LOCATION OF FACILITIES ON A STOCHASTIC NETWORK

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JOINT WORKING PAPER

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FOREWORD

The research project, "Innovative Resource Planning in Urban Public Safety Systems," is a multidisciplinary activity, supported by the National Science Foundation, and involving faculty and students from the M.I.T Schools of Engineering, Architecture and Urban Planning, and Management. The administrative home for the project is the M.I.T. Operations Research Center. The research focuses on three areas: 1) evaluation criteria, 2) analytical tools, and 3) impacts upon traditional methods, standards, roles, and operating procedures. The work reported in this working paper is associated primarily with category 2, in which a set of analytical and simulation models are developed that should be useful as planning, research, and management tools for urban public safety systems in many cities.

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Richard C. Larson
Principal Investigator

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ABSTRACT

This paper examines the location of medians in a weighted nonoriented network when the weights attached to the links are stochastic. It is well-known that at least one set of absolute medians exists at the nodes in the network when the link weights are deterministic. A similar result is proven for the stochastic case.

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1. Introduction

In several location problems which are abstractly analyzed in terms of networks[1] the important results of Hakimi[2,3] come to mind. His results deal with location of <u>absolute medians</u>* in a network. In the next section we describe his problem and review his result. In the succeeding sections we extend his results to stochastic networks, giving first a motivation why the problem of location on stochastic networks is important.

2. Hakimi's Results

In many "locations on a network" problems there is a "demand" h_i associated with each node v_i**, such as incident rate at the node in an urban service network, or the number of messages originating at the node in a communication network. Associated with each link (i,j)*** is a weight $b(i,j)$ which may, for example, represent travel time from node i to node j in an urban network or a distance or cost of the communication link (i,j) in a communication network. The problem then arises of locating facilities on a network in order to minimize total "cost" of the system. In urban service systems this implies minimizing the expected response time and in the communication systems this implies minimizing total length or cost of communications lines. We shall now introduce some graph theoretic notation and define the "location of medians" problem.

Let the (nonoriented) network, having n nodes and ℓ links, be denoted

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* Definition of absolute median is presented in the next section. Absolute medians are sometimes referred to as weighted medians.

** v_i, 1 = 1,2,...,n denotes the nodes in a network with n nodes.

*** (i,j) denotes link connecting node i to node j.

by G, its set of nodes by N and its set of links by L. For each $v_i \in$ N there is a node weight h_i . For each link $(i,j) \in$ L there is a link weight $b(i,j)$. Arbitrary points in G (which could be either nodes or interior points on the links) will be denoted by x, y, and z, and will be subscripted sometimes. The shortest distance between $x \in$ G and $y \in$ G will be denoted by $d(x,y)$. Since the network is nonoriented, note that $d(x,y) = d(y,x)$. Also, since nodes have integer labels, for convience we let $d(i,x)$, instead of $d(v_i,x)$, denote the shortest distance between node $v_i \in$ N and point $x \in$ G.

Definition 2.1

A point $x^* \in$ G is an absolute median of G, if for every $x \in$ G

$$\sum_{i=1}^{n} h_i d(i,x^*) \leq \sum_{i=1}^{n} h_i d(i,x) \tag{1}$$

It is clear from the definition that locating a facility on an absolute median, when only a single facility is to be located, minimizes expected response time in an urban service system and minimizes total length or cost of lines in a communication system. Hakimi[2] proved that there is always an absolute median which is at a node in the network

Definition 2.2

Let X_K be a set of K points, $x_1, x_2, \ldots, x_K$, in G. The set of K points X_K^* is a set of absolute K-medians if for every X_K in G,

$$\sum_{i=1}^{n} h_i d(i,X_K^*) \leq \sum_{i=1}^{n} h_i d(i,X_K) \tag{2}$$

where $d(i,X_K) = \min[d(i,x_1), d(i,x_2), \ldots, d(i,x_K)]$.

It is clear from the above definition that, when K facilities are to be located, locating the facilities on the absolute K-medians minimize the expected response time in urban service systems or the total cost or length of lines in a communication system. Hakimi[3] also proved that a set of absolute K-medians on a graph exists on the nodes in the network. That is, some $X_K^* \subset N$.

