

## XXVIII. NEUROPHYSIOLOGY

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### 1. BASAL OPTIC PROJECTION IN THE FROG (Rana pipiens)

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We have investigated the physiology and anatomy of the basal optic projection (BOP) using single-unit extracellular recording and HRP injections.

With the animal on its back and using a ventral penetration through the upper mouth, we have been able to record routinely single units in the BOP which are driven primarily by the contralateral eye. We have found 3 classes of units: (1) those responsive to stimuli moving in a vertical direction, (2) those responsive to stimuli moving in a horizontal direction, and (3) those responsive to changes in ambient light but not to moving stimuli. All directional units have significant maintained activity. Vertical units increase firing to either slow upward movement or slow downward movement; stimulus motion in the opposite direction gives a reduced response. Horizontal units increase firing either to slow naso-temporal movement or to slow temporo-nasal movement. They, too, give reduced responses in the opposite direction. All these units yield broad tuning curves of response vs direction. A wide variety of sizes and shapes of stimuli elicit responses. The best response is obtained for most units when stimuli are moved with angular velocity in the range of  $0.1^\circ$  to  $1^\circ/\text{sec}$ . Those units which did not respond to

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moving stimuli had maintained activity that was greater in the dark than in the light.

For HRP injections, electrophysiological recordings were first made to ascertain location and depth of the BOP in each animal. Since the target region is very small, we injected HRP solution in volumes of 1/2 nl or less. We have followed efferent fibers from the BOP through the posterior commissure to the opposite side. We have also followed a set of stained fibers which course caudally through the ipsilateral ventral medulla. We have not been able to trace the terminal fields of either set of fibers. We have found ipsilateral efferent projections to the BOP from three principal areas: the posterolateral tegmental field, the posterior nucleus of the thalamus, and a wide extent of the ventral thalamus, particularly in the anterior part. HRP-filled cells in the retina are primarily found in the ganglion-cell layer; however, in keeping with Montgomery et al. (1979), some large displaced ganglion cells are also stained.

## 2. OLD AND NEW DIRECTIONS IN THE THEORY OF COLOR CONSTANCY

Bell Laboratories (Grant)

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It has recently been shown that any visual-processing algorithm aimed at attaining color constancy will, in fact, attain it only for restricted spectral classes of illuminant spectral power distributions (SPDs) and reflectances.<sup>1</sup> Because color constancy is attributed to chromatic adaptation, it is therefore appropriate to ask what physically reasonable spectral classes insure that an accepted model of chromatic adaptation — that of Von Kries — will restore object colors in human tristimulus space to illuminant-invariant positions in the space.

That Von Kries adaptation is responsible for color constancy was suggested by Judd.<sup>2</sup> In Judd's model, chromatic adaptation compensates for changes in illuminant SPD  $I(\lambda)$  by scaling each tristimulus function  $q_j(\lambda)$  in a canonical basis ( $j = 1,2,3$ ) so that a matte-white reflectance maps to a particular tristimulus vector (e.g., [1,1,1]). This is mathematically (although not mechanistically) equivalent to Land's Retinex ratio computations<sup>3</sup>

$$\frac{Q_j(r,I)}{Q_j(w,I)} = \frac{\int I(\lambda) r(\lambda) q_j(\lambda) d\lambda}{\int I(\lambda) w(\lambda) q_j(\lambda) d\lambda} \equiv \phi_j(r,I),$$

where  $r$  is a colored-object matte spectral reflectance,  $w$  is a reference matte-white reflectance, and  $Q_j(r,I), Q_j(w,I)$  are the tristimulus values for light  $I$  reflecting from  $r$  and  $w$ , respectively. Clearly, if  $r = w$ , the adapted tristimulus vector  $\underline{\phi}$  is  $(1,1,1)$ , and is independent of  $I(\lambda)$ . In summary and extension of our previous work<sup>7</sup> with G. West, we now present spectral conditions under which  $\underline{\phi}$  is illuminant-invariant even when  $r \neq w$ , following empirical observations of the invariance.<sup>4-6</sup> Related invariants are also presented, which may be useful in developing interdependent lighting and pigment standards for partially controlled viewing environments. Finally, the relation of the present formulation with other models of color constancy is discussed.

