Prof. H. B. Lee
Prof. W. C. Schwab
J. Anderson
V. K. Prabhu
R. S. Smith

A. BOUNDS ON THE NATURAL FREQUENCIES OF LC STRUCTURES

The purpose of this report is to prove the following theorem.

THEOREM: The smallest (nonzero) natural frequency that one can realize from a set of positive capacitors $C_1, \dots C_m$ and a set of positive inductors $L_1, \dots L_n$ results when one

1. connects the capacitors in parallel to produce a capacitance $C_p = C_1 \dots + C_m$;
2. connects the inductors in a series to produce an inductance $L_s = L_1 \dots + L_n$; and
3. connects C_p and L_s in parallel.

Similarly, the largest (finite) natural frequency that one can realize results when one

1. connects the capacitors in a series to produce a capacitance $C_s = \left(C_1^{-1} \dots + C_m^{-1}\right)^{-1}$;
2. connects the inductors in parallel to produce an inductance $L_p = \left(L_1^{-1} \dots + L_n^{-1}\right)^{-1}$;
3. connects C_s and L_p in parallel.

The theorem is useful as it provides the following bounds on the natural frequencies of any transformerless LC structure

$$\frac{1}{\sqrt{L_pC_s}} \geq \omega_\nu \geq \frac{1}{\sqrt{L_sC_p}}. \tag{1}$$

The proof rests upon the fact that every natural oscillation is self-exciting.

PROOF: Let N_{LC} be any transformerless LC network constructed from $C_1, \dots C_m$ and $L_1, \dots L_n$. Assume that N_{LC} executes a natural oscillation at the frequency $s = j\omega_\nu$, and let the (complex) capacitor voltages of N_{LC} be designated respectively as the e_{Ck}.

The currents in the capacitors of N_{LC} are given, respectively, by the quantities $j\omega_\nu C_k e_{Ck}$. If the capacitors of N_{LC} are replaced by current sources which deliver the capacitor currents $j\omega_\nu C_k e_{Ck}$, then the network behavior remains unchanged. Thus let this be done and let the resulting netowrk be designated as N_{LI}.

The inductor currents and voltages of N_{LC} may now be determined by analyzing the current-driven inductor network N_{LI}. If $A_{\ell k}$ denotes the current transfer ratio from the k^{th} current source to the ℓ^{th} inductor of N_{LI}, then the ℓ^{th} inductor current is given by

$$i_{L\ell} = \sum_k j\omega_\nu C_k e_{Ck} A_{\ell k} \tag{2}$$

where the summation extends over any independent set of the current sources of N_{LI}. The corresponding branch voltages are given by the expressions

$$e_{L\ell} = j\omega_\nu L_\ell i_{L\ell}$$

$$= -\omega_\nu^2 L_\ell \sum_k C_k e_{Ck} A_{\ell k}. \tag{3}$$

If next the current sources of N_{LI} are replaced by the original capacitors and the inductors of N_{LI} are replaced by voltage sources supplying the inductor voltages $e_{L\ell}$, then the network behavior once again remains unchanged. Thus let this be done, and let the resulting network be designated as N_{VC}.

The capacitive voltages e_{Cj} now can be calculated by analyzing the voltage driven capacitor network N_{VC}. If $V_{j\ell}$ denotes the voltage transfer ratio from the ℓ^{th} voltage source of N_{VC} to the j^{th} capacitor, then the j^{th} capacitor voltage is given by

$$e_{Cj} = \sum_\ell V_{j\ell} e_{L\ell} \tag{4}$$

where the summation extends over any independent set of voltage sources of N_{VC}. Substitution of (3) into (4) yields the self-excitation condition

$$e_{Cj} = \sum_k \sum_\ell -\omega_\nu^2 L_\ell C_k A_{\ell k} V_{j\ell} e_{Ck}. \tag{5}$$

The following bound can be inferred for the left-hand member of (5)

$$|e_{Cj}| \leq \omega_\nu^2 \sum_k \sum_\ell L_\ell C_k |A_{\ell k}| |V_{j\ell}| |e_{Ck}|. \tag{6}$$

