COMMUNICATION SCIENCES

AND

ENGINEERING

X. STATISTICAL COMMUNICATION THEORY*

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A. WORK COMPLETED

1. Oven Controller Using a Two-State Modulation System

This study has been completed by S. Greenblatt. It was submitted as a thesis in partial fulfillment of the requirements for the degree of Master of Science, Department of Electrical Engineering, M. I. T., August 1964.

A. G. Bose

2. Optimum Pre-emphasis with a Bandwidth Constraint

This study has been completed by C. J. Boardman. It was submitted as a thesis in partial fulfillment of the requirements for the degree of Master of Science, Department of Electrical Engineering, M. I. T., August 1964.

H. L. Van Trees

3. Simulation of Phase-Locked Loop Demodulators

This study has been completed by M. A. Rich. It was submitted as a thesis in partial fulfillment of the requirements for the degree of Master of Science, Department of Electrical Engineering, M. I. T., August 1964.

H. L. Van Trees

B. CHARACTERIZATION OF A CLASS OF NONLINEAR SYSTEMS

A generalization of the principle of superposition to encompass a class of nonlinear systems has been discussed previously.[1,2] A detailed investigation of this class, referred to as the class of homomorphic systems, appears in the author's doctoral thesis.[3] The inputs and outputs associated with a homomorphic system are represented as vectors in a vector space under appropriate definitions of vector addition and scalar multiplication. With this representation, the associated transformation can be interpreted as a linear transformation between

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vector spaces, thereby permitting the notation and theorems of linear algebra to be applied to the study of homomorphic systems. The investigation of homomorphic systems centered around the determination of a representation for systems in the class and a study of the generality of the class.

The class of homomorphic systems has been shown to include all invertible systems, that is, all systems for which each output has associated with it only one input. More generally, a necessary and sufficient condition for a system to be homomorphic is that the system inputs can be divided into sets with the properties that

1. Each input in a given set produces the same output and all inputs producing the same output belong to the same set.

2. The sets are the cosets in a quotient space associated with the vector space of inputs.

The application of the theory of homomorphic systems to specific areas in which nonlinear systems arise is now being studied.

A. V. Oppenheim

References

1. A. V. Oppenheim, Generalized superposition, Quarterly Progress Report No. 72, Research Laboratory of Electronics, M. I. T., January 15, 1964, pp. 176-183.

2. A. V. Oppenheim, Superposition in a class of nonlinear systems, IEEE International Convention Record Part I, March 1964, pp. 171-177.

3. A. V. Oppenheim, Superposition in a Class of Nonlinear Systems, Sc. D. Thesis, Department of Electrical Engineering, M. I. T., June 1964.

C. TRANSFORM ANALYSIS OF NONLINEAR SYSTEMS WITH SAMPLED INPUT AND OUTPUT

For continuous nonlinear systems, George[1] demonstrated a transform analysis using a multidimensional Laplace transform. This transform analysis is applicable to a wide class of nonlinear systems. Once the techniques of the method are understood, the analysis of systems of this class is straightforward. In this report we present a multidimensional z-transform and show how it may be applied to the analysis of nonlinear systems with sampled input and output. Use of this transform facilitates the analysis of nonlinear sampled data systems in the same way that the multidimensional Laplace or Fourier transforms facilitate the analysis of continuous nonlinear systems.

