18.06 Hour Exam I   8 March 1996   Prof. S. Lee

Your Recitation Leader:

Your Name: **SOLUTIONS**

Your Recitation Time:

Grading

1. **26**
2. **20**
3. **30**
4. **24**

TOTAL: 100

Do all your work on these 7 pages. No calculators or notes.
Point values (total of 100) are marked on the left margin.

[4] **1a.** Are the vectors $\underbrace{\begin{bmatrix} -2 \\ -1 \\ 3 \\ 4 \end{bmatrix}}_{v_1}$ and $\underbrace{\begin{bmatrix} -8 \\ 2 \\ -2 \\ 1 \end{bmatrix}}_{v_2}$ linearly independent?

Are these vectors perpendicular to each other? Explain.

`Linearly independent? Yes.`
`Two vectors are linearly dependent when one vector is a scalar multiple of the other; otherwise, they are linearly independent.`

`Perpendicular? No,` $v_1^T v_2 = 12 \neq 0$.

[4] **1b.** Are the vectors $\underbrace{\begin{bmatrix} -2 \\ -1 \\ 3 \\ 4 \end{bmatrix}}_{v_1}, \underbrace{\begin{bmatrix} 8 \\ 2 \\ 2 \\ 1 \end{bmatrix}}_{v_2}, \underbrace{\begin{bmatrix} 10 \\ 1 \\ 1 \\ 6 \end{bmatrix}}_{v_3}, \underbrace{\begin{bmatrix} -2 \\ -1 \\ 3 \\ 4 \end{bmatrix}}_{v_4}$ a basis for $\mathbf{R}^4$? Explain.

`No, the vectors` v_1, v_2, v_3, v_4 `are not linearly independent.`

`For example,` $c_1 v_1 + c_2 v_2 + c_3 v_3 + c_4 v_4 = \mathbf{0}$ `for` $c = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}$.

[8] **1c.** Are the vectors $\underbrace{\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}}_{v_1}, \underbrace{\begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}}_{v_2}, \underbrace{\begin{bmatrix} -4 \\ -2 \\ 2 \\ 1 \end{bmatrix}}_{v_3}$ a basis for the subspace described by the 3-dimensional plane $x_1 + 2x_2 + 3x_3 + 6x_4 = 0$? Explain.

No, the vectors are <u>not</u> a basis for the given subspace.
The vectors are linearly independent.
However, the vectors do not span the plane because v_3 is not in the plane (v_3 does not satisfy the 3D plane equation).

[10] **1d.** Find the value for q for which the vectors $\begin{bmatrix} 1 \\ 4 \\ 6 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 12 \\ 10 \end{bmatrix}, \begin{bmatrix} q \\ 3 \\ 1 \end{bmatrix}$ do **not** span $\mathbf{R}^3$.

Set up the vectors as columns of a matrix, and perform Gaussian elimination.

$$\begin{bmatrix} 1 & 0 & -1 & q \\ 4 & 2 & 12 & 3 \\ 6 & 2 & 10 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -1 & q \\ 0 & 2 & 16 & 3-4q \\ 0 & 2 & 16 & 1-6q \end{bmatrix} \rightarrow \begin{bmatrix} \boxed{1} & 0 & -1 & q \\ 0 & \boxed{2} & 16 & 3-4q \\ 0 & 0 & 0 & \boxed{\texttt{-2-2q}} \end{bmatrix}.$$

Need $-2-2q = 0$ in last row so that the number of nonzero pivots = $r < 3$.
(Recall that r is the dimension of the column space.)
For $q = -1$, the four vectors span an $r = 2$ dimensional subspace of $\mathbf{R}^3$.

2. Consider the following 3×3 matrix, $A = \begin{bmatrix} 1 & 3 & 0 \\ 2 & 5 & 1 \\ -3 & -9 & -1 \end{bmatrix}$.

[10] **(a)** Find the missing entries of $A^{-1} = \begin{bmatrix} 4 & 3 & 3 \\ \alpha_{21} & -1 & -1 \\ \alpha_{31} & \alpha_{32} & -1 \end{bmatrix}$.

i. $$\underbrace{\begin{bmatrix} 4 & 3 & 3 \\ \alpha_{21} & -1 & -1 \\ \alpha_{31} & \alpha_{32} & -1 \end{bmatrix}}_{A^{-1}} \begin{bmatrix} 1 & 3 & 0 \\ 2 & 5 & 1 \\ -3 & -9 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \boxed{1} & 0 \\ \boxed{0} & 0 & 1 \end{bmatrix}$$

ii. $$\begin{bmatrix} 1 & 3 & 0 \\ 2 & 5 & 1 \\ -3 & -9 & -1 \end{bmatrix} \underbrace{\begin{bmatrix} 4 & 3 & 3 \\ \alpha_{21} & -1 & -1 \\ \alpha_{31} & \alpha_{32} & -1 \end{bmatrix}}_{A^{-1}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \boxed{1} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

From i, (row 2 of A^{-1}) $\cdot$ (column 2 of A) equals 1:
$3\alpha_{21} - 5 + 9 = 1$; $\boxed{\alpha_{21} = -1}$.

From ii, (row 2 of A) $\cdot$ (column 2 of A^{-1}) equals 1:
$6 - 5 + \alpha_{32} = 1$; $\boxed{\alpha_{32} = 0}$.