Because the purely inductive network N_{LI} cannot exhibit current gain and the purely capacitive network N_{VC} cannot exhibit voltage gain,

$$|A_{\ell k}| \leq 1 \qquad \text{and} \qquad |V_{j\ell}| \leq 1 \tag{7}$$

Use of inequalities (7) in (6) yields

$$|e_{Cj}| \leq \omega_\nu^2 \sum_k \sum_\ell L_\ell C_k |e_{Ck}|$$

$$\leq \omega_\nu^2 \max\left[|e_{Ck}|\right] \sum_k \sum_\ell L_\ell C_k$$

$$\leq \omega_\nu^2 \max\left[|e_{Ck}|\right] L_s C_p. \tag{8}$$

If the index j in (8) is chosen so as to maximize $|e_{Cj}|$, there results

$$\max_j \left[\left| e_{Cj} \right| \right] \leqslant \omega_\nu^2 \max_k \left[\left| e_{Ck} \right| \right] L_s C_p$$

or, equivalently,

$$1 \leqslant \omega_\nu^2 L_s C_p \tag{9}$$

From (9) it is evident that

$$\omega_\nu \geqslant \frac{1}{\sqrt{L_s C_p}}. \tag{10}$$

Equation 10 proves the first assertion of the theorem, since ω_ν can represent any natural frequency of N_{LC} and N_{LC} can be any network constructed from the given components.

The second assertion of the theorem can be deduced from the first assertion (now proved) by frequency transformation. Thus, let N'_{LC} denote the network derived from N_{LC} by the frequency transformation $s \to \frac{1}{s}$. The following relationships hold between the parameters of N'_{LC} and N_{LC} (primed quantities refer to N'_{LC}):

$$\omega'_\nu = \frac{1}{\omega_\nu}$$

$$L'_s = C_s^{-1}$$

$$C'_p = L_p^{-1}. \tag{11}$$

Application of (10) to N'_{LC} yields

$$\omega'_\nu \geqslant \frac{1}{\sqrt{L'_s C'_p}}. \tag{12}$$

Use of (11) in (12) yields

$$\frac{1}{\omega_\nu} \geqslant \frac{1}{\sqrt{C_s^{-1} L_p^{-1}}} \tag{13}$$

or, equivalently,

$$\frac{1}{\sqrt{L_p C_s}} \geqslant \omega_\nu. \tag{14}$$

The second assertion of the theorem follows from (14) just as the first assertion followed from (10).

H. B. Lee

B. BOUNDS ON IMPEDANCE FUNCTIONS OF R, ±L, ±C, T NETWORKS

1. Introduction

It is well known that for a one-port network N containing positive resistances, ideal transformers, and one reactive element, the locus of the driving-point

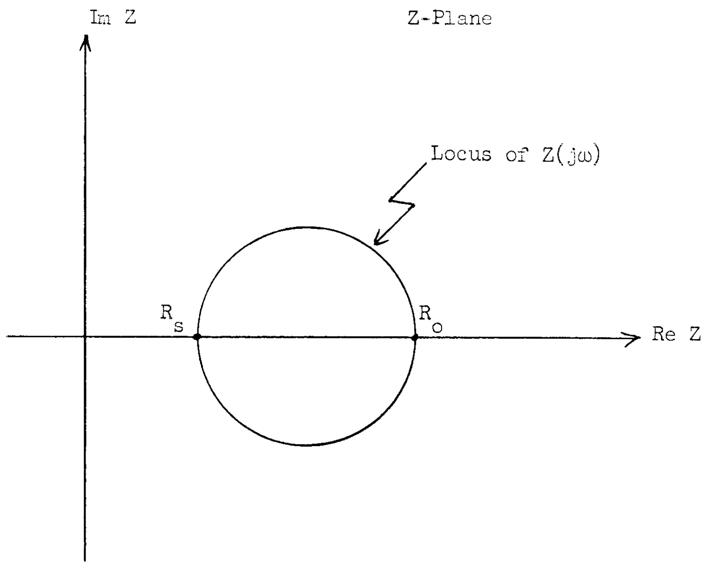


Fig. XVI-1.

impedance Z(jω), as ω varies from −∞ to +∞, is a circle in the complex Z-plane (see Fig. XVI-1). The equation of the circle is

$$\left| Z - \frac{R_o + R_s}{2} \right| = \frac{R_o - R_s}{2},$$

where R_o is the driving-point impedance of N when the reactive element is open circuited, and R_s is the driving-point impedance of N when the reactive element is short circuited.

