10 April 1996 | Prof. S. Lee

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

Hour Exam II for Course 18.06: Linear Algebra

Recitation Instructor:

Your Name: **SOLUTIONS**

Recitation Time:

Lecturer:

Grading

1. **24**
2. **16**
3. **30**
4. **30**

TOTAL: 100

Do all your work on these pages.
No calculators or notes.
Please work carefully, and check your intermediate results whenever possible.
Point values (total of 100) are marked on the left margin.

[6] **1a.** Give a 3 by 3 matrix A with the properties: $A^T = A^{-1}$ and $\det A = 1$.
(A is not allowed to be a diagonal matrix.)

`Hint:` `An orthogonal matrix with` $\det A = 1$.

`Permutation matrices such as` $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ `or` $\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ `have this property.`

`Or,` $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$.

[6] **1b.** Give a 3 by 3 matrix A with the properties: $A^T = A$ and $A^2 = A$ and $\text{rank}\,(A) = 1$.
(A is not allowed to be a diagonal matrix.)

`Hint:` `A projection matrix of rank 1.`

`Example:` `Use` $A = V(V^TV)^{-1}V^T$ `where` $V = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$: $A = \frac{1}{14}\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$.

[6] **1c.** Suppose A is a 5 by 3 matrix with orthonormal columns. Evaluate the following determinants:

$\det(A^TA) =$ ___1___

$\det(AA^T) =$ ___0___

$\det(A(A^TA)^{-1}A^T) =$ ___0___

$\boxed{\texttt{det}\,(A^TA) = \texttt{det}\,(I) = 1\,.}$

AA^T must have dependent columns, det $= 0$, because $A(A^Ty) = \mathbf{0}$ for any nonzero vector y in the nullspace of A^T.
The 3 by 5 matrix A^T has 3 linearly independent (orthonormal!) rows and a nontrivial nullspace of dimension $5 - r = 5 - 3 = 2$.
$\boxed{\texttt{det}\,(AA^T) = 0\,.}$

$$\boxed{\texttt{det}\,(A\,\underbrace{(A^TA)^{-1}}_{I}\,A^T) = \texttt{det}\,(AA^T) = 0\,.}$$

[6] **1d.** What value(s) of x give $\det A = 0$, where

$$A = \begin{bmatrix} x & 2 & 3 \\ -x & x & 0 \\ 3 & 2 & 5 \end{bmatrix}?$$

$\texttt{det}\,(A) = 5x^2 - 6x + 0 - 9x + 10x - 0 = 5x^2 - 5x = 5x(x-1) = 0\,.$
Therefore, $x = 0$, $x = 1$.

2. Suppose $Ax = b$ has the complete solution $x = \underbrace{\begin{bmatrix} 1 \\ -4 \\ 0 \end{bmatrix}}_{x_{\text{particular}}} + \underbrace{x_3 \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}}_{x_{\text{homogeneous}}}$,

where $b = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

[8] **2a.** How many linearly independent rows does A have? Explain.

A has r independent columns and r independent rows.

The vector $\begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$ is a basis for the nullspace of A.

This means the nullspace of A is 1-dimensional; dim $\mathcal{N}(A) = 1$.
Given dim $\mathcal{N}(A) = 1 = n - r = 3 - r$, we can solve for r
to find that A has $r = 2$ linearly independent rows.

[8] **2b.** Give a basis for the row space of A.

Dimension of the row space: $r = 2$.
The row space contains all vectors perpendicular to the nullspace of A.
We can obtain 2 linearly independent vectors in the row space (i.e., a basis)
by solving $2x_1 - 3x_2 + 1x_3 = 0$.

For example, $\begin{bmatrix} 3/2 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -1/2 \\ 0 \\ 1 \end{bmatrix}$ form a basis for the row space of A.

3. The projection matrix

$$P = \frac{1}{50}\begin{bmatrix} 34 & -20 & -12 \\ -20 & 25 & -15 \\ -12 & -15 & 41 \end{bmatrix}$$

projects vectors onto a 2-dimensional plane in $\mathbf{R}^3$.