#### a. Conditions for Invariance of $\underline{\phi}$

The spectral sensitivities  $q_j(\lambda)$  of the eye's photoreceptors can be taken as the primaries of Judd,<sup>8</sup> Vos and Walraven,<sup>9</sup> or Brown and Wald.<sup>10</sup> Let the reference white have a nonselective reflectance spectrum  $w(\lambda) = 1$  over the visible wavelength range. Several natural pigments have approximately this property.<sup>11</sup>

Suppose the illuminant SPD is constrained to be a linear combination of  $N$  basis functions  $s_k(\lambda)$  with coefficients  $a_k$ :

$$I(\lambda) = \sum_{k=1}^N a_k s_k(\lambda), \quad (1)$$

where the  $a_k$  are constrained so that  $I(\lambda) \geq 0$  for all  $\lambda$ . The basis functions are derivable by a principal-components analysis on the SPDs of natural illuminants such as black-body spectra and Abbott-Gibson daylights.<sup>12</sup>

Let each spectral reflectance  $r(\lambda)$  be expressed as an expansion

$$r(\lambda) = \sum_{n=0}^{\infty} \rho_n r_n(\lambda), \quad (2)$$

where the coefficients  $\rho_n$  are characteristic of a particular reflectance, and the

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basis functions  $r_n(\lambda)$  are universal to all reflectances in the "illuminant-invariant" class. Let the reflectance-basis functions be constrained by the conditions:

$$r_0(\lambda) = 1 \quad (3a)$$

and

$$\int \xi_\ell(\lambda) r_k(\lambda) d\lambda = \delta_{k\ell} \quad (1 \leq k \leq \infty, 1 \leq \ell \leq 3N), \quad (3b)$$

where  $\xi_\ell(\lambda) = q_j(\lambda) s_i(\lambda)$  and  $\ell = (j-1)N + i$ .

Up to this point, we have not constrained  $r(\lambda)$ , but only represented it as an expansion in a complete set of functions. Now, we define  $\rho_\ell'' = \int \xi_\ell(\lambda) d\lambda \equiv \langle \xi_\ell \rangle$  for  $1 \leq \ell \leq 3N$ , and constrain the coefficients  $\rho_n$  ( $1 \leq n \leq 3N$ ) such that  $\rho_{(j-1)N+k} = \rho_{(j-1)N+1}' \rho_{(j-1)N+k}''$ . Then  $r(\lambda)$  can be written

$$\begin{aligned} r(\lambda) = & \rho_0 r_0(\lambda) + \rho_1' \sum_{k=1}^N \rho_k'' r_k(\lambda) \\ & + \rho_{N+1}' \sum_{k=N+1}^{2N} \rho_k'' r_k(\lambda) \\ & + \rho_{2N+1}' \sum_{k=2N+1}^{3N} \rho_k'' r_k(\lambda) + \sum_{k=3N+1}^{\infty} \rho_k r_k(\lambda). \end{aligned} \quad (4)$$

Here, the numbers  $\rho_0, \rho_1', \rho_{N+1}', \rho_{2N+1}', \rho_k (k > 3N)$  are the signature of the reflectance in question, constrained only by the condition  $0 \leq r(\lambda) \leq 1$  for all  $\lambda$ .

Applying all the conditions above, one can show that the Von-Kries-adapted tristimulus values for reflectance  $r$  under illuminant  $I$  are

$$\phi_j(r, I) = \rho_0 + \rho_{(j-1)N+1}' \quad (5)$$

This expression is independent of the illuminant coefficients  $a_k$ , and hence is manifestly illuminant-invariant. It also happens to be independent of all reflectance coefficients except  $\rho_0, \rho_1', \rho_{N+1}'$ , and  $\rho_{2N+1}'$ .