In this report we prove two theorems which can be considered as generalizations of the above-mentioned result. The theorems are as follows:

THEOREM 1: Let $Z_{ii}(s)$ be a driving-point impedance of an R, ±L, ±C, T two-port network N (that is, a network containing positive resistances, positive and negative inductances, positive and negative capacitances, and ideal transformers). Then, as ω varies from −∞ to +∞, the locus of $Z_{ii}(j\omega)$ lies within the closed circular disk of the Z-plane defined by (see Fig. XVI-2)

$$\left| Z_{ii}(j\omega) - \frac{R_{iio} + R_{iis}}{2} \right| \leq \frac{R_{iio} - R_{iis}}{2}, \tag{1}$$

where R_{iio} is the driving-point impedance Z_{ii} of N when all reactive elements are open circuited, and R_{iis} is the driving-point impedance Z_{ii} of N when all reactive elements are short circuited.

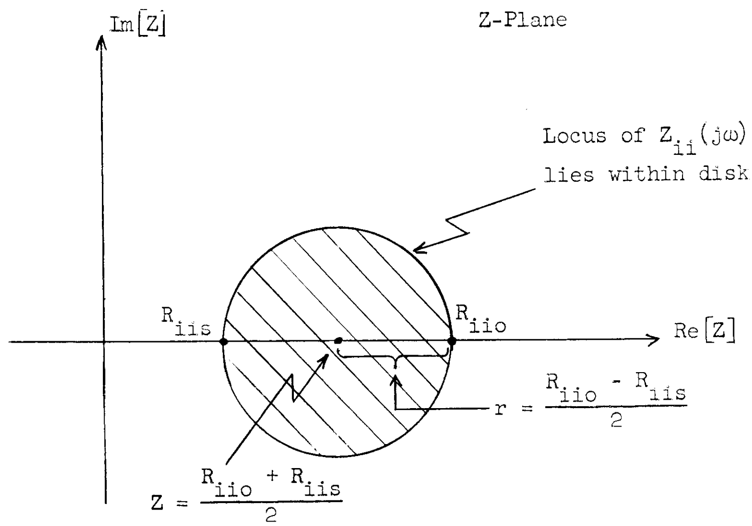


Fig. XVI-2.

THEOREM 2: Let $Z_{12}(j\omega)$ be the transfer impedance of any R, ±L, ±C, T two-port network N. As ω varies from $-\infty$ to $+\infty$, the locus of $Z_{12}(j\omega)$ lies within the closed

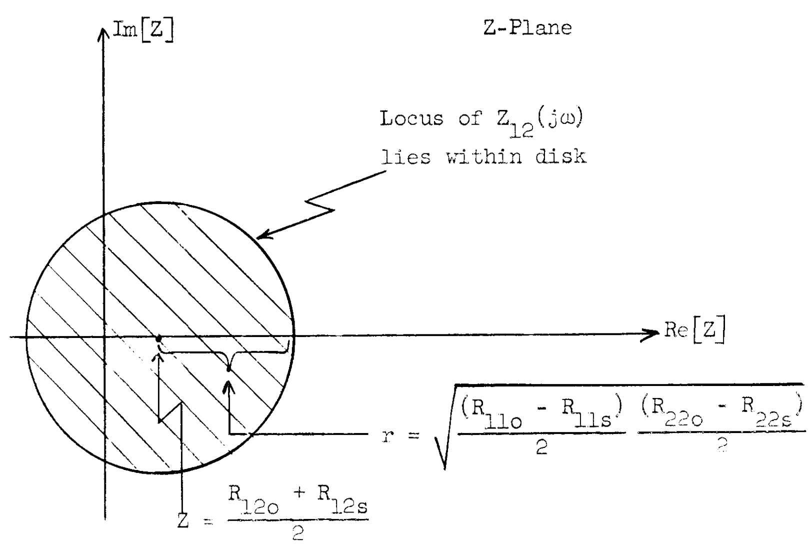


Fig. XVI-3.

circular disk of the Z-plane defined by (see Fig. XVI-3)

$$\left| Z_{12}(j\omega) - \frac{R_{12o} + R_{12s}}{2} \right| \leqslant \sqrt{\frac{(R_{11o}-R_{11s})}{2} \frac{(R_{22o}-R_{22s})}{2}},$$

where R_{12o} is the transfer impedance Z_{12} of N when all reactive elements are open circuited, and R_{12s} is the transfer impedance Z_{12} of N when all reactive elements are short circuited.

The foregoing two theorems can conveniently be summarized in the following single theorem.

