_{r,s=1}^{p}\sum k_{rs}x_r x_s^* \text{ must be positive real} \qquad (7.4.10)$$

The double sum is just the Brune form that corresponds to the matrix of residues

$$K = \begin{bmatrix} k_{11} & k_{12} & \cdots & k_{1p} \\ k_{21} & k_{22} & & \vdots \\ \vdots & & & \\ k_{p1} & & \cdots & k_{pp} \end{bmatrix}$$

Equation 7.4.10 states that this Brune form must be positive real. This can only be true if

$$k_{rs} = k_{sr}^* \qquad (7.4.11)$$

Thus the matrix of residues must be a positive hermitian matrix.

Property 1c follows directly from Property 1b.

We will prove Property 2 for RC-LLF:R networks and the proof for RL-LLF:R networks will follow by analogy. The networks referred to in Property 2 are properly designated as RC-LLFB:R and RL-LLFB:R. We note first that if the R-LLF devices have symmetric parameter matrices then

$$F_n \equiv 0$$

When inductances are absent

$T_o = 0$

Thus

$$Z(s) = \sum_{r,s=1}^{p}\sum z_{rs} i_r i_s^* = \frac{V_o}{s} + F_b \tag{7.4.12}$$

If we assume that $|i_1| = 1$ and $p = 1$ we obtain the following expression for z_{11},

$$z_{11} = \frac{V_o}{s} + F_b \; ; \; |i_1| = 1 \tag{7.4.13}$$

This impedance will have zeroes when

$$\frac{V_o}{s} + F_b = 0 \tag{7.4.14}$$

The function V_o will not be zero when $|i_1| = 1$ except in trivial situations. Thus values of s which satisfy Equation 7.4.14 are given by

$$s = -\frac{V_o}{F_b} \tag{7.4.15}$$

Since V_o and F_b are real, it follows that the zeroes of z_{11} must lie on the σ axis. Of course, F_b may be negative so that positive real axis zeroes of z_{11} are permissible. By using the dual expression to Equation 7.4.13 we find that

$$y_{11} = s\overline{V}_o + \overline{F}_b \tag{7.4.16}$$

from which we deduce that the zeroes of driving point admittances are also on the real axis. The analogous derivation for the RL-LLFB:R network is clear. Thus Property 2 is demonstrated.

The following network properties may be derived from Property 2.

Property 2a: A network containing positive capacitances (or inductances plus positive and negative resistances, must have its natural frequencies constrained to the σ axis.

Property 2b: A network containing positive capacitances, negative inductances, and positive and negative resistances must have its natural frequencies constrained to the σ axis.

Property 2c: A network containing negative capacitances, positive inductances, and positive and negative resistances must have its natural frequencies constrained to the σ axis.

Property 2a is obvious and Property 2c is the dual of Property 2b. Thus, we will demonstrate the truth of Property 2b only. This is proven from Property 2 by noting that a negative inductance may be obtained with a positive capacitance and positive and negative resistances. The method of obtaining a negative inductance from a positive capacitance and resistances has already been discussed in Section 4.2.4. The network which does this is shown in Figure 4.2.6.

Before closing this chapter it should be noted that the Properties 1a, 1b, and 1c are deducible from basic physical considerations. In fact, a special case of 1b and 1c have been demonstrated by Carlin*. The general statement of property 1 is new and the method of proof of Properties 1a, 1b, and 1c is also new. As far as Properties 2, 2a, 2b, and 2c are concerned, they are entirely new.

* Reference 24, page 27.

BIOGRAPHICAL NOTE

Phillip Bello was born on October 22, 1929 in Lynn, Mass. After attending schools in Boston and vicinity he entered Northeastern University in September, 1948. In June, 1953, he graduated from Northeastern University with the degree of B.S. in Electrical Engineering. Immediately following this he accepted a position as Research Assistant in the Research Laboratory of Electronics at M.I.T. In February, 1955, he received the degree of S.M. in Electrical Engineering from M.I.T. The title of his Master's Thesis was "Analysis of Target Noise In Phase Comparison Radar Systems". After leaving M.I.T. in February, 1955, he accepted a position as Research Associate at Northeastern University and was promoted to Assistant Professor in 1956. He taught "Transients in Linear System" and "Filtering and Prediction" in the evening graduate school at Northeastern University. His research work at Northeastern University was concerned with various phases of Statistical Communication Theory in classified areas of application. In February of 1957, Mr. Bello left Northeastern University to accept a position with Dunn Engineering Associates of Cambridge, Massachusetts. His work there was also classified. He was in charge of various study programs associated with the analysis and design of conventional and future type radar systems.

Mr. Bello entered M.I.T. on September, 1957, to commence the Doctorate program in Electrical Engineering. During the September, 1957 - June, 1958 academic year, he was the recipient of a B.T.L. fellowship. In the summer of 1958, he was employed by the Applied Research Laboratory of Sylvania, Waltham, Massachusetts. During the

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