8 May 1996 **Prof. S. Lee**

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

Hour Exam III for Course 18.06: Linear Algebra

Recitation Instructor:

Your Name: **SOLUTIONS**

Recitation Time:

Lecturer:

Grading

1. **22**
2. **22**
3. **22**
4. **34**

TOTAL: 100

Do all your work on these pages.
No calculators or notes.
Please work carefully, and check your intermediate results whenever possible.
Point values (total of 100) are marked on the left margin.

[6] **1a.** Give a 2 by 2 matrix B that is similar to $A = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}$. (Do not use $B = A$.)

B is any 2 by 2 matrix with $\lambda_1 = 3$, $\lambda_2 = 2$.
For example,

$$B = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$$

or any other matrix with det $(B) = 6$ and $\operatorname{trace}(B) = 5$.

[6] **1b.** Give a 2 by 2 matrix B that is similar to $A = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$. (Do not use $B = A$.)

B is any 2 by 2 matrix with $\lambda_1 = 3$, $\lambda_2 = 3$, and rank $(B - 3I) = 1$.
For example,

$$B = \begin{bmatrix} 5/2 & 1/2 \\ -1/2 & 7/2 \end{bmatrix}$$

or any other matrix with det $(B) = 9$, $\operatorname{trace}(B) = 6$, and
rank $(B - 3I) = 1$.

[10] **1c.** Let $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 5 & 4 \\ 2 & 4 & 4 \end{bmatrix}$. Is $x^T Ax > 0$ for all nonzero vectors x?

If not, find a nonzero vector x for which $x^T Ax \leq 0$.

A is **not** positive definite ($x^T Ax > 0$) because: det$(A) = 0$, there is a zero eigenvalue, there is a zero pivot, etc.

From elimination, $\begin{bmatrix} 1 & 2 & 2 \\ 2 & 5 & 4 \\ 2 & 4 & 4 \end{bmatrix} \rightarrow \begin{bmatrix} \boxed{1} & 2 & 2 \\ 0 & \boxed{1} & 0 \\ 0 & 0 & \boxed{0} \end{bmatrix}$. Pivots: 1, 1 and 0.

Find a nonzero vector x for which $x^T Ax \leq 0$.
Hint: $x^T Ax = 0$ for $Ax = \mathbf{0}$.

Reduce $Ax = \mathbf{0}$ to $Ux = \mathbf{0}$, and solve:

$$\underbrace{\begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{U} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{so } x = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}, \text{ a nonzero vector}$$

for which $x^T Ax = 0$.

2. Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$:

[4] **2a.** Find the eigenvalues of A.

Solve $\det(A - \lambda I) = 0$:
$(1-\lambda)(4-\lambda) - 4 = 4 - 5\lambda + \lambda^2 - 4 = \lambda^2 - 5\lambda = \lambda(\lambda - 5) = 0$.
Therefore, $\lambda_1 = 5$ and $\lambda_2 = 0$.

[6] **2b.** Give a factorization $A = Q\Lambda Q^T$ where Q has orthonormal columns and Λ is a diagonal matrix.

$A = Q\Lambda Q^T$ is an eigenvalue-eigenvector factorization of a symmetric matrix.
Λ is a diagonal matrix containing the eigenvalues of A, and
Q is a 2 by 2 matrix whose orthonormal columns are eigenvectors of A.
For example,

$$\Lambda = \begin{bmatrix} 5 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad Q = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}.$$

[12] **2c.** As $t \to \infty$, what is the limit of $u(t)$ for $\frac{du}{dt} = -Au$ given the initial condition $u(0) = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$?

Note that the eigenvalues and eigenvectors of $-A$ need to be used.

$$u(t) = c_1 e^{-\lambda_1 t} x_1 + c_2 e^{-\lambda_2 t} x_2 = c_1 e^{-5t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 e^{0t} \begin{bmatrix} 2 \\ -1 \end{bmatrix}.$$

$u(0) = \begin{bmatrix} 3 \\ 1 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ so $c_1 = 1$ and $c_2 = 1$.

As $t \to \infty$, $u(t) = e^{-5t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 2 \\ -1 \end{bmatrix} \to \begin{bmatrix} 2 \\ -1 \end{bmatrix}$.

3. Let $A = \begin{bmatrix} 3 & 4 \\ 1 & 3 \end{bmatrix}$:

[8] **3a.** Find the eigenvalues and eigenvectors of the matrix A.

`Solve det` $(A - \lambda I) = 0$:
$(3-\lambda)(3-\lambda) - 4 = 9 - 6\lambda + \lambda^2 - 4 = \lambda^2 - 6\lambda + 5 = (\lambda - 5)(\lambda - 1) = 0$.