[8] **3a.** $PP^T =$ ____(give the 3 by 3 matrix)____.

`Properties of (orthogonal) projection matrices:` $P^2 = P$ `and` $P^T = P$.
`For the above projection matrix, we use these properties to obtain`
$PP^T = PP = P^2 = P$.

$$PP^T = P = \frac{1}{50}\begin{bmatrix} 34 & -20 & -12 \\ -20 & 25 & -15 \\ -12 & -15 & 41 \end{bmatrix}.$$

[10] **3b.** Give a nonzero vector c such that $Pc = c$.

P projects all vectors onto a 2-dimensional plane in $\mathbf{R}^3$.
$Pc = c$ for any vector c already in this plane.

$$\text{For example, } P\begin{bmatrix}1\\0\\0\end{bmatrix} = \underbrace{\begin{bmatrix}34/50\\-20/50\\-12/50\end{bmatrix}}_{c} \text{ or } P\begin{bmatrix}0\\1\\0\end{bmatrix} = \underbrace{\begin{bmatrix}-20/50\\25/50\\-15/50\end{bmatrix}}_{c}.$$

[12] **3c.** Find a matrix A such that $A(A^TA)^{-1}A^T = P$.

The columns of A are a basis for the 2-dimensional plane onto which P projects.
To obtain a basis, we need 2 linearly independent vectors in this plane.

$$\text{From 3b, } \begin{bmatrix}34/50\\-20/50\\-12/50\end{bmatrix} \text{ and } \begin{bmatrix}-20/50\\25/50\\-15/50\end{bmatrix} \text{ are independent and in the 2D plane.}$$

$$A = \frac{1}{50}\begin{bmatrix}34 & -20\\-20 & 25\\-12 & -15\end{bmatrix}.$$

4. The vectors $\underbrace{\begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}}_{v_1}$ and $\underbrace{\begin{bmatrix} -10 \\ 0 \\ 1 \end{bmatrix}}_{v_2}$ are linearly independent and span the left nullspace of a matrix A.

[8] **4a.** Give an orthonormal basis for the left nullspace of A.

`Make the vectors` v_1, v_2 `orthogonal to each other via Gram-Schmidt:`

$$w_1 = v_1 = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}.$$

$$w_2 = v_2 - \left(\frac{v_1^T v_2}{v_1^T v_1}\right) v_1 = \begin{bmatrix} -10 \\ 0 \\ 1 \end{bmatrix} - \left(\frac{-30}{10}\right) \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix}.$$

`Normalize these vectors to obtain an orthonormal basis:`

$$q_1 = \frac{1}{\sqrt{10}} \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} \text{ and } q_2 = \frac{1}{\sqrt{11}} \begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix}.$$

[10] **4b.** Find the vector p (in the column space of A) that is closest to $b = \begin{bmatrix} -2 \\ 6 \\ -20 \end{bmatrix}$.

`Check: Is` b `perpendicular to the left nullspace of` A`? Yes.`
`Why? Because` $b^T v_1 = 0$ `and` $b^T v_2 = 0$.

`Therefore,` b `is already in the column space of` A`;` $p = b$.

$$p = \begin{bmatrix} -2 \\ 6 \\ -20 \end{bmatrix}.$$

[12] **4c.** Find the projection matrix P that projects all vectors onto the column space of A.

Dimension of left nullspace = $2 = m - r = 3 - r$.
By solving for r, we find that the column space is 1-dimensional.
We can construct P by first finding a basis for the column space of A and then computing $P = V(V^TV)^{-1}V^T$.
Note: the columns of V are a basis for the column space of A.

From 4b, we have $b = \begin{bmatrix} -2 \\ 6 \\ -20 \end{bmatrix}$ already in the column space of A;
it is a basis because it is linearly independent and spans the 1-dimensional column space.

So, $V = \begin{bmatrix} -2 \\ 6 \\ -20 \end{bmatrix}$ leads to

$$P = V(V^TV)^{-1}V^T = \frac{1}{440}\begin{bmatrix} 4 & -12 & 40 \\ -12 & 36 & -120 \\ 40 & -120 & 400 \end{bmatrix}.$$