Hakimi's results are important because the problem of locating facili-ties on a network reduces to examining the "cost" at discrete points rather than all the points in the network. Location of K-medians becomes a com-binatorial optimization problem instead of a complex non-linear optimization problem. Several algorithms have been proposed[4,5] which use this fact and examine only nodes in the network to determine the K-medians in a network.

3. Stochastic Networks

The stochastic networks that we are considering are networks where travel times from node to node are not deterministic. In other words, the travel time b(i,j) through link (i,j) is random and has an associated probability distribution. We also let our stochastic networks have infinite expected travel time in some of the links (that is, the links may be "block-ed" or inoperative). However, the main assumption that we require is that the nodes with non-zero demands are always connected. The other minor assumption that we need is that speed of travel on a link is uniform. That is, time to travel a $\Theta < 1$ fraction of link (i,j) is $\Theta b(i,j)$. Hakimi's results dealt with deterministic networks where the travel time in link (i,j) was a deterministic b(i,j). It may be argued that when the network is stochastic, one may use the expected values of b(i,j), $\bar{b}(i,j)$, to prove that the "expected medians" on a network are at the nodes. However, the fact that

d(x,y) is the shortest-distance between x and y and that the "minimum" operator is nonlinear, the absolute median obtained using $\bar{b}(i,j)$ does not in general minimize <u>expected</u> <u>response</u> time, and "expected medians"[*] are not obtained this way. To illustrate, consider the network shown in Figure 1.
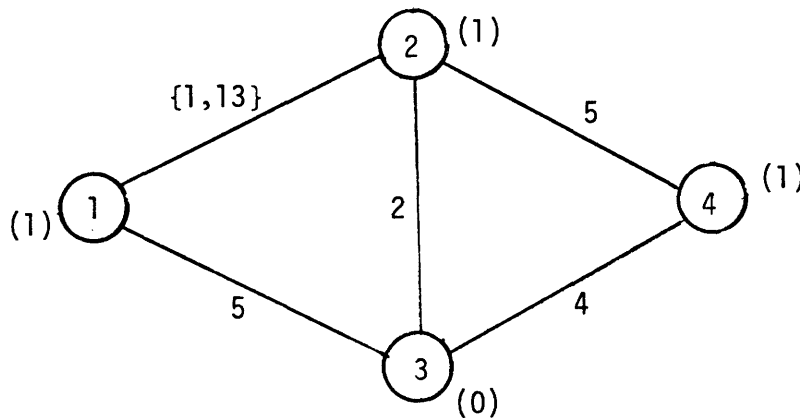

Figure 1: Example of stochastic network. Link (1,2) as 0.5 probability of 1 unit in travel time and 0.5 probability of 13 units in travel time. Nodes 1, 2, and 4 have demands of 1 each and node 3 has zero demand.

Links (1,3), (2,3), (2,4), (3,4) have deterministic travel times of 5, 2, 5, and 4 respectively, and link (1,2) has a 50-50 chance of having a travel time of either 1 or 13 units. The expected travel time in link (1,2) is 7. Using expected travel times on links the absolute median turns out to be node 3, with the expected response time of $3\frac{2}{3}$ units. However, if the facility was located at node 2 then time to respond to incident at node 1 is either 1 or 7 (via node 3, when travel time on (1,2) is 13) and thus the expected travel time is 4. The expected response time to incidents when the facility is located at node 2 becomes 3 units, better than the expected response time

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* Like absolute medians are in a deterministic network, "expected medians are defined to minimize expected response time in a stochastic network. Formal definition of "expected medians" is given in the succeeding section.

at node 3.