From i again, (row 3 of A^{-1}) $\cdot$ (column 1 of A) equals 0:
$\alpha_{31} + 2\alpha_{32} + 3 = 0$; $\boxed{\alpha_{31} = -3}$.

[10] **(b)** Using the result in part(a), find the inverse of $\begin{bmatrix} 3 & 0 & 1 \\ 5 & 1 & 2 \\ -9 & -1 & -3 \end{bmatrix}$.

The new matrix is A with permuted <u>columns</u>.
The new inverse will be A^{-1} with permuted <u>rows</u> since
$(AP)^{-1} = P^{-1}A^{-1} = P^TA^{-1}$.

$$P = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \; P^T = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \text{ so } P^TA^{-1} = \begin{bmatrix} -1 & -1 & -1 \\ -3 & 0 & -1 \\ 4 & 3 & 3 \end{bmatrix}.$$

3. Let $A = \begin{bmatrix} 1 & 5 & 0 & -3 \\ 2 & 10 & 1 & -4 \\ -1 & -5 & 1 & 5 \end{bmatrix}$.

[6] **(a)** Give an LU-factorization of A.

$$A = \begin{bmatrix} 1 & 5 & 0 & -3 \\ 2 & 10 & 1 & -4 \\ -1 & -5 & 1 & 5 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 5 & 0 & -3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} \boxed{1} & 5 & 0 & -3 \\ 0 & 0 & \boxed{1} & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} = U.$$

l_{21}: `row` $\underline{2}$ $-$ (2) $\times$ `row` $\underline{1}$.
l_{31}: `row` $\underline{3}$ $-$ $(-1)\times$ `row` $\underline{1}$.
l_{32}: `row` $\underline{3}$ $-$ (1) $\times$ `row` $\underline{2}$.

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix}.$$

[6] **(b)** Give a basis for the column space of A.

Pivot columns in A: column 1, column 3.

$\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ form a basis for the column space of A.

[6] **(c)** Give a basis for the row space of A.

Pivot rows in U: row 1, row 2.

$\begin{bmatrix} 1 \\ 5 \\ 0 \\ -3 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 0 \\ 1 \\ 2 \end{bmatrix}$ form a basis for the row space of A.

[6] **(d)** Give a basis for the nullspace of A.

`The special solutions to` $Ux = \mathbf{0}$ `yield a basis for the nullspace of` A:

$$x = \begin{bmatrix} -5 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad x = \begin{bmatrix} 3 \\ 0 \\ -2 \\ 1 \end{bmatrix}.$$

[6] **(e)** Give the complete solution (general solution) to the equation $Ax = b$ in the form $x = x_{\text{particular}} + x_{\text{homogeneous}}$ where

$$b = \begin{bmatrix} 6 \\ 4 \\ -14 \end{bmatrix}.$$

$x_{\text{particular}}$ `has the form` $[x_1, 0, x_3, 0]^T$, `where` x_1 `and` x_3 `are pivot variables.`

$$Ax_{\text{particular}} = x_1 \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ -14 \end{bmatrix}; \; x_1 = 6, \text{ so } x_3 = -8.$$

`The complete solution has the form` $x_{\text{particular}} + x_{\text{homogeneous}}$:

$$x = \begin{bmatrix} 6 \\ 0 \\ -8 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -5 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 3 \\ 0 \\ -2 \\ 1 \end{bmatrix}.$$

4. The following information is known about A:

$$A\underbrace{\begin{bmatrix}1\\-2\\3\\1\end{bmatrix}}_{v}=\begin{bmatrix}-6\\3\end{bmatrix}\quad\text{and}\quad A\underbrace{\begin{bmatrix}3\\-1\\1\\2\end{bmatrix}}_{w}=\begin{bmatrix}-18\\9\end{bmatrix}.$$

In fact, Ax is always some multiple of $\begin{bmatrix}-2\\1\end{bmatrix}$.

`The column space (always some multiple of` $\begin{bmatrix}-2\\1\end{bmatrix}$`) is a 1-dimensional subspace in` $\mathbf{R}^2$`; the dimension of the column space is` $r=1$.

`Note that the matrix` A `maps vectors in` $\mathbf{R}^4$ `to vectors in` $\mathbf{R}^2$`;`
A `is an` $m=2$ `by` $n=4$ `matrix.`

[6] **(a)** What is the dimension of the nullspace of A?

`Dimension of the nullspace:` $n-r=4-1=3$.

[4] **(b)** What is the dimension of the row space of A?

`Dimension of the row space:` $r=1$.

[6] **(c)** What is the dimension of the left nullspace of A?

`Dimension of the left nullspace:` $m-r=2-1=1$.

[8] **(d)** Give a nonzero solution to $Ax=\begin{bmatrix}0\\0\end{bmatrix}$.

$$-3Av+Aw=\begin{bmatrix}18\\-9\end{bmatrix}+\begin{bmatrix}-18\\9\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}.$$

$$-3Av+Aw=A(-3v)+Aw=A(-3v+w)=\begin{bmatrix}0\\0\end{bmatrix}.$$

$$x=-3v+w=\begin{bmatrix}-3\\6\\-9\\-3\end{bmatrix}+\begin{bmatrix}3\\-1\\1\\2\end{bmatrix}=\begin{bmatrix}0\\5\\-8\\-1\end{bmatrix}.$$