THEOREM 3: Let $Z_{ij}(s)$ be any open-circuit impedance of an R, ±L, ±C, T two-port network. As ω varies from $-\infty$ to $+\infty$, the locus of $Z_{ij}(j\omega)$ remains within the closed circular disk of the Z-plane defined by

$$\left| Z_{ij}(j\omega) - \frac{R_{ijo} + R_{ijs}}{2} \right| \leqslant \sqrt{\frac{(R_{iio}-R_{iis})}{2} \frac{(R_{jjo}-R_{jjs})}{2}}. \tag{2}$$

The quantities R_{ijo} and R_{ijs} which appear in the preceding theorems are easy to calculate, because they are the impedances of resistance networks. Thus the theorems provide a simple means for bounding the magnitude, the phase angle, and the real and imaginary parts of $Z_{ij}(j\omega)$.

2. Proof of Theorem 1

We begin by considering three lemmas.

LEMMA 1: If a ±R, T network N (that is, a network containing positive and negative resistances and ideal transformers) is simultaneously excited by complex current sources I_o, I_1, ..., I_m, and complex voltage sources E_1, ..., E_n, then the total complex power supplied to N can be expressed as follows:

$$P = P_I + P_E,$$

where P_I equals the complex power supplied by the current sources acting together, with the voltage sources set to zero, and P_E equals the complex power supplied by the voltage sources acting together, with the current sources set to zero.

The reader is referred to Guillemin[1] for a proof of Lemma 1. Guillemin intends his proof to apply to the case of identical time-varying sources and instantaneous power. After some obvious modifications, however, his proof applies equally well to the case described above.

LEMMA 2: When an R, T network N is excited by complex current sources I_o, I_1, ..., I_m, the complex power P supplied to the network is such that

$$\text{Re}\,[P] \geqslant R_s |I_o|^2,$$

where R_s denotes the driving-point impedance seen by the source I_o when all other sources are short-circuited.

PROOF: Let N be excited by the current sources I_o, I_1, ..., I_m, and let V_o, V_I, ..., V_m denote, respectively, the voltages developed across these sources. Replace each current source I_j (j=1, 2, ..., m) by a voltage source of value V_j. Observe that this substitution does not affect the network behavior and, in particular, does not affect the complex power supplied to N.

Application of Lemma 1 to the network thus obtained shows that the complex power can be calculated as follows:

$$P = P_I + P_E,$$

where P_I is the complex power supplied to N by I_o with the V_j (j=1, 2, ..., m) set to zero, and P_E is the complex power supplied to N by the V_j (j=1, ..., m) acting together, with I_o set to zero. It follows that

$$\text{Re}\,[P] = \text{Re}\,[P_I] + \text{Re}\,[P_E].$$

But

$$\text{Re}\,[P_I] = |I_o|^2 R_s \qquad \text{and} \qquad \text{Re}\,[P_E] \geq 0.$$

Thus

$$\text{Re}\,[P] \geq |I_o|^2 R_s.$$

Q. E. D.

LEMMA 3: The real part of the complex power supplied by the current sources of Fig. XVI-4a is non-negative (the resistance R_s is defined as shown in Fig. XVI-4b).

PROOF: Let I_o denote the current flowing through $-R_s$. The real part of the complex power P supplied by the sources is

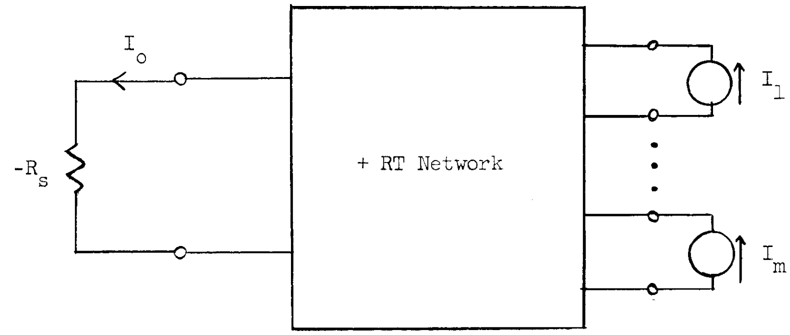
$$\text{Re}\,[P] = \text{Re}\left[P_{-R_s}\right] + \text{Re}\,[P_{box}],$$

where P_{-R_s} denotes the complex power supplied to $-R_s$, and P_{box} denotes that supplied to the box. It follows that

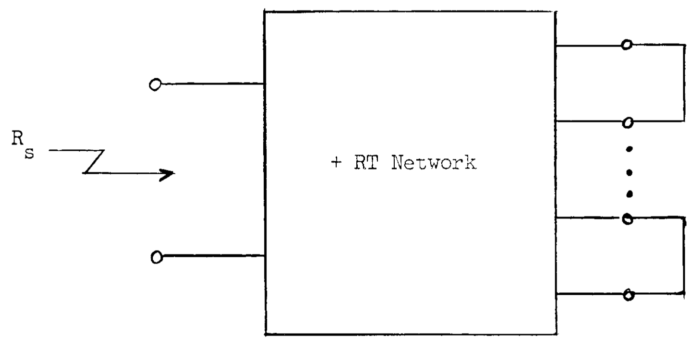
$$\text{Re}\left[P_{-R_s}\right] = -|I_o|^2 R_s.$$