`The eigenvalues of` A `are:` $\lambda_1 = 5$ `and` $\lambda_2 = 1$.

$$\underbrace{\begin{bmatrix} -2 & 4 \\ 1 & -2 \end{bmatrix}}_{A-5I} \underbrace{\begin{bmatrix} 2 \\ 1 \end{bmatrix}}_{x_1} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}; \quad \underbrace{\begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix}}_{A-1I} \underbrace{\begin{bmatrix} -2 \\ 1 \end{bmatrix}}_{x_2} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

`The vectors` x_1, x_2 `are eigenvectors of` A.

[14] **3b.** Let $u_{k+1} = A^{-1}u_k$. What is the limit of u_k as $k \to \infty$ given the initial condition $u_0 = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$?

`Note that the eigenvalues and eigenvectors of` A^{-1} `need to be used.`

$$u_k = c_1 \left(\frac{1}{\lambda_1}\right)^k x_1 + c_2 \left(\frac{1}{\lambda_2}\right)^k x_2 = c_1 \left(\frac{1}{5}\right)^k \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 (1)^k \begin{bmatrix} -2 \\ 1 \end{bmatrix}.$$

$u_0 = \begin{bmatrix} 0 \\ 2 \end{bmatrix} = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ `so` $c_1 = 1$ `and` $c_2 = 1$.

`As` $k \to \infty$, $u_k = \left(\frac{1}{5}\right)^k \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \begin{bmatrix} -2 \\ 1 \end{bmatrix} \to \begin{bmatrix} -2 \\ 1 \end{bmatrix}$.

4. Suppose the following information is known about A:

$$A\begin{bmatrix}1\\2\\1\end{bmatrix} = 6\begin{bmatrix}1\\2\\1\end{bmatrix}, \quad \text{and} \quad A\begin{bmatrix}1\\-1\\1\end{bmatrix} = 3\begin{bmatrix}1\\-1\\1\end{bmatrix}, \quad \text{and} \quad A\begin{bmatrix}2\\-1\\0\end{bmatrix} = 3\begin{bmatrix}1\\-1\\1\end{bmatrix}.$$

[9] **4a.** Find the eigenvalues of A.

The first two results show that 6 and 3 are eigenvalues of A.
The last two results show that there are (at least) two different solutions

to $Ax = \begin{bmatrix}3\\-3\\3\end{bmatrix}$.

In other words, A has a nontrivial nullspace -- which means

$Ax = \begin{bmatrix}0\\0\\0\end{bmatrix} = \lambda_3 x$ where x is a nonzero vector and λ_3 must be zero.

$\boxed{\lambda_1 = 6,\ \lambda_2 = 3 \text{ and } \lambda_3 = 0.}$

[9] **4b.** Find the eigenvectors of A.

The first two results in 4. show that $\underbrace{\begin{bmatrix}1\\2\\1\end{bmatrix}}_{x_1}$ and $\underbrace{\begin{bmatrix}1\\-1\\1\end{bmatrix}}_{x_2}$ are eigenvectors of A.

We can find a third eigenvector that satisfies $Ax = 0x$ via

$$A\begin{bmatrix}1\\-1\\1\end{bmatrix} - A\begin{bmatrix}2\\-1\\0\end{bmatrix} = \begin{bmatrix}3\\-3\\3\end{bmatrix} - \begin{bmatrix}3\\-3\\3\end{bmatrix}$$

$$A\left(\begin{bmatrix}1\\-1\\1\end{bmatrix} - \begin{bmatrix}2\\-1\\0\end{bmatrix}\right) = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$A\underbrace{\begin{bmatrix}-1\\0\\1\end{bmatrix}}_{x_3} = \begin{bmatrix}0\\0\\0\end{bmatrix}.$$

[16] **4c.** In each of these questions, you must give a correct reason (based on eigenvalues, eigenvectors, etc.) to get full credit.

(i) Is A a diagonalizable matrix?

`Yes.` A `has distinct eigenvalues.`
`Another reason: the eigenvectors are linearly independent.`

(ii) Is A an invertible matrix?

`No. One of the eigenvalues is zero.`

(iii) Is A a Markov matrix?

`No. At least two of the eigenvalues are larger than one in magnitude.`

(iv) Is A a projection matrix?

`No. Each eigenvalue of a projection matrix is either` 0 `or` 1.

(v) Is A a symmetric matrix?

`Yes.` A `has real eigenvalues` <u>`and`</u> `orthogonal eigenvectors.`