6 November 1996 — Profs. S. Lee and M. Hopkins

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

Hour Exam II for Course 18.06: Linear Algebra

Recitation Instructor:

Your Name: **SOLUTIONS**

Recitation Time:

Lecturer:

Grading

1. **16**
2. **28**
3. **36**
4. **20**

TOTAL: 100

Do all your work on these pages.
No calculators or notes.
Please work carefully, and check your intermediate results whenever possible.
Point values (total of 100) are marked on the left margin.

1. Let $A = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 2 \\ 0 & 3 & 3 & 3 \\ 4 & 4 & 4 & 4 \end{bmatrix}$.

[8] **1a.** $\det(A) =$ ____ 24 ________.

`Obtain an upper triangular matrix via two, row exchanges.`
`Exchange rows 1 and 4; exchange rows 2 and 3.`
`Two row exchanges:  determinant does` <u>`not`</u> `change sign.`
`Determinant of the upper triangular matrix is` $(1)(2)(3)(4) = 24$.

[8] **1b.** $\det(\frac{1}{2}A) =$ ____ $\frac{3}{2}$ ________.

$\det(\frac{1}{2}A) = \left(\frac{1}{2}\right)^4 \det(A) = \frac{24}{16}$.

2. Let $A = \begin{bmatrix} 13 & 12 \\ 12 & 13 \end{bmatrix}$.

[8] **2a.** Find the eigenvalues of A.

$$\det(A - \lambda I) = (13 - \lambda)(13 - \lambda) - 144 = \lambda^2 - 26\lambda + 25 = (\lambda - 25)(\lambda - 1) = 0\,.$$

$$\boxed{\lambda_1 = 25\,,\ \ \lambda_2 = 1\,.}$$

[8] **2b.** Find the eigenvectors of A.

Solve $(A - 25I)x_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and $(A - 1I)x_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

$$\boxed{x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \ \ x_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.}$$

[12] **2c.** Find a matrix B such that $B^2 = A$.

There are four, different answers for B.
$B = S\Lambda^{1/2}S^{-1}$ where the columns of S contain the eigenvectors of A;
$S = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$.

For $\Lambda^{1/2} = \begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 5 & 0 \\ 0 & -1 \end{bmatrix}$, $\begin{bmatrix} -5 & 0 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} -5 & 0 \\ 0 & -1 \end{bmatrix}$ we obtain

$$\boxed{B = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix},\ B = \begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix},\ B = \begin{bmatrix} -2 & -3 \\ -3 & -2 \end{bmatrix},\ \text{or } B = \begin{bmatrix} -3 & -2 \\ -2 & -3 \end{bmatrix},}$$

respectively.

3. Let $A = \begin{bmatrix} 1 & 4 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}$.

[12] **3a.** Give a factorization $A = QR$ where R is an upper triangular matrix and Q is a matrix with orthonormal columns.

$A = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. `Note: Do not confuse this vector` A `with the matrix` A `above.`

$$B = \underbrace{\begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix}}_{b} - \left(\frac{A^T b}{A^T A}\right) \underbrace{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}_{A} = \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix} - \left(\frac{6}{3}\right) \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}.$$

`Normalize the vectors` A `and` B `to obtain` q_1 `and` q_2`, respectively.`
`Then, obtain` R `via` $R = Q^T A$.

$$\boxed{Q = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{6}} \end{bmatrix} \text{ and } R = \begin{bmatrix} \sqrt{3} & 2\sqrt{3} \\ 0 & \sqrt{6} \end{bmatrix}.}$$

[8] **3b.** Find the least squares solution to $Ax = b$, where $b = \begin{bmatrix} 4 \\ 8 \\ 6 \end{bmatrix}$.

$(A^T A)\bar{x} = A^T b$ `gives` $\begin{bmatrix} 3 & 6 \\ 6 & 18 \end{bmatrix} \bar{x} = \begin{bmatrix} 18 \\ 30 \end{bmatrix}$.

$$\boxed{\bar{x} = \begin{bmatrix} 8 \\ -1 \end{bmatrix}.}$$

[16] **3c.** The projection matrix $P = A\left(A^TA\right)^{-1}A^T$ projects all vectors onto the column space of A.

Find a vector b <u>not</u> in the column space of A, and which gives $Pb = \underbrace{\begin{bmatrix} 1 \\ 4 \\ 4 \end{bmatrix}}_{p}$.

The vector p is in the column space of A.
The error vector $e = b - p$ is orthogonal to the column space of A, which means e is in the left nullspace of A.

We obtain a basis for the left nullspace by finding the special solution(s) to $\underbrace{\begin{bmatrix} 1 & 1 & 1 \\ 4 & 1 & 1 \end{bmatrix}}_{A^T} y = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, which gives $y = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$.

Note that the question requires that b is <u>not</u> in the column space of A.

$$\boxed{b = \begin{bmatrix} 1 \\ 4 \\ 4 \end{bmatrix} + t \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \text{ for } t \neq 0.}$$

4. Suppose P is a rank one projection matrix and that the vector Px is always orthogonal to the vectors $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}$.

[8] **4a.** Give an orthonormal basis for the left nullspace of P.

`Vectors in the column space,` Px`, are always orthogonal to vectors in the left nullspace.`
`Use the Gram-Schmidt process to obtain an orthonormal basis using the linearly independent vectors in the left nullspace.`

$$A = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

$$B = \underbrace{\begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}}_{b} - \left(\frac{A^T b}{A^T A}\right) \underbrace{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}_{A} = \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix} - \left(\frac{2}{2}\right) \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}.$$

`Normalize the vectors` A `and` B `to obtain` q_1 `and` q_2`, respectively.`

$$\boxed{q_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad q_2 = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}.}$$

[12] **4b.** The projection matrix P is ______(give the 3 by 3 matrix)______.

$P = A\left(A^TA\right)^{-1}A^T$, where the columns of A are a basis for the column space of P.

Recall that the column space of P is orthogonal to its left nullspace.

By finding the special solution(s) to $\begin{bmatrix} 1 & 1 & 0 \\ 2 & 0 & 2 \end{bmatrix} x = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$,

we obtain a basis $A = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$ that is orthogonal to the left nullspace of P.

$$\boxed{P = A\left(A^TA\right)^{-1}A^T = \frac{1}{3}\begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 1 \\ -1 & 1 & 1 \end{bmatrix}.}$$