Moreover, Lemma 2 assures that

$$\text{Re}\,[P_{box}] \geq |I_o|^2 R_s,$$



(a)



(b)

Fig. XVI-4.

if it is observed that $-R_s$ can be replaced by a current source of value I_o for the purpose of computing P_{box}. Thus

$$\text{Re}\,[P] \geqslant 0. \qquad \text{Q.E.D.}$$

Theorem 1 now can be proved as follows:

PROOF OF THEOREM 1: Let N be any R, ±L, ±C, T two-port network and let $Z_{ii}(s)$ be one of N's driving-point impedances. Let R_{iio} and R_{iis} be the resistances defined in Theorem 1. Finally, let a resistance $-R_{iis}$ be placed in series at port i of N to create a one-port N' which has the impedance $Z'(s) = Z_{ii}(s) - R_{iis}$.

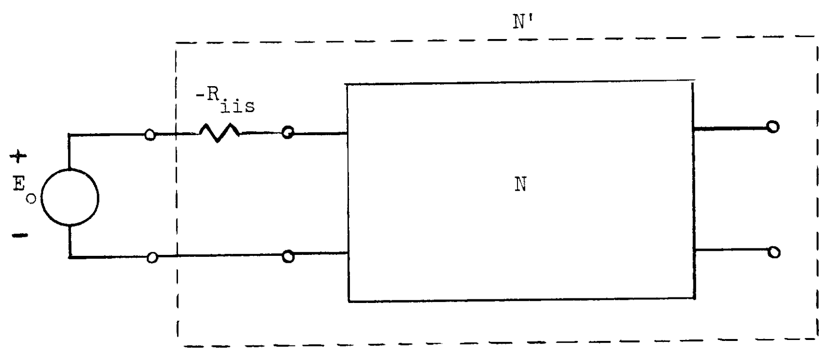
Consider the admittance of N'

$$Y'(s) = \frac{1}{Z_{ii}(s) - R_{iis}}.$$

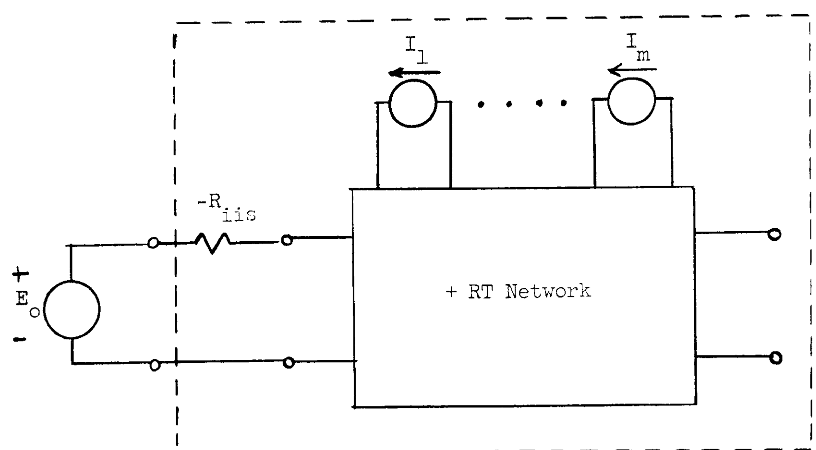
The real part of $Y'(j\omega)$ is given by

$$\mathrm{Re}\,[Y'(j\omega)] = \frac{\mathrm{Re}\,[P]}{|E_o|^2}, \tag{3}$$

where P denotes the complex power supplied by the voltage source E_o in the experiment shown in Fig. XVI-5a. For the purpose of calculating Re [P] the reactive elements of N'



(a)



(b)

Fig. XVI-5.