Most real networks are stochastic in such a sense. For example in urban service networks, the travel time in most links is not constant, and in some cases due to extreme traffic conditions the links are "blocked" or inoperative. The same holds for communication networks; depending on the load of the system, travel time on the links vary and sometimes links "fail" due to external factors. When there is a loss of connectivity between the demand points and the service facilities we say that the system is in crisis[6] and this problem of optimal location becomes meaningless. This is the basis for our assumption in the succeeding development, that the nodes with non-zero demands are always connected. Since most real networks are stochastic and the existing results and algorithms are not directly applicable to such stochastic networks it is a worthwhile effort to develop such results and algorithms for stochastic networks. It is not obvious where the optimal locations on a network are which minimize expected response time and how these locations are determined. In this paper we prove that a set of expec-ted medians exists at the nodes in the network and in a later paper we propose an algorithm to determine these expected medians.

4. Expected Medians

We first consider the problem of locating a single facility on the stochastic network. For better exposition and easier comparison with Hakimi's results we first consider the case when travel times on the links have a discrete probability distribution over a finite set of values. We then will generalize to the case of continuous probability distributions.

Given the stated assumption we have a finite number of possible travel times on each link and hence finite number of network "states." Each network

state differs from the other by a difference in the travel time of a link. (When the link is inoperative the travel time on that link is infinite.) That is, we have divided the event space into a finest-grain sample space with mutually exclusive and collectively exhaustive listing of all possible network states. Each of the network states has an easily calculable probability of occurrence. Let there be m network states, denoted by $G_1, G_2, \ldots G_m$, with probabilities of occurrence $P_1, P_2, \ldots P_m$ respectively. Let the shortest-distance between points x and y in state G_j be denoted by $d_j(x,y)$, $j = 1, 2, \ldots m$. Let G denote the (nonoriented) stochastic network having a set of n nodes, N, and a set of ℓ links, L. Let $J(x) = \sum_{j=1}^{m} P_j \sum_{i=1}^{n} h_i d_j(i,x)$ denote the "cost" or the objective function for locating a facility at $x \varepsilon G$.

Definition 4.1

A point $x^* \varepsilon G$ is an expected median in G if for every $x \varepsilon G$

$$J(x^*) \leq J(x) \tag{3}$$

when G has discrete network states as defined above.

Lemma 4.1

An expected median cannot be located on a link which has a non-zero probability of failure when G has discrete network states.

Proof (by contradiction): Since the nodes with non-zero demands are always connected, all of the m possible states contain a connected subgraph spanning the non-zero demand nodes. Let this subgraph be denoted by G'. Then for any point $x' \varepsilon G'$

$$J(x') = \sum_{j=1}^{m} P_j \sum_{i=1}^{n} h_i d_j(i,x') \leq \sum_{j=1}^{m} \sum_{i=1}^{n} h_i d_j(i,x') < \infty . \tag{4}$$

The last inequality follows from the fact that for all $h_i > 0$ $d_j(i,x')$ is finite from our definition of x'.

Now assume that the expected median x^* is located on link (p,q) which has a non-zero probability of failure. Let G_k be a state in which link (p,q) is failed and $P_k > 0$. Then

$$J(x^*) = \sum_{j=1}^{m} P_j \sum_{i=1}^{n} h_i d_j(i,x^*) \geq P_k \sum_{i=1}^{n} h_i d_k(i,x^*) = \infty . \qquad (5)$$

The last equality follows from the fact when the link fails the distance between x^* and $i \varepsilon N$ is infinite. From (4) and (5) we have

$$J(x^*) > J(x') \qquad (6)$$

which contradicts the definition of x^*, and thus proves that the expected median cannot be located on a link with non-zero probability of failure. ◆

_Theorem 4.1

   At least one expected median is at a node when G has discrete network states.

Proof: Let $b_i(p,x)$ denote the travel time from x to p on link (p,q) in G_i. Let the expected median be at a point x interior on (p,q) and let $\dfrac{b_i(p,x)}{b_i(p,q)} = \Theta$, $0 < \Theta < 1$.

   $J(x)$ is finite because of Lemma 4.1.

   Let N_{jp} be the set of nodes which communicate most efficiently with point x via v_p when the state of the system is G_j, $j = 1,2,\ldots,m$, (i.e., the shortest path from x to $v_i \varepsilon N_{jp}$ in G_j is through v_p).

