MASSACHUSETTS INSTITUTE OF TECHNOLOGY

Practice Final for 18.06: Linear Algebra

Recitation Instructor: Your Name: **SOLUTIONS**

Recitation Time: Lecturer: **S. Lee**

Grading

1. **16**
2. **18**
3. **20**
4. **20**
5. **24**
6. **24**
7. **24**
8. **18**
9. **24**
10. **12**

TOTAL: 200

Point values (total of 200) are marked on the left margin.
No calculators or notes. Please work carefully, and check your intermediate results whenever possible.

1. Let $A = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 4 & 6 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$.

`***` Elimination: $\underbrace{\begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 4 & 6 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}}_{A} \to \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} \to \underbrace{\begin{bmatrix} 0 & \boxed{1} & 2 & 3 & 4 \\ 0 & 0 & 0 & \boxed{1} & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_{U}$.

[4] **1a.** Give a basis for the row space of A.

$$\boxed{\begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 2 \end{bmatrix}.}$$

`Pivot rows in` U`:  row` 1`, row` 2.

[4] **1b.** Give a basis for the nullspace of A.

$$\boxed{\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}.}$$

`Special solutions to` $Ux = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

[4] **1c.** Give a basis for the column space of A.

$$\boxed{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix}.}$$

`Pivot columns in` A`:  column 2, column 4.`

[4] **1d.** Do the solutions to $Ax = \begin{bmatrix} 5 \\ 7 \\ 2 \end{bmatrix}$ form a subspace?

If so, what is the dimension of this subspace.

$\boxed{\texttt{No.}}$

`Every subspace of` $\mathbf{R}^3$ `must include the zero vector.`

`However,` $x = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ `is` <u>`not`</u> `a solution to` $Ax = \begin{bmatrix} 5 \\ 7 \\ 2 \end{bmatrix}$.

2. Write down a matrix with the specified properties, or explain why no such matrix exists:

[6] **2a.** A matrix whose column space is spanned by $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$,

and row space is spanned by $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

$$\boxed{\texttt{Example:} \quad A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}.}$$

`For other examples,` A `must have the following properties:`
A `has 3 rows and 2 columns,` $\operatorname{rank}(A) = 2$, `and the second row of` A `is zero.`

[6] **2b.** A matrix whose column space has a basis $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and row space has a basis $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$.

$$\boxed{A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & 1 \end{bmatrix}, \text{ or any nonzero multiple of it.}}$$

[6] **2c.** A matrix whose row space contains $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$, nullspace contains $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$, and whose determinant is -1.

$\boxed{\texttt{Not possible.}}$

$$A \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{ means } \det(A) = 0, \text{ because } A \text{ has linearly dependent columns.}$$

[20] **3.** In $\mathbf{R}^3$, what point on the plane described by $x_1+x_2-x_3=0$ is closest to $b=\begin{bmatrix}2\\1\\0\end{bmatrix}$?

$$\boxed{\texttt{The point } p=\begin{bmatrix}1\\0\\1\end{bmatrix} \texttt{ is closest to } b=\begin{bmatrix}2\\1\\0\end{bmatrix}.}$$

Every point on the plane described by $1x_1+1x_2-1x_3=0$ is a solution to $\begin{bmatrix}1 & 1 & -1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\end{bmatrix}$.

The special solutions $\begin{bmatrix}-1\\1\\0\end{bmatrix}$, $\begin{bmatrix}1\\0\\1\end{bmatrix}$ are a basis for the 2-dimensional plane of points in $\mathbf{R}^3$.

The least squares solution $\bar{x}$ to $\underbrace{\begin{bmatrix}-1 & 1\\1 & 0\\0 & 1\end{bmatrix}}_{A}x=b$ gives a point $p=A\bar{x}$ that is closest to b.

In particular, $\underbrace{\begin{bmatrix}2 & -1\\-1 & 2\end{bmatrix}}_{A^TA}\bar{x}=\underbrace{\begin{bmatrix}-1\\2\end{bmatrix}}_{A^Tb}$ gives $\bar{x}=\begin{bmatrix}0\\1\end{bmatrix}$ and $p=A\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}1\\0\\1\end{bmatrix}$.

