4 December 1996 **Profs. S. Lee and M. Hopkins**

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

Hour Exam III for Course 18.06: Linear Algebra

Recitation Instructor:

Your Name: **SOLUTIONS**

Recitation Time:

Lecturer:

Grading

1. **20**
2. **24**
3. **32**
4. **24**

TOTAL: 100

Do all your work on these pages.
No calculators or notes.
Please work carefully, and check your intermediate results whenever possible.
Point values (total of 100) are marked on the left margin.

[10] **1a.** If possible, find an invertible matrix M such that $M^{-1}\underbrace{\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}}_{A} M = \underbrace{\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 2 \end{bmatrix}}_{B}$.

If it is not possible, briefly state why M cannot exist.

$\boxed{\texttt{Not possible.}}$

$B = M^{-1}AM$ means that B is similar to A.
For similar matrices, all of the following properties are the same:
eigenvalues, determinant, <u>trace</u>, <u>rank</u>, and Jordan form.

Note that $\mathrm{trace}(A) = 3$ but $\mathrm{trace}(B) = 5$.
Also, $\mathrm{rank}(A) = 1$ but $\mathrm{rank}(B) = 2$.

[10] **1b.** For what real values of c (if any) is $A = \begin{bmatrix} -1 & c & 2 \\ c & -4 & -3 \\ 2 & -3 & 4 \end{bmatrix}$ a symmetric positive definite matrix?

$\boxed{\texttt{Not possible. We cannot find } c \texttt{ such that } A \texttt{ is symmetric positive definite.}}$

`For symmetric positive definite matrices, all upper left determinants are greater than zero.`
`Note that the` 1 `by` 1 `upper left determinant is` -1.

2. Let $A = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}$, where $i = \sqrt{-1}$.

[8] **2a.** Find the eigenvalues and eigenvectors of A.

$$\boxed{\lambda_1 = 1\,, \;\; \lambda_2 = -1\,; \;\; x_1 = \begin{bmatrix} i \\ 1 \end{bmatrix}, \;\; x_2 = \begin{bmatrix} -i \\ 1 \end{bmatrix}.}$$

`For eigenvalues:` $\det(A - \lambda I) = \lambda^2 - 1 = 0\,.$
`The eigenvalues are` $\lambda_1 = 1\,,\;\; \lambda_2 = -1\,.$

`For eigenvectors:` $\underbrace{\begin{bmatrix} -1 & i \\ -i & -1 \end{bmatrix}}_{A-1I} \underbrace{\begin{bmatrix} i \\ 1 \end{bmatrix}}_{x_1} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}; \;\; \underbrace{\begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix}}_{A-(-1)I} \underbrace{\begin{bmatrix} -i \\ 1 \end{bmatrix}}_{x_2} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$

`The eigenvectors are` $x_1 = \begin{bmatrix} i \\ 1 \end{bmatrix}, \;\; x_2 = \begin{bmatrix} -i \\ 1 \end{bmatrix}.$

[8] **2b.** Find A^{-1}.

$$\boxed{A^{-1} = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}.}$$

Let Λ be a diagonal matrix whose entries are the eigenvalues λ_1, λ_2 of A.
Let S be a matrix whose columns are the eigenvectors x_1, x_2 of A.

$$A^{-1} = S \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}^{-1}}_{\Lambda^{-1}=\Lambda} S^{-1} = S\Lambda S^{-1} = A = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}.$$

[8] **2c.** Find A^{100}.

$$\boxed{A^{100} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.}$$

Let S be a matrix whose columns are the eigenvectors x_1, x_2 of A.

$$A^{100} = S \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}^{100}}_{\Lambda^{100}=I} S^{-1} = SS^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

3. Let $A = \begin{bmatrix} -1 & 4 \\ 1 & -4 \end{bmatrix}$.

[8] **3a.** Find the eigenvalues and eigenvectors of A.

$$\boxed{\lambda_1 = -5\,,\ \ \lambda_2 = 0\,;\ \ x_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix},\ \ x_2 = \begin{bmatrix} 4 \\ 1 \end{bmatrix}.}$$

`For eigenvalues:` $\det(A - \lambda I) = \lambda(\lambda + 5) = 0$.
`The eigenvalues of` A `are` $\lambda_1 = -5$, $\lambda_2 = 0$.