   Let N_{jq} be the set of nodes which communicate most efficiently with

point via v_q when the state of the system is G_j, $j = 1, 2, \ldots m$. Note $N_{jp} \cup N_{jq} = N$, $N_{jp} \cap N_{jq} = \phi$. Then

$$J(x) = \sum_{j=1}^{m} P_j \left[\sum_{i \in N_{jp}} h_i (d_j(i,p) + b_j(p,x)) + \sum_{i \in N_{jq}} h_i (d_j(i,q) + b_j(q,x)) \right]. \quad (7)$$

Let $\quad \displaystyle\sum_{j=1}^{m} P_j \sum_{i \in N_{jp}} h_i d_j(i,p) = D_p$, $\quad \displaystyle\sum_{j=1}^{m} P_j \sum_{i \in N_{jq}} h_i d_j(i,q) = D_q$

$$\sum_{j=1}^{m} P_j \sum_{i \in N_{jp}} h_i b_j(p,q) = \alpha_p \quad, \quad \text{and} \quad \sum_{j=1}^{m} P_j \sum_{i \in N_{jq}} h_i b_j(p,q) = \alpha_q \ .$$

Then we can write (7) as

$$J(x) = D_p + D_q + \Theta \alpha_p + (1 - \Theta) \alpha_q \ . \quad (8)$$

Since $J(x)$ is linear in Θ, if $\alpha_p \neq \alpha_q$ we can move the point x, (that is, vary Θ between 0 and 1), keeping the same communicating assignment of nodes as before, and decrease the value of $J(x)$. If $\alpha_p > \alpha_q$ we could decrease Θ to 0, and if $\alpha_p > \alpha_q$ we could increase Θ to 1 to decrease $J(x)$. Thus, for the case $\alpha_p \neq \alpha_q$, x cannot be on (p,q) because we can do better.

If $\alpha_p = \alpha_q$, x at v_p or v_q is as good as x on (p,q). In fact, if N_{jp} and N_{jq} are regrouped to have most efficient communication assignment when x has been moved up to v_p or v_q (i.e., Θ to 0 or 1) we may do better but in no way worse.

Hence, there is at least one expected median on a node. ◆

We now generalize these results to general stochastic networks where links can have any probability distribution, discrete, continuous or mixed. Let the random variable w denote the state of the network, and let the distribution of w be denoted by $F_w(w_0)$. Let the travel time on link (i,j), when $w = w_0$, be denoted $b_{w_0}(i,j)$ and the travel time from $x \varepsilon G$ to $y \varepsilon G$ by $d_{w_0}(x,y)$. For general stochastic networks, link (i,j) is defined to "fail stochastically" if $\int_w dF_w(w_0)b_{w_0}(i,j) = \infty$. Also, in this case, the assumption that the nodes with non-zero demands are always connected will imply that $\int_w dF_w(w_0)d_{w_0}(i,j) < \infty$ where v_i and v_j are any two nodes with non-zero demands. Note also that this assumption is weaker than the assumption that the shortest-distance between any two non-zero demand points is bounded .

Definition 4.2

A point $x^* \varepsilon G$ is an expected median in G if for every $x \varepsilon G$

$$\int_w dF_w(w_0) \sum_{i=1}^{n} h_i d_{w_0}(i,x^*) \leq \int_w dF_w(w_0) \sum_{i=1}^{n} h_i d_{w_0}(i,x) \qquad (9)$$

where G is a general stochastic network.

We now give without proof Lemma 4.2 which is proved similarly as Lemma 4.1 and is a special case of Lemma 5.1.

Lemma 4.2

An expected median cannot be located on a link which fails stochastically when G is a general stochastic network.

We now generalize Theorem 4.1.

Theorem 4.2

At least one expected median is at a node when G is a general stochastic network.

The proof of Theorem 4.2 is similar to that of Theorem 4.1. In fact, Theorem 4.2 is a special case of Theorem 5.1 with K = 1, and hence, its proof will be omitted.