can be replaced by current sources which carry the reactive currents. The network thus obtained is indicated in Fig. XVI-5b. According to Lemma 1

$$\text{Re}\,[P] = \text{Re}\,[P_E] + \text{Re}\,[P_I].$$

When the current sources in Fig. XVI-5b are set to zero, the source E_o sees the impedance $R_{iio} - R_{iis} > 0$. Thus

$$\text{Re}\,[P_E] = \frac{|E_o|^2}{R_{iio} - R_{iis}}.$$

Lemma 3 ensures that

$$\text{Re}\,[P_I] \geq 0.$$

It follows that

$$\text{Re}\,[P] \geq \frac{|E_o|^2}{R_{iio} - R_{iis}}. \tag{4}$$

Substitution of (4) into (3) yields

$$\text{Re}\,[Y'(j\omega)] \geq \frac{1}{R_{iio} - R_{iis}}.$$

The foregoing inequality shows that the locus of $Y'(j\omega)$ lies within the closed half of the Y'-plane defined by $\text{Re}\,[Y'] \geq \frac{1}{R_{iio} - R_{iis}}$. It follows that the locus of the reciprocal function $Z'(j\omega)$ lies within the closed circular disk of the Z'-plane defined by

$$\left| Z' - \frac{R_{iio} - R_{iis}}{2} \right| \leq \frac{R_{iio} - R_{iis}}{2}.$$

But $Z_{ii}(j\omega) = R_{iis} + Z'(j\omega)$. Therefore the locus of $Z_{ii}(j\omega)$ lies within the closed circular disk of the Z-plane defined by (1). Q.E.D.

3. Proof of Theorem 2

It is well known that the quadratic form

$$Z(s) = x_1^2 Z_{11}(s) + 2x_1 x_2 Z_{12}(s) + x_2^2 Z_{22}(s) \tag{5}$$

of the impedance matrix of any R, ±L, ±C, T two-port network can be interpreted as the driving-point impedance of a related R, ±L, ±C, T one-port network.[2] Thus Theorem 1 can be applied to the quadratic form (5). This observation underlies the following proof of Theorem 2.

PROOF: Consider the quadratic form (5) for the network N. Application of Theorem 1 to (5) shows that

$$\left|x_1^2U_{11}+2x_1x_2U_{12}+x_2^2U_{22}\right| \leq x_1^2V_{11} + 2x_1x_2V_{12} + x_2^2V_{22}, \tag{6}$$

where

$$U_{ij} = Z_{ij}(j\omega) - \frac{R_{ijo} + R_{ijs}}{2} \qquad \text{and} \qquad V_{ij} = \frac{R_{ijo} - R_{ijs}}{2} \qquad (i, j=1, 2).$$

Substitution of $-x_1$ for x_1 in (5) yields the companion inequality

$$\left|x_1^2U_{11}-2x_1x_2U_{12}+x_2^2U_{22}\right| \leq x_1^2V_{11} - 2x_1x_2V_{12} + x_2^2V_{22}. \tag{7}$$

Addition of (6) and (7) leads to

$$\left\{\begin{array}{l} \left|x_1^2U_{11}+2x_1x_2U_{12}+x_2^2U_{22}\right| \\ +\left|x_1^2U_{11}-2x_1x_2U_{12}+x_2^2U_{22}\right| \end{array}\right\} \leq 2x_1^2V_{11} + 2x_2^2V_{22}.$$

Use of the triangle inequality $|A-B| \leq |A| + |B|$ in the left-hand member shows that

$$4|x_1||x_2||U_{12}| \leq 2x_1^2V_{11} + 2x_2^2V_{22}.$$

This expression implies that

$$4x_1x_2|U_{12}| \leq 2x_1^2V_{11} + 2x_2^2V_{22},$$

or equivalently

$$0 \leq x_1^2V_{11} - 2x_1x_2|U_{12}| + x_2^2V_{22}. \tag{8}$$

Because (8) holds for all real values of x_1 and x_2, the quadratic form

$$F(x_1, x_2) = x_1^2V_{11} - 2x_1x_2|U_{12}| + x_2^2V_{22}$$

is positive semidefinite, and the following relationships obtain:

$$V_{11} \geq 0 \tag{9a}$$

$$V_{22} \geq 0 \tag{9b}$$

$$V_{11}V_{22} \geq |U_{12}|^2. \tag{9c}$$

Inequality 9c shows that

$$|U_{12}| \leq \sqrt{V_{11}V_{12}},$$

or equivalently

$$\left| Z_{12}(j\omega) - \frac{R_{12o} + R_{12s}}{2} \right| \leq \sqrt{\frac{(R_{11o}-R_{11s})}{2} \frac{(R_{22o}-R_{22s})}{2}}.$$

Q. E. D.

It is interesting to note that the radius of the bounding disk for $Z_{12}(j\omega)$ is the geometric mean of the radii of the bounding disks for $Z_{11}(j\omega)$ and $Z_{22}(j\omega)$. This means that the bounding disk for $Z_{12}(j\omega)$ is smaller than one of the bounding disks for $Z_{11}(j\omega)$ and $Z_{22}(j\omega)$, and larger than the other.