To find a set of reflectance-basis functions  $\{r_k\}$  satisfying Eq. 3 is

straightforward. Choose any complete set of independent functions  $\{p_m(\lambda)\}$ . Take as the first  $3N$  reflectance-basis functions linear combinations of the first  $3N$  of the functions  $p_m(\lambda)$  such that

$$\sum_{m=1}^{3N} b_{\ell m} \langle p_m(\lambda) \xi_j(\lambda) \rangle = \delta_{i\ell}, \quad (6)$$

where  $(b_{\ell m})$  is the  $3N \times 3N$  transformation matrix from  $\{p_m\}_1^{3N}$  to  $\{r_\ell\}_1^{3N}$ . Clearly,  $(b_{\ell m})$  is the inverse of the matrix  $(\langle p_m \xi_j \rangle)$ , and can be computed by standard methods if  $(\langle p_m \xi_j \rangle)$  is not singular. (This nonsingularity is a necessary constraint on  $\{p_m\}_1^{3N}$ , and in return requires the set  $\{\xi_i\}_1^{3N}$  to be linearly independent.) Then, for  $1 \leq k \leq 3N$ ,

$$r_k(\lambda) = \sum_{m=1}^{3N} b_{km} p_m(\lambda). \quad (7)$$

Once these basis functions are found, more  $\{r_k\}$  — the orthogonal complement of  $\{\xi_\ell\}$  — can be found by the Gram-Schmidt method, but may be unnecessary if the first  $3N$  functions  $\{p_m\}_1^{3N}$  are chosen as the first  $3N$  principal-component reflectance spectra for a large ensemble of natural pigments. (In fact, Cohen's analysis<sup>13</sup> on 433 Munsell pigments showed that they are adequately represented by only three principal components — about zero rather than about the mean spectrum as is usually done.)

#### b. Related Illuminant Invariants

Having established conditions for the illuminant invariance of Von-Kries-adapted tristimulus values, we now enumerate (in Table XXVIII-1) some related functions of tristimulus values  $Q_j(r,I)$ ,  $Q_j(w,I)$  that have come to our attention as illuminant invariants, together with spectral conditions sufficient for each invariance.

Within Table XXVIII-1,  $\phi_j$ ,  $\psi_j$ , and  $H$  form a hierarchy of invariants. When illuminant invariance of  $\phi_j$  fails, the adapted chromaticity coordinates  $\psi_j$  (functions of  $\phi_k$ ) may still be invariant. If this fails,  $H$  (a hue analogue and

Table XXVIII-1. Miscellaneous illuminant invariants.

Illuminant Invariant	Sufficient Spectral Conditions for Invariance		
	$r(\lambda)$	$I(\lambda)$	$q_j(\lambda)$
$h = \frac{Q_3(r, I) - Q_1(r, I)}{Q_2(r, I) - Q_1(r, I)}$	Eqs. 2, 3, and k-invariance of coefficients $\rho_k - \rho_{k+2N} + \rho_0 \left( \frac{\rho_k'' - \rho_{k+2N}''}{\rho_{k+N}'' - \rho_{k+2N}''} \right), \quad k = 1, \dots, N$	Eq. 1	any
$\phi_j(r, I) = \frac{Q_j(r, I)}{Q_j(w, I)}$	Eqs. 2, 3, and 4	Eq. 1	any
$\psi_j = \phi_j / \sum_{k=1}^3 \phi_k$	B (a constant)	b e <sup>-cλ</sup>	e <sup>-(λ-λ<sub>j</sub>)<sup>2</sup></sup>
$H = (\psi_3 - \psi_1) / (\psi_2 - \psi_1)$	B e <sup>-Cλ</sup>		
$S = (\lambda_2 - \lambda_3)R_1 + (\lambda_3 - \lambda_1)R_2 + (\lambda_1 - \lambda_2)R_3$ where $R_j = \ln \phi_j$	B e <sup>-Cλ</sup> + A		
$T = (R_2 - R_1)^2 - SG(\lambda)(R_2 + R_1)$ where $G(\lambda) = \frac{2(\lambda_2 - \lambda_1)^2}{[(\lambda_2 - \lambda_3)\lambda_1^2 + (\lambda_3 - \lambda_1)\lambda_2^2 + (\lambda_1 - \lambda_2)\lambda_3^2]}$	B e <sup>-[\lambda - \lambda_0 / \sigma]^2</sup>		
$\det(R), \text{ where } (R) \text{ has components } R_{jk} = \ln \phi_j(r_k, I) \quad (j, k=1, 2, 3)^*$			