The use of the multidimensional z-transform in the analysis of discrete nonlinear systems has been suggested by Alper.[2]

1. Systems with Sampled Input and Output

A nonlinear system with sampled input and output is shown in Fig. X-1. The sampling interval of the input sampler is T_i and that of the output sampler is T_o. The

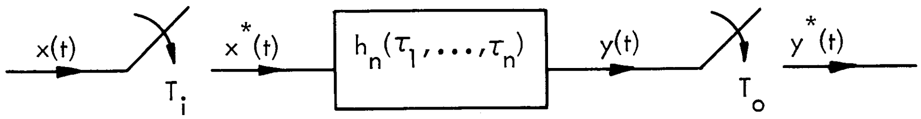

Fig. X-1. Nonlinear system with sampled input and output.

input is x(t) and the sampled input is $x^*(t)$; the output is y(t) and the sampled output is $y^*(t)$. The nonlinear system is assumed to be representable by the n^{th} term of a Volterra series and hence is characterized by the n^{th} degree kernel $h_n(\tau_1, \dots, \tau_n)$. The input-output relationship of the nonlinear system is given by

$$y(t) = \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} h_n(\tau_1, \dots, \tau_n)\, x(t-\tau_1) \dots x(t-\tau_n)\, d\tau_1 \dots d\tau_n. \tag{1}$$

We will assume that the output of each sampler is a sequence of impulses with areas equal to the amplitude of the input to the sampler at the sampling time, so that

$$x^*(t) = \sum_{k=-\infty}^{\infty} x(kT)\, u_0(t-kT). \tag{2}$$

We will assume that $h_n(\tau_1, \dots, \tau_n)$ contains no impulsive components. Then substitution of Eq. 2 into Eq. 1 gives for y(t)

$$y(t) = \sum_{k_n=-\infty}^{\infty} \dots \sum_{k_1=-\infty}^{\infty} x(k_1T_i) \dots x(k_nT_i)\, h_n(t-k_1T_i, \dots, t-k_nT_i).$$

We will assume initially that the input and output samplers operate synchronously and with the same sampling interval $T_i = T_o = T$. Then we have for $y^*(t)$ a sequence of impulses of areas

$$y(pT) = \sum_{k_n=-\infty}^{\infty} \dots \sum_{k_1=-\infty}^{\infty} x(k_1T) \dots x(k_nT)\, h_n(pT-k_1T, \dots, pT-k_nT). \tag{3}$$

We note that only the values of the kernel at isolated multidimensional sample points determine the output sample values; the remainder of the kernel is not important.

2. Multidimensional z-Transforms

The multidimensional z-transform of a function $f(\tau_1, \dots, \tau_n)$ may be defined as

$$F(z_1, \dots, z_n) = \sum_{k_n=-\infty}^{\infty} \dots \sum_{k_1=-\infty}^{\infty} f(k_1 T, \dots, k_n T)\, z_1^{-k_1} \dots z_n^{-k_n}.$$

Convergence properties and other questions of mathematical rigor will not be discussed in this report.

The sample values $f(k_1 T, \dots, k_n T)$ may be obtained from $F(z_1, \dots, z_n)$ by expanding $F(z_1, \dots, z_n)$ in a Taylor's series in $z_1, \dots, z_n$. Note that the coefficients in the expansion are the sample values. If $F(z_1, \dots, z_n)$ is a ratio of polynomials, this expansion can be carried out by division of numerator by the denominator. The sample values can also be obtained from $F(z_1, \dots, z_n)$ by means of the integral

$$f(k_1 T, \dots, k_n T) = \left(\frac{1}{2\pi j}\right)^n \oint_{\Gamma_n} \dots \oint_{\Gamma_1} z_1^{k_1 - 1} \dots z_n^{k_n - 1}\, F(z_1, \dots, z_n)\, dz_1 \dots dz_n$$

where Γ_i is an appropriate contour in the z_i-plane.