5. Expected K-medians

When there are two or more facilities on a stochastic network, the facility serving a particular demand point will depend on the state of the network. To illustrate this, consider the case of two facilities on a simple network shown in Figure 2.
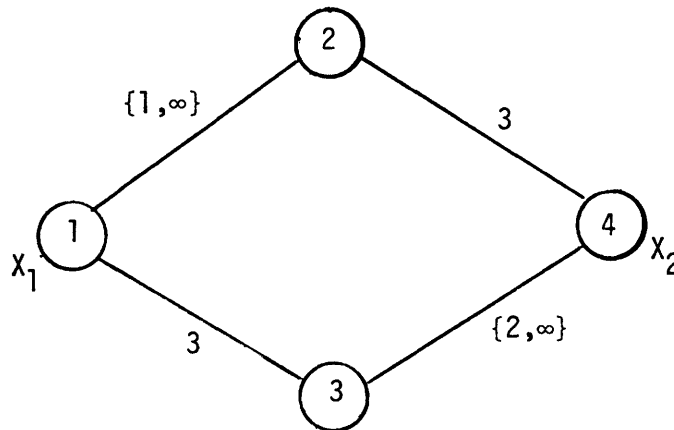

Figure 2: Example of a stochastic network; there is a 0.5 probability that link (1,2) is inoperative and also a 0.5 probability that link (3,4) is inoperative.

Let the two facilities be located at nodes 1 and 4, and let these locations be denoted by x_1 and x_2. When all the links are operative, demand at node 2 is serviced by the facility at x_1 and demand at node 3 is serviced by the facility at x_2. However, there is a 0.25 probability that both the links (1,2) and (3,4) are inoperative. When the links (1,2) and (3,4) are inoperative, facility at x_1 services demands at node 3 while the facility at x_2 services demands at node 2. The existance of such a flip-flop nature in the assignment of demand nodes to the facilities raises the question whether the optimal

locations (to minimize expected response times) of the facilities can still be at nodes in the network?

Definition 5.1

Let X_K be a set of points, $x_1, x_2, \ldots x_K$, is a general stochastic network G. The set of K points X_K^* is a set of expected K-medians if for every $X_K \in G$

$$J(X_K^*) \equiv \int_W dF_W(w_o) \sum_{i=1}^{n} h_i d_{w_o}(i, X_K^*) \leqq \int_W dF_W(w_o) \sum_{i=1}^{n} h_i d_{w_o}(i, X_K) \equiv J(X_K) \qquad (12)$$

where $d_{w_o}(i, X_K) = \min[d_{w_o}(i, x_1), d_{w_o}(i, x_2), \ldots d_{w_o}(i, x_K)]$.

Lemma 5.1

Not all expected medians can be located on links which fail stochastically when G is a general stochastic network.

Proof (by contradiction): Let the set of nodes with non-zero demands be denoted by $N' \in N$. Since the nodes with non-zero demands are always connected by a subgraph $G' \in G$, then for any point $x' \in G'$

$$\int_W dF_W(w_o) \sum_{i=1}^{n} h_i d_{w_o}(i, x') \leq \max_{v_i \in N'} (h_i) \int_W dF_W(w_o) \sum_{v_i \in N'} d_{w_o}(i, x') < \infty .$$

Let X_K' denote facility locations such that the first facility is located at x' and the other $K - 1$ facilities at arbitrary points in G. Then

$$J(X_K') = \int_W dF_W(w_o) \sum_{i=1}^{n} h_i d_{w_o}(i, X_K') < \infty . \qquad (13)$$

Now assume that all the K-medians X_K^* are located on links which can fail stochastically. Thus, from definition of stochastic failure of links

for general stochastic networks

$$\int_W dF_w(w_0) \sum_{v_i \in N'} d_{w_0}(i, X_K^*) = \infty \; .$$

Hence,

$$J(X_K^*) = \int_W dF_w(w_0) \sum_{i=1}^{n} h_i d_{w_0}(i, X_K^*) \geq \min_{v_i \in N'} (h_i) \int_W dF_w(w_0) \sum_{v_i \in N'} d_{w_0}(i, X_K^*) = \infty. \quad (14)$$

From (13) and (14) we have

$$J(X_K^*) > J(X_K^!)$$

which contradicts the definition of X_K^* and thus proves Lemma 5.1. ◆

Theorem 5.1

At least one set of expected K-medians is on the nodes of the network G, when G is a general stochastic network. That is, some $X_K^* \subset N$.