\* In this last entry (column 1), the  $r_k$  are arbitrary nonwhite reflectances (not the basis functions of sections b and c).

function of  $\psi_j$ ) may still be invariant. The hierarchical invariance of  $\phi_j$ ,  $\psi_j$ , and  $H$  is illustrated in Table XXVIII-1 for exponential illuminant SPDs and equal-spread Gaussian tristimulus spectra. In this development, there is clearly a tradeoff between degree of invariance and number of invariants: Whereas a reflectance is represented by three numbers  $\phi_j$ , it is represented by only two numbers  $\psi_j$  and by only one number  $H$ .

Another aspect of classification of the invariants in Table XXVIII-1 is the number of tristimulus vectors required to compute each invariant. The entry  $h$  is the only one-point invariant (requiring only one object color in the visual field). All the others are two-point invariants (requiring one nonwhite reflectance and a matte white) except  $\det(R)$ , which is a four-point invariant (requiring three non-white reflectances and a matte white).  $\det(R)$  is the most difficult invariant to verify in Table XXVIII-1, but it was arrived at first.<sup>14</sup>

It may be noted that, subject to the given spectral conditions, the quantity  $S$  is proportional to  $1/(1 + \sigma^2)$  (where  $\sigma$  is the spread of a Gaussian reflectance spectrum), and hence  $S$  is an indicator of object-color saturation.  $T$  does not have such a straightforward interpretation, but is included for completeness: All functions of  $\phi_j$  that are two-point invariants under the given spectral assumptions are functions of  $S$  and  $T$ . Since  $S$  and  $T$  are functions of only one vector  $\underline{\phi}$ , they are actually one-point invariants in  $\underline{\phi}$ -space.

Since  $H$ ,  $S$ ,  $T$ , and  $\det(R)$  are all functions of  $\phi_j$ , they also partake of the invariance properties of  $\phi_j$  presented at the outset of this report. The new invariance depends on the tristimulus functions being equal-spread Gaussians — a tolerably good approximation that has been used for heuristic purposes previously.<sup>15</sup> Interpreting  $\lambda$  as a monotonic function of wavelength (as opposed to wavelength itself) may improve the approximation, and does not otherwise affect the formalism.

### c. Connection with Other Theories of Color Constancy

In the last ten years, several models of color constancy have emerged that incorporated explicit spectral constraints. Of these, the models of Weinberg<sup>16</sup> and Buchsbaum<sup>17</sup> proceed from a general methodology hereafter called the "method of inverses":

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Suppose the illuminant spectrum is a function of three parameters and  $\lambda$ , where the form of the function is known but the parameters are allowed to vary in the illuminant-invariant paradigm. Then

$$I(\lambda) = f(a,b,c;\lambda),$$

where the form of  $f$  is known a priori, and  $a,b,c$  are unknown.

Suppose the reflectance is likewise constrained:

$$R(\lambda) = g(A,B,C;\lambda),$$

where the form of  $g$  is known a priori, but  $A,B,C$  are unknown.