[20] **4.** Find a matrix B such that $B^TB = \underbrace{\begin{bmatrix} 1 & 2 & 2 \\ 2 & 8 & 8 \\ 2 & 8 & 12 \end{bmatrix}}_{A}$.

$$\boxed{B = \begin{bmatrix} 1 & 2 & 2 \\ 0 & 2 & 2 \\ 0 & 0 & 2 \end{bmatrix}.}$$

Find the $A = LDL^T = CC^T$ factorization, where $C = L\sqrt{D}$.

$$\underbrace{\begin{bmatrix} 1 & 2 & 2 \\ 2 & 8 & 8 \\ 2 & 8 & 12 \end{bmatrix}}_{A} \to \begin{bmatrix} 1 & 2 & 2 \\ 0 & 4 & 4 \\ 0 & 4 & 8 \end{bmatrix} \to \underbrace{\begin{bmatrix} 1 & 2 & 2 \\ 0 & 4 & 4 \\ 0 & 0 & 4 \end{bmatrix}}_{U} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}}_{L^T}.$$

$$C = L\sqrt{D} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 0 \\ 2 & 2 & 2 \end{bmatrix}.$$

Then let $B^T = C$ and (consequently) $B = C^T$, so that $CC^T = B^TB = A$.

[24] **5.** Give an orthonormal basis for the column space of $A = \begin{bmatrix} 1 & 3 & 8 \\ 1 & 3 & 0 \\ 1 & -1 & 0 \\ 1 & -1 & 0 \end{bmatrix}$.

$$\boxed{q_1 = \tfrac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad q_2 = \tfrac{1}{2}\begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}, \quad q_3 = \tfrac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}.}$$

`Use the Gram-Schmidt process:`

$$A = \underbrace{\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}}_{a}.$$

$$B = \underbrace{\begin{bmatrix} 3 \\ 3 \\ -1 \\ -1 \end{bmatrix}}_{b} - \left(\tfrac{A^T b}{A^T A}\right)\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ -1 \\ -1 \end{bmatrix} - \left(\tfrac{4}{4}\right)\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ -2 \\ -2 \end{bmatrix}.$$

$$C = \underbrace{\begin{bmatrix} 8 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{c} - \left(\tfrac{A^T c}{A^T A}\right)\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \left(\tfrac{B^T c}{B^T B}\right)\begin{bmatrix} 2 \\ 2 \\ -2 \\ -2 \end{bmatrix} = \begin{bmatrix} 8 \\ 0 \\ 0 \\ 0 \end{bmatrix} - \left(\tfrac{8}{4}\right)\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \left(\tfrac{16}{16}\right)\begin{bmatrix} 2 \\ 2 \\ -2 \\ -2 \end{bmatrix} = \begin{bmatrix} 4 \\ -4 \\ 0 \\ 0 \end{bmatrix}.$$

q_1, q_2, q_3 are then obtained by normalizing the vectors A, B, C, respectively.

6. Let $A = \begin{bmatrix} 1 & -2 & v_1 \\ -2 & 1 & v_2 \\ -2 & -2 & v_3 \end{bmatrix}$, where the third column of A is the vector $v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$.

If possible, find a vector v that gives A the following properties.

`***` Let $V = \begin{bmatrix} 1 & -2 \\ -2 & 1 \\ -2 & -2 \end{bmatrix}$ contain the first two columns of A.

[6] **6a.** A vector v that makes A a singular matrix.

$v = \begin{bmatrix} 1 \\ -2 \\ -2 \end{bmatrix}$ or $v = \begin{bmatrix} -2 \\ 1 \\ -2 \end{bmatrix}$ or any other vector in the column space of V.

[6] **6b.** A vector v that makes A an invertible matrix.

Example: $v = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

Choose a vector v that is <u>not</u> in the column space of V.