The z-transform has a multidimensional convolution property which makes it useful for the analysis of systems with sampled input and output. If $F(z_1, \dots, z_n)$ is the transform of $f(\tau_1, \dots, \tau_n)$ and $G(z_1, \dots, z_n)$ is the transform of $g(\tau_1, \dots, \tau_n)$, then

$$H(z_1, \dots, z_n) = F(z_1, \dots, z_n)\, G(z_1, \dots, z_n)$$

is the transform of $h(\tau_1, \dots, \tau_n)$ with sample values given by

$$h(p_1 T, \dots, p_n T) = \sum_{k_n=-\infty}^{\infty} \dots \sum_{k_1=-\infty}^{\infty} f(k_1 T, \dots, k_n T)\, g(p_1 T - k_1 T, \dots, p_n T - k_n T) \tag{4}$$

3. System Transform Relations

Although Eq. 3 is not quite a multidimensional convolution sum of the form of Eq. 4, we may make use of an artifice suggested by George for continuous systems, introducing the auxiliary function

$$y_{(n)}(p_1 T, \dots, p_n T) = \sum_{k_n=-\infty}^{\infty} \dots \sum_{k_1=-\infty}^{\infty} x(k_1 T) \dots x(k_n T)\, h_n(p_1 T - k_1 T, \dots, p_n T - k_n T)$$

We can then obtain the output sample values as

$$y(pT) = y_{(n)}(p_1T, \ldots, p_nT)\Big|_{p_1=\ldots=p_n=p}$$

Using the convolution property, the transform of the auxiliary function is given by

$$Y_{(n)}(z_1, \ldots, z_n) = H_n(z_1, \ldots, z_n)\, X(z_1) \ldots X(z_n)$$

where $H_n(z_1, \ldots, z_n)$ is the multidimensional z-transform of the system kernel, and $X(z)$ is the one-dimensional z-transform of the system input.

The one-dimensional z-transform of the output, $Y(z)$, can be obtained from $Y_{(n)}(z_1, \ldots, z_n)$ by use of the following integral.

$$Y(z) = \left(\frac{1}{2\pi j}\right)^{n-1} \oint_{\Gamma_{n-1}} \ldots \oint_{\Gamma_1} (w_1w_2\ldots w_{n-1})^{-1}\, Y_{(n)}\left(w_1, \frac{w_2}{w_1}, \ldots, \frac{z}{w_{n-1}}\right) dw_1 \ldots dw_{n-1}.$$

This procedure is analogous to the association of variables used by George for the continuous case and, as in the continuous case, can be carried out as an inspection technique for simple $Y_{(n)}(z_1, \ldots, z_n)$.

If the output sampling interval is not equal to the input sampling interval, Eq. 3 becomes

$$y(pT_o) = \sum_{k_n=-\infty}^{\infty} \ldots \sum_{k_1=-\infty}^{\infty} x(k_1T_i) \ldots x(k_nT_i)\, h_n(pT_o-k_1T_i, \ldots, pT_o-k_nT_i).$$

Let $T_i = rT_o$, where r is a positive integer. Then we have

$$y(pT_o) = \sum_{k_n=-\infty}^{\infty} \ldots \sum_{k_1=-\infty}^{\infty} x(k_1rT_o) \ldots x(k_nrT_o)\, h_n(pT_o-k_1rT_o, \ldots, pT_o-k_nrT_o).$$

Forming the auxiliary function $y_{(n)}(p_1T_o, \ldots, p_nT_o)$ and subsequently the transform $Y_{(n)}(z_1, \ldots, z_n)$ yields now

$$Y_{(n)}(z_1, \ldots, z_n) = H_n(z_1, \ldots, z_n)\, X\left(z_1^r\right) \ldots X\left(z_n^r\right).$$

To study the behavior between sampling instants, we may use a multidimensional modified z-transform. The modified z-transform of a function $f(\tau_1, \ldots, \tau_n)$ may be defined as

$$F(z_1, m_1, \ldots, z_n, m_n) = \sum_{k_n=-\infty}^{\infty} \ldots \sum_{k_1=-\infty}^{\infty} f[k_1T-(1-m_1)T, \ldots, k_nT-(1-m_nT)]\, z_1^{-k_1} \ldots z_n^{-k_n}$$

where $0 \leq m_i < 1$, $i = 1, \ldots, n$. The inverse modified z-transform and the system relations with the modified z-transform may be developed similarly.

A. M. Bush

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