Let the tristimulus functions be  $q_j(\lambda)$  ( $j=1,2,3$ ), and let the eye be given two sets of tristimulus values, one set  $Q_{jw}$  for the reference reflectance (whose parameters  $A_0, B_0, C_0$  are known) and one set  $Q_j$  for the test reflectance ( $A,B,C$  are unknown). Both reflectances are under the same light (with parameters  $a,b,c$ ).

Then

$$Q_{jw} = \int d\lambda q_j(\lambda) f(a,b,c;\lambda) g(A_0, B_0, C_0; \lambda)$$

and

$$Q_j = \int d\lambda q_j(\lambda) f(a,b,c;\lambda) g(A,B,C;\lambda).$$

These six equations in the six unknowns  $a,b,c,A,B,C$  can, in principle, be solved at least in local regions of the parameter space, if  $f$  and  $g$  are sufficiently well behaved.

Buchsbaum uses the method of inverses with

$$I(\lambda) = \underline{a} \cdot \underline{S}(\lambda), \quad R(\lambda) = \underline{A} \cdot \underline{r}(\lambda).$$

Weinberg's theory uses

$$I(\lambda) = e^{\underline{a}} \cdot \underline{v}(\lambda), \quad R(\lambda) = e^{\underline{A}} \cdot \underline{v}(\lambda)$$

and, in particular, cites the example of Gaussians,  $\underline{v}(\lambda) = (1, \lambda, \lambda^2)$ . (Here  $\lambda$  is



a monotonic function of wavelength and  $\underline{a} = (a,b,c)$ , etc.)

Properties of Buchsbaum's and Weinberg's models differ in several ways:

(i) Buchsbaum's model allows additional degrees of freedom from illuminant and reflectance basis functions whose products are always metameric (the eye is blind to them). For such degrees of freedom, see Brill<sup>18</sup> and the end of Brill<sup>19</sup> (more realistically the latter, since human tristimulus functions are not trigonometric). Of course, Von Kries's model allows such degrees of freedom also, as is apparent from Brill and West<sup>7</sup> and section a of this report. On the other hand, Weinberg's theory does not allow additional degrees of freedom in  $I(\lambda), R(\lambda)$ : No reflectance can be a metameric black when it has to hide from a probing delta function in illuminant, and delta functions at all visible wavelengths are members of the class of spectra assumed by Weinberg's eye ("protomers," to use his terminology).

(ii) Weinberg's model provides a structure within which illuminant-invariant computations can proceed given two unknown reflectances, not only when a known and an unknown reflectance are given. To see this, note that the tristimulus values for a reflected light  $k$  ( $k = 1,2$ ) are

$$Q_{jk} = \int d\lambda q_j(\lambda) \exp[(\underline{a} + \underline{A}_k) \cdot \underline{v}(\lambda)].$$

To compute three illuminant-invariant quantities from two reflectances  $k = 1,2$ , solve separately the triads of equations in the unknowns  $\underline{a} + \underline{A}_k$  (in general, a very difficult task) and then subtract the solutions vectorially. The result,  $\underline{A}_2 - \underline{A}_1$ , is illuminant-invariant.

In the above process, six equations suffice to solve for three of the nine unknowns  $\underline{a}, \underline{A}_1, \underline{A}_2$ . This is possible in Weinberg's theory because the operation of illuminant on reflectance (more appropriately conceived as the operation of reflectance on illuminant because most generally the reflectance acts as the kernel in the linear operation — e.g., for fluorescent materials) is a representation of the translation group in three dimensions. To import illuminant-invariant computations between unknown reflectances to Buchsbaum's theory, one must use the tristimulus volume ratio as an illuminant-invariant quantity. That introduces the requirement of at least four reflectances in the visual field, but totally

unconstrains the illuminant (except in singular cases).<sup>19</sup>

It should be noted that, subject to the enumerated spectral constraints, all the entries except H in Table XXVIII-1 are illuminant-invariant even when the reflectance  $w$  is not a known reference, whether or not the action of illuminant on reflectance has a group-theoretic interpretation.

The pattern-recognition use of invariants in the absence of a "ground-truth" reference is at present problematic, and will be the subject of future study.