[6] **6c.** A vector v that makes A a matrix with orthogonal columns.

$v = \begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix}$, or any nonzero multiple of it.

v is any nonzero vector in the left nullspace of V.

[6] **6d.** A vector v that makes A a symmetric positive definite matrix.

Not possible.

The upper left 2 by 2 determinant of A is less than zero.
By definition, A cannot be symmetric positive definite.

7. Let $A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$. The eigenvalues of A are $\lambda_1 = 4$ and $\lambda_2 = 2$.

`***` `Eigenvectors:` $x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ `and` $x_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

[6] **7a.** Find $u(t)$ given that $\frac{du}{dt} = Au$ and $u(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

$$\boxed{u(t) = \tfrac{1}{2}e^{4t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} - \tfrac{1}{2}e^{2t}\begin{bmatrix} -1 \\ 1 \end{bmatrix}.}$$

`General solution:` $u(t) = c_1 e^{\lambda_1 t} x_1 + c_2 e^{\lambda_2 t} x_2 = c_1 e^{4t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{2t}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

`Initial condition:` $u(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \underbrace{c_1}_{1/2}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \underbrace{c_2}_{-1/2}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

`Solution:` $u(t) = \frac{1}{2}e^{4t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} - \frac{1}{2}e^{2t}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

[6] **7b.** Find $u(t)$ given that $\frac{du}{dt} = Au$ and $u(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

$$\boxed{u(t) = e^{4t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}.}$$

`General solution (from 7a.):` $u(t) = c_1 e^{\lambda_1 t} x_1 + c_2 e^{\lambda_2 t} x_2 = c_1 e^{4t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{2t}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

`Initial condition:` $u(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \underbrace{c_1}_{1}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \underbrace{c_2}_{0}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

`Solution:` $u(t) = e^{4t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

[12] **7c.** Find e^{At}.

$$\boxed{e^{At} = \begin{bmatrix} \frac{1}{2}e^{4t} + \frac{1}{2}e^{2t} & \frac{1}{2}e^{4t} - \frac{1}{2}e^{2t} \\ \frac{1}{2}e^{4t} - \frac{1}{2}e^{2t} & \frac{1}{2}e^{4t} + \frac{1}{2}e^{2t} \end{bmatrix}.}$$

$$\begin{aligned} e^{At} = Se^{\Lambda t}S^{-1} &= \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^{4t} & 0 \\ 0 & e^{2t} \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}^{-1} \\ &= \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^{4t} & 0 \\ 0 & e^{2t} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \\ &= \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2}e^{4t} & \frac{1}{2}e^{4t} \\ -\frac{1}{2}e^{2t} & \frac{1}{2}e^{2t} \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{2}e^{4t} + \frac{1}{2}e^{2t} & \frac{1}{2}e^{4t} - \frac{1}{2}e^{2t} \\ \frac{1}{2}e^{4t} - \frac{1}{2}e^{2t} & \frac{1}{2}e^{4t} + \frac{1}{2}e^{2t} \end{bmatrix}. \end{aligned}$$

8. Suppose that A has eigenvalues 0, 1, 2, with respective eigenvectors u, v, w.

[6] **8a.** Describe the nullspace, column space, and row space of A in terms of u, v, w.

nullspace: all multiples of u.
column space: all linear combinations of v and w.
row space: all vectors orthogonal to u.

[6] **8b.** Find all solutions to $Ax = v - w$ in terms of u, v, w.

$\boxed{A(v - \tfrac{1}{2}w + ku) = v - w\text{, where } k \text{ is any real number.}}$

Using the properties of linearity and the given eigenvalue, eigenvector information, we find all solutions to $Ax = v - w$ via:

$$v - w = \underbrace{Av}_{v} - \tfrac{1}{2}\underbrace{Aw}_{2w} + k\underbrace{Au}_{\mathbf{0}} = A(v - \tfrac{1}{2}w + ku).$$

[6] **8c.** Prove that A is not an orthogonal matrix.