4. Corollaries of Theorem 1

We next list some useful corollaries of Theorem 1. Unless otherwise specified, the corollaries follow directly from Fig. XVI-2.

COROLLARY 1:

$$R_{iis} \leq \text{Re}\,[Z_{ii}(j\omega)] \leq R_{iio} \qquad \text{for } -\infty < \omega < \infty.$$

COROLLARY 2:

$$|\text{Im}\,[Z_{ii}(j\omega)]| \leq \frac{R_{iio} - R_{iis}}{2} \qquad \text{for } -\infty < \omega < \infty.$$

COROLLARY 3:

$$R_{iis} \leq |Z_{ii}(j\omega)| \leq R_{iio} \qquad \text{for } -\infty < \omega < \infty.$$

COROLLARY 4:

$$|\angle Z_{ii}(j\omega)| \leq \sin^{-1} \frac{R_{iio} - R_{iis}}{R_{iio} + R_{iis}} \qquad \text{for } -\infty < \omega < \infty.$$

COROLLARY 5: Let $Z_{ii}(s)$ be an RCT or RLT driving-point impedance. As ω varies from $-\infty$ to $+\infty$, the locus of $Z_{ii}(j\omega)$ lies within the closed circular disk of the Z-plane defined by

$$\left| Z - \frac{Z_{ii}(0) + Z_{ii}(\infty)}{2} \right| \leqslant \left| \frac{Z_{ii}(0) - Z_{ii}(\infty)}{2} \right|.$$

PROOF: Corollary 5 follows from Theorem 1 by observing that for an RCT network $R_{iio} = Z_{ii}(0)$ and $R_{iis} = Z_{ii}(\infty)$; and for an RLT network $R_{iio} = Z_{ii}(\infty)$ and $R_{iis} = Z_{ii}(0)$.

5. Corollaries of Theorem 2

In completely analogous fashion we list the following corollaries of Theorem 2 (see Fig. XVI-3). Each of these corollaries employs the shorthand

$$c = \frac{1}{2}(R_{12o}+R_{12s})$$

and

$$r = \sqrt{\frac{(R_{11o}-R_{12s})}{2}\,\frac{(R_{22o}-R_{22s})}{2}}.$$

COROLLARY 1:

$$c - r \leqslant \mathrm{Re}\,[Z_{12}(j\omega)] \leqslant c + r \qquad \text{for } -\infty < \omega < \infty$$

COROLLARY 2:

$$|\mathrm{Im}[Z_{12}(j\omega)]| \leqslant r \qquad \text{for } -\infty < \omega < \infty$$

COROLLARY 3:

$$||c|-r| \leqslant |Z_{12}(j\omega)| \leqslant |c| + r \qquad \text{for } -\infty < \omega < \infty$$

COROLLARY 4:

$$|\angle Z_{12}(j\omega)| \begin{cases} \leqslant \sin^{-1}\frac{r}{c} & \text{if } r < c \\ \geqslant \sin^{-1}\frac{r}{c} & \text{if } r < -c \end{cases} \qquad \text{for } -\infty < \omega < \infty$$

COROLLARY 5: Let $Z_{12}(s)$ be the transfer impedance of an RCT or RLT two-port network. As ω varies from $-\infty$ to $+\infty$, the locus of $Z_{12}(j\omega)$ remains within the closed circular disk of the Z-plane defined by

$$\left| Z - \frac{Z_{12}(0) + Z_{12}(\infty)}{2} \right| \leqslant \sqrt{\frac{Z_{11}(0) - Z_{11}(\infty)}{2}\,\frac{Z_{22}(0) - Z_{22}(\infty)}{2}}.$$

PROOF: See the proof of the corresponding corollary of Theorem 1.

It should be noted that for a general R, ±L, ±C, T network N, the R_{ijo} and the R_{ijs}

are not properties of the $Z_{ij}(s)$; rather, the R_{ijo} and the R_{ijs} are properties of N. Thus, in general, our bounds on the $Z_{ij}(j\omega)$ cannot be determined directly from the $Z_{ij}(s)$ but must be determined from some network realization of the $Z_{ij}(s)$. In the special cases of RCT and RLT impedances, however, the R_{ijo} and the R_{ijs} are properties of the $Z_{ij}(s)$. In these cases the bounds on the $Z_{ij}(j\omega)$ can be determined directly from the $Z_{ij}(s)$ (Corollary 5 of Theorems 1 and 2).