#### References

1. G. West, "Color Perception and the Limits of Color Constancy," *J. Math. Biol.* 8, 47-53 (1979).
2. D.B. Judd, "Hue, Saturation, and Lightness of Surface Colors with Chromatic Illumination," *J. Opt. Soc. Am.* 30, 2-32 (1940).
3. E.H. Land and J.J. McCann, "Lightness and Retinex Theory," *J. Opt. Soc. Am.* 61, 1-11 (1971).
4. J.J. McCann, S.P. McKee, and T.H. Taylor, "Quantitative Studies in Retinex Theory," *Vision Res.* 16, 445-458 (1976).
5. R. Corth, "The Basis for a New System of Colorimetry," *J. Illum. Engineering Soc.*, 155-161 (April 1979).
6. M.H. Brill, "Computer Simulations of Object Color Recognizers," *J. Opt. Soc. Am.* 69, 1405A (1979).
7. M.H. Brill and G. West, "Contributions to the Theory of Invariance of Color under the Condition of Varying Illumination," *J. Math. Biol.* (in press).
8. D.B. Judd, "Standard Response Functions for Protanopic and Deuteranopic Vision," *J. Opt. Soc. Am.* 35, 199-221 (1945).
9. J.J. Vos and P.L. Walraven, "On the Derivation of the Foveal Receptor Primaries," *Vision Res.* 11, 799-818 (1971).
10. P.K. Brown and G. Wald, "Visual Pigments in Single Rods and Cones of the Human Retina," *Science* 144, 45-52 (1964).
11. G. Wyszecki and W.S. Stiles, Color Science (John Wiley and Sons, New York, 1967).
12. D.B. Judd, D.L. MacAdam, and G. Wyszecki, "Spectral Distribution of Typical Daylight as a Function of Correlated Color Temperature," *J. Opt. Soc. Am.* 54, 1031-1040, 1382 (1964).
13. J. Cohen, "Dependency of the Spectral Reflectance Curves of the Munsell Color Chips," *Psychonomic Science* 1, 369-370 (1964).
14. M.H. Brill, "Illuminant Invariants and a Plausible Model of Human Vision" (unpublished, 1974).

15. W. Thornton, "A Simple Picture of Matching Lights," J. Illum. Engineering Soc., 78-85 (January 1979).
16. J. Weinberg, "The Geometry of Colors," General Relativity and Gravitation 7, 135-169 (1976).
17. G. Buchsbaum, "A Spatial Processor Model for Object Colour Perception," J. Franklin Inst. 310, 1-26 (1980).
18. M.H. Brill, "A Device Performing Illuminant-Invariant Assessment of Chromatic Relations," J. Theor. Biol. 71, 473-478 (1978).
19. M.H. Brill, "Further Features of the Illuminant-Invariant Trichromatic Photosensor," J. Theor. Biol. 78, 305-308 (1979).

3. TRISTIMULUS SPACES IN WHICH SCHROEDINGER OBJECT COLORS  
ARE NOT OPTIMAL

Bell Laboratories (Grant)

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Schroedinger<sup>1</sup> and later MacAdam<sup>2</sup> deduced — from barycentric arguments for lights in tristimulus space — the spectral reflectances constituting the envelope (optimal colors) of the object-color solid in tristimulus space.

A reflectance spectrum on the surface of optimal colors has, at every visible wavelength, a value of 1 or 0. Furthermore, the spectrum undergoes, at most, two transitions between 1 and 0 over the visible wavelength range.

In human tristimulus space, the transition wavelengths generate two twofolds, one parametrizing the pass-band colors and the other the stop-band colors. The intersection of these twofolds consists of the "end colors," each of which has only one spectral transition and therefore can be interpreted as either pass- or stop-band. Except for this ambiguity, the Schroedinger representation of optimal colors is unique. Also, Schroedinger's condition of optimality is illuminant-invariant (although the locus of the optimal colors in tristimulus space is not).