An orthogonal matrix has linearly independent columns.

However, the columns of A are linearly dependent because $Au = \mathbf{0}$.

9. Let $A = \begin{bmatrix} 0.4 & 0.3 \\ 0.6 & 0.7 \end{bmatrix}$.

[4] **9a.** Find the eigenvalues of A.

$\boxed{\lambda_1 = 1 \texttt{ and } \lambda_2 = 0.1\,.}$

`det` $(A - \lambda I) = (\lambda - 1)(\lambda - 0.1) = 0$ `so that` $\lambda_1 = 1$ `and` $\lambda_2 = 0.1$.

`Or, recall that a positive Markov matrix has` $\lambda_1 = 1$.

`Therefore` $\lambda_2 = 0.1$ `because` $\lambda_1 + \lambda_2 =$ `trace` $(A) = 1.1$.

`The eigenvectors of` A `turn out to be` $x_1 = \begin{bmatrix} 1/2 \\ 1 \end{bmatrix}$ `and` $x_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

[10] **9b.** Find the steady-state limit of the vector $u_k = A^k \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, as $k \to \infty$.

$$\boxed{u_k \to \begin{bmatrix} 1/3 \\ 2/3 \end{bmatrix} \texttt{ as } k \to \infty\,.}$$

`General solution:` $u_k = c_1 \lambda_1^k x_1 + c_2 \lambda_2^k x_2 = c_1 \begin{bmatrix} 1/2 \\ 1 \end{bmatrix} + c_2 (0.1)^k \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

`Initial condition:` $u_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \underbrace{c_1}_{2/3} \begin{bmatrix} 1/2 \\ 1 \end{bmatrix} + \underbrace{c_2}_{1/3} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

`Solution:` $u_k = \frac{2}{3} \begin{bmatrix} 1/2 \\ 1 \end{bmatrix} + \frac{1}{3} (0.1)^k \begin{bmatrix} -1 \\ 1 \end{bmatrix} \to \begin{bmatrix} 1/3 \\ 2/3 \end{bmatrix}$ `as` $k \to \infty$.

[10] **9c.** Find the limit of A^k, as $k \to \infty$.

$$\boxed{A^k \to \begin{bmatrix} 1/3 & 1/3 \\ 2/3 & 2/3 \end{bmatrix} \quad \texttt{as} \; k \to \infty\,.}$$

$$A^k = S\Lambda^k S^{-1} = \begin{bmatrix} 1/2 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0.1 \end{bmatrix}^k \begin{bmatrix} 1/2 & -1 \\ 1 & 1 \end{bmatrix}^{-1}.$$

$$A^k \to \begin{bmatrix} 1/2 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 2/3 & 2/3 \\ -2/3 & 1/3 \end{bmatrix} = \begin{bmatrix} 1/2 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2/3 & 2/3 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1/3 & 1/3 \\ 2/3 & 2/3 \end{bmatrix}$$

as $k \to \infty$.

10. Let $A = \begin{bmatrix} i & -1 \\ 1 & i \end{bmatrix}$, where $i = \sqrt{-1}$.

[6] **10a.** Give the LU factorization of A.

$$\boxed{L = \begin{bmatrix} 1 & 0 \\ -i & 1 \end{bmatrix} \text{ and } U = \begin{bmatrix} i & -1 \\ 0 & 0 \end{bmatrix}.}$$

`Elimination:` $\underbrace{\begin{bmatrix} i & -1 \\ 1 & i \end{bmatrix}}_{A} \to \underbrace{\begin{bmatrix} i & -1 \\ 0 & 0 \end{bmatrix}}_{U}.$

l_{21} : row $\underline{2} - (-i) \times$ row $\underline{1}$.

[6] **10b.** If possible, find a real matrix B that is similar to A.

$\boxed{\texttt{Not possible.}}$

`Similar matrices have the same trace.`
`However, trace`$(A) = 2i$ `and trace`$(B) \neq 2i$ `because the sum of the (real) diagonal entries of` B `must be a real number.`