Proof: Let the expected K-medians, X_K, be at points $x_1, x_2, \ldots x_K$ in G. Let $N_{w_0 j}$ be the set of nodes assigned to facility at x_j when the $w = w_0$.

Then the objective function $J(X_K)$, which is finite from Lemma 5.1, can be written as

$$J(X_K) = \int_W dF_w(w_0) \sum_{j=1}^{K} \sum_{i \in N_{w_0 j}} h_i d_{w_0}(i, x_j) \; . \qquad (15)$$

Let the expected K-median x_j be on link (p_j, q_j) and let $\dfrac{b_{w_0}(p_j, x_j)}{b_{w_0}(p_j, q_j)} = \Theta_j$, $0 < \Theta_j < 1$.

Let $N_{w_0 p_j} \varepsilon N_{w_0 j}$ be the set of nodes which communicate most efficiently with x_j via v_{p_j} when $w = w_0$.

Let $N_{w_o q_j} \in N_{w_o j}$ be the set of nodes which communicate most efficiently with x_j via v_{q_j} when $w = w_o$. Then $J(X_K)$ can be written as

$$J(X_K) = T_j + \int_w dF_w(w_o) \left[\sum_{i \in N_{w_o p_j}} h_i (d_{w_o}(i,p_j) + b_{w_o}(p_j,x_j)) \right.$$

$$\left. \sum_{i \in N_{w_o q_j}} h_i (d_{w_o}(i,q_j) + b_{w_o}(q_j,x_j)) \right] \qquad (16)$$

where

$$T_j = \int_w dF_w(w_o) \sum_{\substack{k=1 \\ k \neq j}}^{K} \sum_{i \in N_{w_o k}} h_i d_{w_o}(i,x_k) .$$

We can write (16) as

$$J(X_K) = T_j + D_{jp} + D_{jq} + \Theta_j \alpha_{jp} + (1 - \Theta_j)\alpha_{jq} \qquad (17)$$

where

$$D_{jp} = \int_w dF_w(w_o) \sum_{i \in N_{w_o p_j}} h_i d_{w_o}(i,p_j)$$

$$D_{jq} = \int_w dF_w(w_o) \sum_{i \in N_{w_o q_j}} h_i d_{w_o}(i,q_j)$$

$$\alpha_{jp} = \int_w dF_w(w_o) \sum_{i \in N_{w_o p_j}} h_i b_{w_o}(p_j,q_j)$$

and

$$\alpha_{jq} = \int_w dF_w(w_o) \sum_{i \in N_{w_o q_j}} h_i b_{w_o}(p_j,q_j) .$$

Equation (17) is linear in Θ_j and similar to (8). Thus, with the same reasoning as in Theorem 4.1, we see that either $\Theta_j = 0$ or $\Theta_j = 1$ will give a minimum value of $J(X_K)$. Hence, with x_j at a node and the other x_k, $k \neq j$, at the same position as before, we obtain at least as good a value of $J(X_K)$. Since x_j was any of the K facilities we can repeat the same argument for $j = 1,2,\ldots,K$. Thus, at least one set of expected K-medians is on the nodes of the network. ◆

6. Conclusions

In this paper we extended Hakimi's results[2,3] to stochastic networks where travel time on links is stochastic. We allowed some of the links in the network to have a non-zero probability of becoming inoperative or to fail stochastically as long as the non-zero demand points were always connected. We defined expected medians on such a stochastic network and showed that at least one set of expected medians exist on the nodes. In urban service systems, expected medians minimize expected response time for a unit at a facility to respond to an incident at a node provided there are no queue formations. It can also be shown[6] that if the utility function for time response is convex, then there exists a set of nodes which maximize expected utility for response times. If $b(i,j)$ denotes the cost of communication lines from node i to node j and it is a random variable, then the expected medians minimize the cost of lines in a communication system.

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