6. Discussion

It has been assumed here that the R_{ijs} are nonzero and the R_{ijo} noninfinite. A review of the proofs of Theorems 1 and 2 shows that these restrictions are unnecessary; the situations depicted in Figs. XVI-2 and XVI-3 remain valid in these limiting cases.

When $R_{iis} = 0$, the allowable disk of Fig. XVI-2 becomes tangent to the imaginary axis at the origin. If $R_{iio} = \infty$, the allowable disk enlarges to become the half plane defined by $\text{Re}\,[Z] \geq R_{iis}$. When both $R_{iis} = 0$ and $R_{iio} = \infty$, the allowable disk enlarges to become the entire right half plane of the Z-plane (including the imaginary axis).

If any of the R_{ijs} equal zero, the situation shown in Fig. XVI-3 continues to hold. If any of the R_{ijo} are infinite this situation also holds, but the radius of the allowable disk becomes infinite and the allowable region becomes the entire Z-plane.

It is interesting to note that three classical types of driving-point impedances require the limiting disks described above. These cases are as follows:

(i) $Z_{ii}(j\omega)$ has a zero at $s = j\omega_o$;

(ii) $Z_{ii}(j\omega)$ has a pole at $s = j\omega_o$;

(iii) $Z_{ii}(j\omega)$ is minimum resistive at $s = j\omega_o$ [that is, $\text{Re}\,[Z_{ii}(j\omega_o)] = 0 \quad \text{but} \quad \text{Im}\,[Z_{ii}(j\omega_o)] \neq 0$].

When $Z_{ii}(j\omega)$ has a j-axis zero, the bounding circle must pass through the origin of the Z-plane to accommodate the zero value of $Z_{ii}(j\omega_o)$. When $Z_{ii}(j\omega)$ has a j-axis pole, the bounding circle must become a vertical line to accommodate the infinite magnitude of $Z_{ii}(j\omega_o)$. When $Z_{ii}(j\omega)$ is minimum resistive, the bounding circle must become the imaginary axis of the Z-plane to accommodate the value $Z_{ii}(j\omega_o) = jX$.

It should be noted that our main theorem is basically a mapping theorem. The theorem states that the impedance function $Z_{ij}(s)$ maps the j-axis of the s-plane into the closed circular disk of the Z-plane defined by (2). In this connection we should like to point out that the following stronger mapping theorem applies if attention is restricted to the driving-point impedances of RLCT networks.

THEOREM 4: Any driving-point impedance $Z_{ii}(s)$ of an RLCT network N maps the right half of the s-plane ($\text{Re}\,[s] \geq 0$) into the closed circular disk of the Z-plane defined by (1).

PROOF: Let N be any RLCT network, and let $Z_{ii}(s)$ be a driving-point impedance of N. Consider the related impedance function $Z'_{ii}(s) = Z_{ii}(s+a)$, where $a \geq 0$. $Z'_{ii}(s)$

can be regarded as the impedance of a new network N' obtained from N (i) by placing a resistor of value aL_m in series with each inductor L_m of N, and (ii) by placing a conductance of value aC_n in parallel with each capacitor C_n of N. Application of Theorem 1 to $Z'_{ii}(j\omega)$ shows that the locus of $Z_{ii}(a+j\omega)$ $[a \geq 0]$ lies within the closed circular disk of the Z-plane defined by

$$\left| Z - \frac{R'_{iio} + R'_{iis}}{2} \right| \leq \frac{R'_{iio} - R'_{iis}}{2}, \tag{10}$$

where R'_{iis} is the impedance Z'_{ii} of N' when all reactive elements are short circuited, and R'_{iio} is the impedance Z'_{ii} of N' when all reactive elements are open circuited.

Now it is evident that $R_{iis} \leq R'_{iis}$ and $R'_{iio} \leq R_{iio}$. This fact shows that the disk of the Z-plane defined by (1) encloses that defined by (10) which in turn encloses the locus of $Z_{ii}(a+j\omega)$ $[a \geq 0]$. Thus the disk defined by (1) encloses the locus of $Z_{ii}(a+j\omega)$ $[a \geq 0]$. Q.E.D.

T. S. Huang, H. B. Lee

References

1. E. A. Guillemin, The Theory of Linear Physical Systems (John Wiley and Sons, Inc., New York, 1963), p. 127.

2. E. A. Guillemin, Synthesis of Passive Networks (John Wiley and Sons, Inc., New York, 1957), p. 7.