The purpose of this note, inspired by a didactic paper of G. West,<sup>3</sup> is to point out that Schroedinger's theorem depends on three accidents of human tristimulus space:

- a. The spectrum-locus twofold is a single-valued mapping from intensity

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and wavelength of the monochromatic lights.

b. The spectrum locus is simply connected.

c. The spectrum locus is convex.

Hypothetical examples of violations of these conditions readily show failure of the Schroedinger colors to be the envelope of the object-color solid. We enumerate three examples below (with visible wavelength range  $\lambda \in [0, 2\pi]$ ):

a. Let the tristimulus functions be  $a(\lambda) = 1$ ,  $b(\lambda) = \sin(n\lambda)$ , and  $c(\lambda) = \cos(n\lambda)$  (with  $n$  a positive integer). The resulting tristimulus space maps  $n$  distinct monochromatic lights into each (nonorigin) point of the spectrum locus. The surface of Schroedinger colors,  $n$  football-shaped figures end-to-end along the  $a$  axis, is clearly not convex, and hence does not circumscribe the object-color solid.

b. Let the tristimulus functions be

$$\left. \begin{aligned} a(\lambda) &= 1 \\ b(\lambda) &= (n-m+1) \sin(n\lambda) \\ c(\lambda) &= (n-m+1) \cos(n\lambda) \end{aligned} \right\} \text{ for } \begin{aligned} &2\pi(m-1) < n\lambda < 2\pi m \text{ and } m \text{ an} \\ &\text{integer with } 0 < m \leq n \end{aligned}$$

The spectrum locus for this space consists of  $n$  concentric cones and is therefore not simply connected. The Schroedinger-color surface consists of  $n$  sequentially thinning footballs end-to-end along the  $a$  axis, a nonconvex locus that cannot circumscribe the object-color solid. Note that, in chromaticity coordinates, the projection of this locus is not contained within all branches of the spectrum locus.

c. Let the tristimulus functions be  $a(\lambda) = 1$ ,  $b(\lambda) = f(\lambda) \sin \lambda$ , and  $c(\lambda) = f(\lambda) \cos \lambda$ , where  $f(\lambda) = 1$  for all  $\lambda$  outside a narrow region about some  $\lambda_0$ , and  $f(\lambda)$  is a small positive fraction of 1 near  $\lambda_0$ . The spectrum locus for this tristimulus space is not convex; there is a notch in the spectrum locus at  $\lambda_0$ , and also in the surface of Schroedinger colors. This latter notch renders the Schroedinger-color surface nonconvex, hence the Schroedinger surface does not circumscribe the object-color solid.

Although these pathologies have not been found in the visual domain, they might be encountered in generalized colorimetry such as is performed by Richards<sup>4</sup>

for other sensory modalities.

Studies uncovering analytic constraints on human tristimulus functions may, on the other hand, help us to better understand color-matching phenomena such as Thornton's modal wavelengths<sup>5</sup> (450, 540, and 610 nm). At these wavelengths, monochromatic light of unit power requires maximum mixed power of the complement to produce white; also, many zero crossings for metameric-black spectra are near these wavelengths.

#### References

1. E. Schroedinger, "Theorie der Pigmente von grösster Leuchtkraft," *Annalen der Physik* 62, 603-622 (1920).
2. D. MacAdam, "The Theory of the Maximum Visual Efficiency of Colored Materials," *J. Opt. Soc. Am.* 25, 249-252 (1935).
3. G. West, "Einfache Modellrechnungen zur Niederen Farbmatrik des Trichomatischen Farbsehens," 1980, unpublished.
4. W. Richards, "Quantifying Sensory Channels: Generalizing Colorimetry to Orientation and Texture, Touch and Tones," to appear in Sensory Processes (in press).
5. W.A. Thornton, "A Simple Picture of Matching Lights," *J. Illum. Engineering Soc.*, 78-85 (January 1979).