`For eigenvectors:` $\underbrace{\begin{bmatrix} 4 & 4 \\ 1 & 1 \end{bmatrix}}_{A-(-5)I} \underbrace{\begin{bmatrix} -1 \\ 1 \end{bmatrix}}_{x_1} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$; $\underbrace{\begin{bmatrix} -1 & 4 \\ 1 & -4 \end{bmatrix}}_{A-0I} \underbrace{\begin{bmatrix} 4 \\ 1 \end{bmatrix}}_{x_2} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

`The eigenvectors of` A `are` $x_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, $x_2 = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$.

[12] **3b.** $\det(e^{At}) =$ ______ $\boxed{e^{-5t}}$ ______.

`Let` S `be a matrix whose columns are the eigenvectors` x_1, x_2 `of` A.

e^{At} `is similar to` $e^{\Lambda t} = \begin{bmatrix} e^{-5t} & 0 \\ 0 & 1 \end{bmatrix}$ `because`

$$e^{At} = Se^{\Lambda t}S^{-1} = S \begin{bmatrix} e^{-5t} & 0 \\ 0 & e^{0t} \end{bmatrix} S^{-1} = S \underbrace{\begin{bmatrix} e^{-5t} & 0 \\ 0 & 1 \end{bmatrix}}_{e^{\Lambda t}} S^{-1}.$$

`Similar matrices have the same determinant.`
$\det(e^{At}) = \det(e^{\Lambda t}) = e^{-5t}$.

[12] **3c.** For the differential equation $\frac{du}{dt} = Au$, give a nonzero initial vector $u(0)$ such that $u(t) \to \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ as $t \to \infty$.

$$\boxed{u(0) = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \text{ or any nonzero multiple of it.}}$$

The general solution is $u(t) = c_1 e^{\lambda_1 t} x_1 + c_2 e^{\lambda_2 t} x_2 = c_1 e^{-5t} \begin{bmatrix} -1 \\ 1 \end{bmatrix} + c_2 e^{0t} \begin{bmatrix} 4 \\ 1 \end{bmatrix}$,

so that $u(t) = c_1 e^{-5t} \begin{bmatrix} -1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 4 \\ 1 \end{bmatrix}$.

If $u(0)$ is any nonzero multiple of $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$, then $c_2 = 0$ and we obtain a solution $u(t) = c_1 e^{-5t} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ that goes to $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ as $t \to \infty$.

4. Suppose the following information is known about a matrix A:

$$A\underbrace{\begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}}_{v} = \begin{bmatrix} 2 \\ 4 \\ -6 \end{bmatrix}, \quad A\underbrace{\begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}}_{w} = \begin{bmatrix} -2 \\ -4 \\ 6 \end{bmatrix} \text{ and } A \text{ is } \underline{\text{symmetric}}.$$

The questions below refer to any matrix A with the above properties.

[8] **4a.** Is A invertible? Explain.

$\boxed{\texttt{No.}}$

`A is not invertible because it has linearly dependent columns.`
`For example, the results for Av and Aw show that`

$$A\underbrace{\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}}_{(v+w)} = \begin{bmatrix} 2 \\ 4 \\ -6 \end{bmatrix} + \begin{bmatrix} -2 \\ -4 \\ 6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

`That is,` $(2 \times \text{column } 1) - \text{column } 2$ `gives the zero vector.`

[8] **4b.** Does A have linearly independent eigenvectors? Explain.

$\boxed{\texttt{Yes.}}$

`The eigenvectors of a symmetric matrix are linearly independent`
`(and can be chosen to be orthonormal).`

[8] **4c.** Give a specific example of a matrix A with the above properties and whose eigenvalues add up to zero.

$$\boxed{A = \begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -3 & -6 & a_{33} \end{bmatrix}, \text{ where } a_{33} = -5.}$$

Av `gives` 2 `times the first column of` A.
Aw `gives` -1 `times the second column of` A.
`Then from symmetry, we have` $a_{13} = a_{31}$ `and` $a_{23} = a_{32}$.
`For` a_{33}, `we have` $\sum \lambda_k = \text{trace}(A) = 1 + 4 + \underbrace{a_{33}}_{-5} = 0$.